

## AN IMPROVEMENT OF SOME INEQUALITIES SIMILAR TO HILBERT'S INEQUALITY

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**ABSTRACT.** We give an improvement of some inequalities similar to Hilbert's inequality involving series of nonnegative terms. The integral analogies of the main results are also given.

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**1. Introduction.** It is well known that the following Hilbert's double series inequality (see [3, page 226]) plays an important role in many branches of mathematics.

**THEOREM 1.1.** *If  $p > 1$ ,  $p' = p/(p-1)$  and  $\sum a_m^p \leq A$ ,  $\sum b_n^{p'} \leq B$ , the summations running 1 to  $\infty$ , then*

$$\sum \sum \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} A^{1/p} B^{1/p'}, \quad (1.1)$$

unless all the sequence  $\{a_m\}$  or  $\{b_n\}$  is null.

The integral analogue of Hilbert's inequality can be stated as follows (see [3, page 226]).

**THEOREM 1.2.** *If  $p > 1$ ,  $p' = p/(p-1)$  and*

$$\int_0^\infty f^p(x) dx \leq F, \quad \int_0^\infty g^{p'}(y) dy \leq G, \quad (1.2)$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} F^{1/p} G^{1/p'}, \quad (1.3)$$

unless  $f \equiv 0$  or  $g \equiv 0$ .

These two theorems were studied extensively and numerous variants, generalizations, and extensions appeared in the literature, see [1, 2, 3, 5, 6, 9] and the references therein.

Recently, Pachpatte [9] gave new inequalities similar to Hilbert's inequalities given in the above theorems, involving a series of nonnegative terms as follows.

**THEOREM 1.3.** *Let  $p \geq 1$ ,  $q \geq 1$ , and let  $\{a_m\}$  and  $\{b_n\}$  be two nonnegative sequences of real numbers defined for  $m = 1, 2, \dots, k$  and  $n = 1, 2, \dots, r$ , where  $k, r$  are the natural*

numbers and define  $A_m = \sum_{s=1}^m a_s$ ,  $B_n = \sum_{t=1}^n b_t$ . Then

$$\sum_{m=1}^k \sum_{n=1}^r \frac{A_m^p B_n^q}{m+n} \leq C(p, q, k, r) \left( \sum_{m=1}^k (k-m+1)(A_m^{p-1} a_m)^2 \right)^{1/2} \times \left( \sum_{n=1}^r (r-n+1)(B_n^{q-1} b_n)^2 \right)^{1/2}, \quad (1.4)$$

unless  $\{a_m\}$  or  $\{b_n\}$  is null, where

$$C(p, q, k, r) = \frac{1}{2} p q \sqrt{kr}. \quad (1.5)$$

An integral analogue of [Theorem 1.3](#) is given in the following theorem.

**THEOREM 1.4.** Let  $p \geq 1$ ,  $q \geq 1$  and  $f(\sigma) \geq 0$ ,  $g(\tau) \geq 0$  for  $\sigma \in (0, x)$ ,  $\tau \in (0, y)$ , where  $x, y$  are positive real numbers and define  $F(s) = \int_0^s f(\sigma) d\sigma$  and  $G(t) = \int_0^t g(\tau) d\tau$ , for  $s \in (0, x)$ ,  $t \in (0, y)$ . Then

$$\int_0^x \int_0^y \frac{F^p(s) G^q(t)}{s+t} ds dt \leq D(p, q, x, y) \left( \int_0^x (x-s)(F^{p-1}(s)f(s))^2 ds \right)^{1/2} \times \left( \int_0^y (y-t)(G^{q-1}(t)g(t))^2 dt \right)^{1/2}, \quad (1.6)$$

unless  $f \equiv 0$  or  $g \equiv 0$ , where

$$D(p, q, x, y) = \frac{1}{2} p q \sqrt{xy}. \quad (1.7)$$

In this paper, we give an improvement of the inequalities given in [Theorems 1.3](#) and [1.4](#) similar to Hilbert's double series inequality and its integral analogue, involving a series of nonnegative terms. In addition, we obtain some new Hilbert type inequalities. These inequalities improve the results obtained by Pachpatte [9].

**2. Main results.** Our main results are given in the following theorems.

**THEOREM 2.1.** Let  $p \geq 1$ ,  $q \geq 1$ ,  $0 < \alpha$ , and let  $\{a_m\}$  and  $\{b_n\}$  be two nonnegative sequences of real numbers defined for  $m = 1, 2, \dots, k$  and  $n = 1, 2, \dots, r$ , where  $k, r$  are the natural numbers and define  $A_m = \sum_{s=1}^m a_s$ ,  $B_n = \sum_{t=1}^n b_t$ . Then

$$\sum_{m=1}^k \sum_{n=1}^r \frac{A_m^p B_n^q}{(m^\alpha + n^\alpha)^{1/\alpha}} \leq C(p, q, k, r; \alpha) \left( \sum_{m=1}^k (k-m+1)(A_m^{p-1} a_m)^2 \right)^{1/2} \times \left( \sum_{n=1}^r (r-n+1)(B_n^{q-1} b_n)^2 \right)^{1/2}, \quad (2.1)$$

unless  $\{a_m\}$  or  $\{b_n\}$  is null, where

$$C(p, q, k, r; \alpha) = \left( \frac{1}{2} \right)^{1/\alpha} p q \sqrt{kr}. \quad (2.2)$$

**PROOF.** By using the following inequality (see [1, 7]),

$$\left(\sum_{m=1}^n z_m\right)^\beta \leq \beta \sum_{m=1}^n z_m \left(\sum_{k=1}^m z_k\right)^{\beta-1}, \tag{2.3}$$

where  $\beta \geq 1$  is a constant and  $z_m \geq 0$ , ( $m = 1, 2, \dots$ ), it is easy to observe that

$$A_m^p \leq p \sum_{s=1}^m a_s A_s^{p-1}, \quad m = 1, 2, \dots, k, \quad B_n^q \leq q \sum_{t=1}^n a_t B_t^{q-1}, \quad n = 1, 2, \dots, r. \tag{2.4}$$

From (2.4) and using the Schwarz inequality and the elementary inequality

$$\left(\prod_{i=1}^n a_i\right)^{1/n} \leq \left(\sum_{i=1}^n \frac{a_i^\alpha}{n}\right)^{1/\alpha}, \quad 0 < \alpha, \text{ (see [4])}, \tag{2.5}$$

(for  $a_i, i = 1, 2, \dots, n$ , nonnegative real numbers) we observe that

$$\begin{aligned} A_m^p B_n^q &\leq pq \left(\sum_{s=1}^m a_s A_s^{p-1}\right) \left(\sum_{t=1}^n a_t B_t^{q-1}\right) \\ &\leq pq(m)^{1/2} \left(\sum_{s=1}^m (a_s A_s^{p-1})^2\right)^{1/2} (n)^{1/2} \left(\sum_{t=1}^n (a_t B_t^{q-1})^2\right)^{1/2} \\ &\leq pq \left(\frac{m^\alpha + n^\alpha}{2}\right)^{1/\alpha} \left(\sum_{s=1}^m (a_s A_s^{p-1})^2\right)^{1/2} \left(\sum_{t=1}^n (a_t B_t^{q-1})^2\right)^{1/2}. \end{aligned} \tag{2.6}$$

Dividing both sides of (2.6) by  $(m^\alpha + n^\alpha)^{1/\alpha}$ , and then taking the sum over  $n$  from 1 to  $r$  first and then the sum over  $m$  from 1 to  $k$  and using the Schwarz inequality and then interchanging the order of the summations (see [7, 8]) we observe that

$$\begin{aligned} &\sum_{m=1}^k \sum_{n=1}^r \frac{A_m^p B_n^q}{(m^\alpha + n^\alpha)^{1/\alpha}} \\ &\leq pq \left(\frac{1}{2}\right)^{1/\alpha} \left\{ \sum_{m=1}^k \left(\sum_{s=1}^m (a_s A_s^{p-1})^2\right)^{1/2} \right\} \left\{ \sum_{n=1}^r \left(\sum_{t=1}^n (b_t B_t^{q-1})^2\right)^{1/2} \right\} \\ &\leq pq \left(\frac{1}{2}\right)^{1/\alpha} (k)^{1/2} \left\{ \sum_{m=1}^k \left(\sum_{s=1}^m (a_s A_s^{p-1})^2\right) \right\}^{1/2} (r)^{1/2} \left\{ \sum_{n=1}^r \left(\sum_{t=1}^n (b_t B_t^{q-1})^2\right) \right\}^{1/2} \\ &= pq\sqrt{kr} \left(\frac{1}{2}\right)^{1/\alpha} \left\{ \sum_{s=1}^k (a_s A_s^{p-1})^2 \left(\sum_{m=s}^k 1\right) \right\}^{1/2} \left\{ \sum_{t=1}^r (b_t B_t^{q-1})^2 \left(\sum_{n=t}^r 1\right) \right\}^{1/2} \\ &= C(p, q, k, r; \alpha) \left\{ \sum_{s=1}^k (a_s A_s^{p-1})^2 (k-s+1) \right\}^{1/2} \left\{ \sum_{t=1}^r (b_t B_t^{q-1})^2 (r-t+1) \right\}^{1/2} \\ &= C(p, q, k, r; \alpha) \left\{ \sum_{m=1}^k (k-m+1) (a_m A_m^{p-1})^2 \right\}^{1/2} \left\{ \sum_{n=1}^r (r-n+1) (b_n B_n^{q-1})^2 \right\}^{1/2}. \end{aligned} \tag{2.7}$$

This completes the proof. □

**REMARK 2.2.** In [Theorem 2.1](#), setting  $\alpha \equiv 1$ , we have [Theorem 1.3](#). If we take  $p = q = 1$  in [Theorem 2.1](#), then the inequality of the result of [Theorem 2.1](#) reduces to the following inequality:

$$\sum_{m=1}^k \sum_{n=1}^r \frac{A_m B_n}{(m^\alpha + n^\alpha)^{1/\alpha}} \leq C(1, 1, k, r; \alpha) \left( \sum_{m=1}^k (k - m + 1)(a_m)^2 \right)^{1/2} \times \left( \sum_{n=1}^r (r - n + 1)(b_n)^2 \right)^{1/2}, \tag{2.8}$$

where  $C(1, 1, k, r; \alpha)$  is obtained by taking  $p = q = 1$  in [\(2.2\)](#).

Our next result deals with further generalization of the inequality obtained in [Remark 2.2](#).

**THEOREM 2.3.** Let  $\{a_m\}$ ,  $\{b_n\}$ ,  $A_m$ , and  $B_n$  be as defined in [Theorem 2.1](#). Let  $\{p_m\}$  and  $\{q_n\}$  be two nonnegative sequences for  $m = 1, 2, \dots, k$  and  $n = 1, 2, \dots, r$ , and define  $P_m = \sum_{s=1}^m p_s$ ,  $Q_n = \sum_{t=1}^n q_t$ . Let  $\phi$  and  $\psi$  be two real-valued, nonnegative, convex, and submultiplicative functions defined on  $\mathbb{R}_+ = [0, \infty)$ . Then

$$\sum_{m=1}^k \sum_{n=1}^r \frac{\phi(A_m)\psi(B_n)}{(m^\alpha + n^\alpha)^{1/\alpha}} \leq M(k, r; \alpha) \left( \sum_{m=1}^k (k - m + 1) \left[ p_m \phi\left(\frac{a_m}{p_m}\right) \right]^2 \right)^{1/2} \times \left( \sum_{n=1}^r (r - n + 1) \left[ q_n \phi\left(\frac{b_n}{q_n}\right) \right]^2 \right)^{1/2}, \tag{2.9}$$

where

$$M(k, r; \alpha) = \left(\frac{1}{2}\right)^{1/\alpha} \left( \sum_{m=1}^k \left[ \frac{\phi(P_m)}{P_m} \right]^2 \right)^{1/2} \left( \sum_{n=1}^r \left[ \frac{\psi(Q_n)}{Q_n} \right]^2 \right)^{1/2}. \tag{2.10}$$

**PROOF.** From the hypotheses of  $\phi$  and  $\psi$  and by using Jensen’s inequality and the Schwarz inequality (see [\[5\]](#)), it is easy to observe that

$$\begin{aligned} \phi(A_m) &= \phi\left(\frac{P_m \sum_{s=1}^m p_s a_s / p_s}{\sum_{s=1}^m p_s}\right) \leq \phi(P_m) \phi\left(\frac{\sum_{s=1}^m p_s a_s / p_s}{\sum_{s=1}^m p_s}\right) \\ &\leq \frac{\phi(P_m)}{P_m} \sum_{s=1}^m p_s \phi\left(\frac{a_s}{p_s}\right) \leq \frac{\phi(P_m)}{P_m} (m)^{1/2} \left( \sum_{s=1}^m \left[ p_s \phi\left(\frac{a_s}{p_s}\right) \right]^2 \right)^{1/2}, \end{aligned} \tag{2.11}$$

and similarly,

$$\psi(B_n) \leq \frac{\psi(Q_n)}{Q_n} (n)^{1/2} \left( \sum_{t=1}^n \left[ q_t \psi\left(\frac{b_t}{q_t}\right) \right]^2 \right)^{1/2}. \tag{2.12}$$

From [\(2.11\)](#) and [\(2.12\)](#) and using the elementary inequality

$$\left( \prod_{i=1}^n a_i \right)^{1/n} \leq \left( \sum_{i=1}^n \frac{a_i^\alpha}{n} \right)^{1/\alpha}, \quad 0 < \alpha, \tag{2.13}$$

(for  $a_i, i = 1, 2, \dots, n$ , nonnegative real numbers) we observe that

$$\begin{aligned} \phi(A_m)\psi(B_n) &\leq \left(\frac{m^\alpha + n^\alpha}{2}\right)^{1/\alpha} \left\{ \frac{\phi(P_m)}{P_m} \left( \sum_{s=1}^m \left[ p_s \phi\left(\frac{a_s}{p_s}\right) \right]^2 \right)^{1/2} \right\} \\ &\quad \times \left\{ \frac{\psi(Q_n)}{Q_n} \left( \sum_{t=1}^n \left[ q_t \psi\left(\frac{b_t}{q_t}\right) \right]^2 \right)^{1/2} \right\}. \end{aligned} \quad (2.14)$$

Dividing both sides of the above inequality by  $(m^\alpha + n^\alpha)^{1/\alpha}$ , and then taking the sum over  $n$  from 1 to  $r$  first and then the sum over  $m$  from 1 to  $k$  and using the Schwarz inequality and then interchanging the order of the summations we observe that

$$\begin{aligned} &\sum_{m=1}^k \sum_{n=1}^r \frac{\phi(A_m)\psi(B_n)}{(m^\alpha + n^\alpha)^{1/\alpha}} \\ &\leq \left(\frac{1}{2}\right)^{1/\alpha} \left[ \sum_{m=1}^k \left\{ \frac{\phi(P_m)}{P_m} \left( \sum_{s=1}^m \left[ p_s \phi\left(\frac{a_s}{p_s}\right) \right]^2 \right)^{1/2} \right\} \right] \\ &\quad \times \left[ \sum_{n=1}^r \left\{ \frac{\psi(Q_n)}{Q_n} \left( \sum_{t=1}^n \left[ q_t \psi\left(\frac{b_t}{q_t}\right) \right]^2 \right)^{1/2} \right\} \right] \\ &\leq \left(\frac{1}{2}\right)^{1/\alpha} \left( \sum_{m=1}^k \left[ \frac{\phi(P_m)}{P_m} \right]^2 \right)^{1/2} \left( \sum_{m=1}^k \left( \sum_{s=1}^m \left[ p_s \phi\left(\frac{a_s}{p_s}\right) \right]^2 \right) \right)^{1/2} \\ &\quad \times \left( \sum_{n=1}^r \left[ \frac{\psi(Q_n)}{Q_n} \right]^2 \right)^{1/2} \left( \sum_{n=1}^r \left( \sum_{t=1}^n \left[ q_t \psi\left(\frac{b_t}{q_t}\right) \right]^2 \right) \right)^{1/2} \\ &= M(k, r; \alpha) \left( \sum_{s=1}^k \left[ p_s \phi\left(\frac{a_s}{p_s}\right) \right]^2 \left( \sum_{m=s}^k 1 \right) \right)^{1/2} \left( \sum_{t=1}^r \left[ q_t \psi\left(\frac{b_t}{q_t}\right) \right]^2 \left( \sum_{n=t}^r 1 \right) \right)^{1/2} \\ &= M(k, r; \alpha) \left( \sum_{s=1}^k \left[ p_s \phi\left(\frac{a_s}{p_s}\right) \right]^2 (k-s+1) \right)^{1/2} \left( \sum_{t=1}^r \left[ q_t \psi\left(\frac{b_t}{q_t}\right) \right]^2 (r-t+1) \right)^{1/2} \\ &= M(k, r; \alpha) \left( \sum_{m=1}^k (k-m+1) \left[ p_m \phi\left(\frac{a_m}{p_m}\right) \right]^2 \right)^{1/2} \left( \sum_{n=1}^r (r-n+1) \left[ q_n \psi\left(\frac{b_n}{q_n}\right) \right]^2 \right)^{1/2}. \end{aligned} \quad (2.15)$$

The proof is complete.  $\square$

**REMARK 2.4.** By applying the elementary inequality (see [4])

$$\left( \prod_{i=1}^n a_i \right)^{1/n} \leq \left( \sum_{i=1}^n \frac{a_i^y}{n} \right)^{1/y}, \quad 0 < y, \quad (2.16)$$

(for  $a_i, i = 1, 2, \dots, n$ , nonnegative real numbers) on the right sides of the inequalities

in Theorems 2.1 and 2.3, we get, respectively, the following inequalities:

$$\begin{aligned} & \sum_{m=1}^k \sum_{n=1}^r \frac{A_m^p B_n^q}{(m^\alpha + n^\alpha)^{1/\alpha}} \\ & \leq C_1 \left( \left( \sum_{m=1}^k (k-m+1)(A_m^{p-1} a_m)^2 \right)^y + \left( \sum_{n=1}^r (r-n+1)(B_n^{q-1} b_n)^2 \right)^y \right)^{1/y}, \end{aligned} \tag{2.17}$$

where  $C_1 = (1/2)^{1/y} C(p, q, k, r; \alpha)$ , and

$$\begin{aligned} & \sum_{m=1}^k \sum_{n=1}^r \frac{\phi(A_m) \psi(B_n)}{(m^\alpha + n^\alpha)^{1/\alpha}} \\ & \leq M_1 \left( \left( \sum_{m=1}^k (k-m+1) \left[ p_m \phi \left( \frac{a_m}{p_m} \right) \right]^2 \right)^y + \left( \sum_{n=1}^r (r-n+1) \left[ q_n \psi \left( \frac{b_n}{q_n} \right) \right]^2 \right)^y \right)^{1/y}, \end{aligned} \tag{2.18}$$

where  $M_1 = (1/2)^{1/y} M(k, r; \alpha)$ , which we believe are new to the literature.

The following theorems deal with slight variants of (2.9) given in Theorem 2.3.

**THEOREM 2.5.** *Let  $\{a_m\}$  and  $\{b_n\}$  be as defined in Theorem 2.1, and define  $A_m = 1/m \sum_{s=1}^m a_s$  and  $B_n = 1/n \sum_{t=1}^n b_t$ , for  $m = 1, 2, \dots, k$  and  $n = 1, 2, \dots, r$ , where  $k, r$  are the natural numbers. Let  $\phi$  and  $\psi$  be two real-valued, nonnegative, and convex functions defined on  $\mathbb{R}_+ = [0, \infty)$ . Then*

$$\begin{aligned} \sum_{m=1}^k \sum_{n=1}^r \frac{mn}{(m^\alpha + n^\alpha)^{1/\alpha}} \phi(A_m) \psi(B_n) & \leq C(1, 1, k, r; \alpha) \left( \sum_{m=1}^k (k-m+1) [\phi(a_m)]^2 \right)^{1/2} \\ & \quad \times \left( \sum_{n=1}^r (r-n+1) [\psi(b_n)]^2 \right)^{1/2}, \end{aligned} \tag{2.19}$$

where  $C(1, 1, k, r)$  is defined by taking  $p = q = 1$  in (2.2).

**PROOF.** From the hypotheses and by using Jensen’s inequality and the Schwarz inequality, it is easy to observe that

$$\begin{aligned} \phi(A_m) & = \phi \left( \frac{1}{m} \sum_{s=1}^m a_s \right) \leq \frac{1}{m} \sum_{s=1}^m \phi(a_s) \leq \frac{1}{m} (m)^{1/2} \left( \sum_{s=1}^m [\phi(a_s)]^2 \right)^{1/2}, \\ \psi(B_n) & = \psi \left( \frac{1}{n} \sum_{t=1}^n b_t \right) \leq \frac{1}{n} \sum_{t=1}^n \psi(b_t) \leq \frac{1}{n} (n)^{1/2} \left( \sum_{t=1}^n [\psi(b_t)]^2 \right)^{1/2}. \end{aligned} \tag{2.20}$$

The rest of the proof can be completed by following the same steps as in the proofs of Theorems 2.1 and 2.3 with suitable changes and hence we omit the details.  $\square$

**THEOREM 2.6.** *Let  $\{a_m\}, \{b_n\}, \{p_m\}, \{q_n\}, P_m$ , and  $Q_n$  be as in Theorem 2.3, and define  $A_m = 1/P_m \sum_{s=1}^m p_s a_s$  and  $B_n = 1/Q_n \sum_{t=1}^n q_t b_t$ , for  $m = 1, 2, \dots, k$  and*

$n = 1, 2, \dots, r$ , where  $k, r$  are the natural numbers. Let  $\phi$  and  $\psi$  be as defined in [Theorem 2.5](#). Then

$$\sum_{m=1}^k \sum_{n=1}^r \frac{P_m Q_n \phi(A_m) \psi(B_n)}{(m^\alpha + n^\alpha)^{1/\alpha}} \leq C(1, 1, k, r; \alpha) \left( \sum_{m=1}^k (k-m+1) [p_s \phi(a_s)]^2 \right)^{1/2} \times \left( \sum_{n=1}^r (r-n+1) [q_t \psi(b_t)]^2 \right)^{1/2}, \quad (2.21)$$

where  $C(1, 1, k, r; \alpha)$  is defined by taking  $p = q = 1$  in [\(2.2\)](#).

**PROOF.** From the hypotheses and by using Jensen's inequality and the Schwarz inequality, it is easy to observe that

$$\begin{aligned} \phi(A_m) &= \phi\left(\frac{1}{P_m} \sum_{s=1}^m p_s a_s\right) \leq \frac{1}{P_m} \sum_{s=1}^m p_s \phi(a_s) \leq \frac{1}{P_m} (m)^{1/2} \left(\sum_{s=1}^m p_s [\phi(a_s)]^2\right)^{1/2}, \\ \psi(B_n) &= \psi\left(\frac{1}{Q_n} \sum_{t=1}^n q_t b_t\right) \leq \frac{1}{Q_n} \sum_{t=1}^n q_t \psi(b_t) \leq \frac{1}{Q_n} (n)^{1/2} \left(\sum_{t=1}^n [q_t \psi(b_t)]^2\right)^{1/2}. \end{aligned} \quad (2.22)$$

The rest of the proof can be completed by following the same steps as in the proofs of [Theorems 2.1](#) and [2.3](#) with suitable changes and hence we omit the details.  $\square$

**3. Integral analogues.** In this section, we present the integral analogues of the inequalities given in [Theorems 2.1, 2.3, 2.5, and 2.6](#), which in fact are motivated by the integral analogue of Hilbert's inequality given in [Theorem 1.2](#).

An integral analogue of [Theorem 2.1](#) is given in the following theorem.

**THEOREM 3.1.** Let  $p \geq 1, q \geq 1, 0 < \alpha \leq 1$  and  $f(\sigma) \geq 0, g(\tau) \geq 0$  for  $\sigma \in (0, x), \tau \in (0, y)$ , where  $x, y$  are positive real numbers, define  $F(s) = \int_0^s f(\sigma) d\sigma$  and  $G(t) = \int_0^t g(\tau) d\tau$ , for  $s \in (0, x), t \in (0, y)$ . Then

$$\begin{aligned} \int_0^x \int_0^y \frac{F^p(s) G^q(t)}{(s^\alpha + t^\alpha)^{1/\alpha}} ds dt &\leq D(p, q, x, y; \alpha) \left( \int_0^x (x-s) (F_f(s))^2 ds \right)^{1/2} \\ &\times \left( \int_0^y (y-t) (G_g(t))^2 dt \right)^{1/2}, \end{aligned} \quad (3.1)$$

unless  $f \equiv 0$  or  $g \equiv 0$ , where  $F_f(s) = F^{p-1}(s)f(s), G_g(t) = G^{q-1}(t)g(t)$ , and

$$D(p, q, x, y; \alpha) = \left(\frac{1}{2}\right)^{1/\alpha} pq\sqrt{xy}. \quad (3.2)$$

**PROOF.** From the hypotheses of  $F(s)$  and  $G(t)$ , it is easy to observe that

$$\begin{aligned} F^p(s) &= p \int_0^s F^{p-1}(\sigma) f(\sigma) d\sigma, \quad s \in (0, x), \\ G^q(t) &= q \int_0^t G^{q-1}(\tau) g(\tau) d\tau, \quad t \in (0, y). \end{aligned} \quad (3.3)$$

From (3.3) and using the Schwarz inequality and the elementary inequality

$$\left(\prod_{i=1}^n a_i\right)^{1/n} \leq \left(\sum_{i=1}^n \frac{a_i^\alpha}{n}\right)^{1/\alpha}, \quad 0 < \alpha, \quad (3.4)$$

(for  $a_i, i = 1, 2, \dots, n$ , nonnegative real numbers) we observe that

$$\begin{aligned} F^p(s)G^q(t) &= pq \left( \int_0^s F^{p-1}(\sigma)f(\sigma)d\sigma \right) \left( \int_0^t G^{q-1}(\tau)g(\tau)d\tau \right) \\ &\leq pq(s)^{1/2} \left( \int_0^s (F^{p-1}(\sigma)f(\sigma))^2 d\sigma \right)^{1/2} (t)^{1/2} \left( \int_0^t (G^{q-1}(\tau)g(\tau))^2 d\tau \right)^{1/2} \\ &\leq pq \left( \frac{s^\alpha + t^\alpha}{2} \right)^{1/\alpha} \left( \int_0^s (F^{p-1}(\sigma)f(\sigma))^2 d\sigma \right)^{1/2} \left( \int_0^t (G^{q-1}(\tau)g(\tau))^2 d\tau \right)^{1/2}. \end{aligned} \quad (3.5)$$

Dividing both sides of the above inequality by  $(s^\alpha + t^\alpha)^{1/\alpha}$ , and then integrating over  $t$  from 0 to  $y$  first and then integrating the resulting inequality over  $s$  from 0 to  $x$  and using the Schwarz inequality we observe that

$$\begin{aligned} &\int_0^x \int_0^y \frac{F^p(s)G^q(t)}{(s^\alpha + t^\alpha)^{1/\alpha}} ds dt \\ &\leq pq \left(\frac{1}{2}\right)^{1/\alpha} \left\{ \int_0^x \left( \int_0^s (F_f(\sigma))^2 d\sigma \right)^{1/2} dt \right\} \left\{ \int_0^y \left( \int_0^t (G_g(\tau))^2 d\tau \right)^{1/2} dt \right\} \\ &\leq pq \left(\frac{1}{2}\right)^{1/\alpha} (xy)^{1/2} \left\{ \int_0^x \left( \int_0^s (F_f(\sigma))^2 d\sigma \right) ds \right\}^{1/2} \left\{ \int_0^y \left( \int_0^t (G_g(\tau))^2 d\tau \right) dt \right\}^{1/2} \\ &= D(p, q, x, y; \alpha) \left( \int_0^x (x-s)(F_f(s))^2 ds \right)^{1/2} \left( \int_0^y (y-t)(G_g(t))^2 dt \right)^{1/2}, \end{aligned} \quad (3.6)$$

where  $F_f(\sigma) = F^{p-1}(\sigma)f(\sigma)$ ,  $G_g(\tau) = G^{q-1}(\tau)g(\tau)$ . This completes the proof.  $\square$

**REMARK 3.2.** In the special case when  $p = q = 1$ , inequality (3.1) in Theorem 3.1 reduces to the following inequality:

$$\begin{aligned} &\int_0^x \int_0^y \frac{F(s)G(t)}{(s^\alpha + t^\alpha)^{1/\alpha}} ds dt \\ &= D(1, 1, x, y; \alpha) \left( \int_0^x (x-s)f^2(s)ds \right)^{1/2} \left( \int_0^y (y-t)g^2(t)dt \right)^{1/2}, \end{aligned} \quad (3.7)$$

where  $D(1, 1, x, y; \alpha)$  is obtained by taking  $p = q = 1$  in (3.2).

The integral analogues of the inequalities in Theorems 2.3, 2.5, and 2.6 are established in the following theorems.



**THEOREM 3.3.** Let  $f, g, F, G$  be as in [Theorem 3.1](#). Let  $p(\sigma)$  and  $q(\tau)$  be two positive functions defined for  $\sigma \in (0, x)$ ,  $\tau \in (0, y)$ , and define  $P(s) = \int_0^s p(\sigma) d\sigma$  and  $Q(t) = \int_0^t q(\tau) d\tau$ , for  $s \in (0, x)$ ,  $t \in (0, y)$ , where  $x, y$  are positive real numbers. Let  $\phi$  and  $\psi$  be as in [Theorem 2.3](#). Then

$$\int_0^x \int_0^y \frac{\phi(Fp(s))\psi(G(t))}{(s^\alpha + t^\alpha)^{1/\alpha}} ds dt \leq L(x, y; \alpha) \left( \int_0^x (x-s) \left[ p(s)\phi\left(\frac{f(s)}{p(s)}\right) \right]^2 ds \right)^{1/2} \times \left( \int_0^y (y-t) \left[ q(t)\psi\left(\frac{g(t)}{q(t)}\right) \right]^2 dt \right)^{1/2}, \tag{3.8}$$

where

$$L(x, y; \alpha) = \left(\frac{1}{2}\right)^{1/\alpha} \left( \int_0^x \left[ \frac{\phi(P(s))}{P(s)} \right]^2 ds \right)^{1/2} \left( \int_0^y \left[ \frac{\psi(Q(t))}{Q(t)} \right]^2 dt \right)^{1/2}. \tag{3.9}$$

**PROOF.** From the hypotheses and by using Jensen's inequality and the Schwarz inequality, it is easy to observe that

$$\begin{aligned} \phi(F(s)) &= \phi\left( \frac{\left[ P(s) \int_0^s P(\sigma) (f(\sigma)/p(\sigma)) d\sigma \right]}{\left[ \int_0^s p(\sigma) d\sigma \right]} \right) \\ &\leq \frac{\phi(P(s))}{P(s)} \int_0^s P(\sigma) \phi\left(\frac{f(\sigma)}{p(\sigma)}\right) d\sigma \\ &\leq \frac{\phi(P(s))}{P(s)} (s)^{1/2} \left( \int_0^s \left[ P(\sigma) \phi\left(\frac{f(\sigma)}{p(\sigma)}\right) \right]^2 d\sigma \right)^{1/2}, \end{aligned} \tag{3.10}$$

and similarly,

$$\psi(G(t)) \leq \frac{\psi(Q(t))}{Q(t)} (t)^{1/2} \left( \int_0^t \left[ q(\tau)\psi\left(\frac{g(\tau)}{q(\tau)}\right) \right]^2 d\tau \right)^{1/2}. \tag{3.11}$$

From [\(3.10\)](#) and [\(3.11\)](#) and using the elementary inequality

$$\left( \prod_{i=1}^n a_i \right)^{1/n} \leq \left( \sum_{i=1}^n \frac{a_i^\alpha}{n} \right)^{1/\alpha}, \quad 0 < \alpha, \tag{3.12}$$

(for  $a_i, i = 1, 2, \dots, n$ , nonnegative real numbers) we observe that

$$\begin{aligned} \phi(F(s))\psi(G(t)) &\leq \left( \frac{s^\alpha + t^\alpha}{2} \right)^{1/\alpha} \left\{ \frac{\phi(P(s))}{P(s)} (s)^{1/2} \left( \int_0^s \left[ P(\sigma) \phi\left(\frac{f(\sigma)}{p(\sigma)}\right) \right]^2 d\sigma \right)^{1/2} \right\} \\ &\quad \times \left\{ \frac{\psi(Q(t))}{Q(t)} (t)^{1/2} \left( \int_0^t \left[ q(\tau)\psi\left(\frac{g(\tau)}{q(\tau)}\right) \right]^2 d\tau \right)^{1/2} \right\}. \end{aligned} \tag{3.13}$$

The rest of the proof can be completed by following the same steps as in the proof of [Theorem 3.1](#) and closely looking at the proof of [Theorem 2.3](#), and hence we omit the details.  $\square$

**THEOREM 3.4.** *Let  $f, g$ , be as in [Theorem 3.1](#), and define  $F(s) = \int_0^s f(\sigma)d\sigma$  and  $G(t) = \int_0^t g(\tau)d\tau$ , for  $s \in (0, x)$ ,  $t \in (0, y)$ , where  $x, y$  are positive real numbers. Let  $\phi$  and  $\psi$  be as in [Theorem 2.5](#). Then*

$$\int_0^x \int_0^y \frac{st}{(s^\alpha + t^\alpha)^{1/\alpha}} \phi(F(s))\psi(G(t))ds dt \leq D(1, 1, x, y; \alpha) \left( \int_0^x (x-s)[\phi(f(\sigma))]^2 ds \right)^{1/2} \left( \int_0^y (y-t)[\psi(g(t))]^2 dt \right)^{1/2}, \quad (3.14)$$

where  $D(1, 1, x, y; \alpha)$  is obtained by taking  $p = q = 1$  in [\(3.2\)](#).

**THEOREM 3.5.** *Let  $f, g, p, q, P$ , and  $Q$  be as in [Theorem 3.3](#), and define  $F(s) = 1/P(s) \int_0^s p(\sigma)f(\sigma)d\sigma$  and  $G(t) = 1/Q(t) \int_0^t q(\tau)g(\tau)d\tau$ , for  $s \in (0, x)$ ,  $t \in (0, y)$ , where  $x, y$  are positive real numbers. Let  $\phi$  and  $\psi$  be as in [Theorem 2.5](#). Then*

$$\int_0^x \int_0^y \frac{P(s)Q(s)\phi(F(s))\psi(G(t))}{(s^\alpha + t^\alpha)^{1/\alpha}} ds dt \leq D(1, 1, x, y; \alpha) \left( \int_0^x (x-s)[p(s)\phi(f(s))]^2 ds \right)^{1/2} \times \left( \int_0^y (y-t)[q(t)\psi(g(t))]^2 dt \right)^{1/2}, \quad (3.15)$$

where  $D(1, 1, x, y; \alpha)$  is obtained by taking  $p = q = 1$  in [\(3.2\)](#).

The proofs of [Theorems 3.4](#) and [3.5](#) can be completed by following the proof of [Theorem 3.3](#) and by closely looking at the proofs of [Theorems 2.5](#) and [2.6](#) and by making use of the integral versions of Jensen's and the Schwarz inequalities. Here, we omit the details.

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