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INTEGRAL MEAN ESTIMATES FOR POLYNOMIALS WHOSE ZEROS ARE WITHIN A CIRCLE

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ABSTRACT. Let $p(z)$ be a polynomial of degree n having all its zeros in $|z| \le k$; $k \le 1$, then for each $r > 0$, $p > 1$, $q > 1$ with $p^{-1} + q^{-1} = 1$, Aziz and Ahemad (1996) recently proved that $n\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\}^{1/r} \le \{\int_0^{2\pi} |1+ke^{i\theta}|^{pr} d\theta\}^{1/pr}\{\int_0^{2\pi} |p'(e^{i\theta})|^{qr} d\theta\}^{1/qr}$. In this paper, we extend the above inequality to the class of polynomials $p(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$; $1 \le \mu \le n$ having all its zeros in $|z| \le k$; $k \le 1$ and obtain a generalization as well as a refinement of the above result.

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1. Introduction and statement of results. Let $p(z)$ be a polynomial of degree n and $p'(z)$ its derivative. If $p(z)$ has all its zeros in $|z| \le 1$, then it was shown by Turan [7] that

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{1.1}$$

Inequality (1.1) is best possible with equality for $p(z) = \alpha z^n + \beta$, where $|\alpha| = |\beta|$. As an extension of (1.1) Malik [4] proved that if $p(z)$ has all its zeros in $|z| \le k$, where $k \le 1$, then

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{1+k} \max_{|z|=1} |p(z)|. \tag{1.2}$$

Malik [5] obtained a generalization of (1.1) in the sense that the right-hand side of (1.1) is replaced by a factor involving the integral mean of $|p(z)|$ on $|z| = 1$. In fact he proved the following theorem.

THEOREM 1.1. *If $p(z)$ has all its zeros in $|z| \le 1$, then for each $r > 0$*

$$n\left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{1/r} \le \left\{\int_0^{2\pi} |1+e^{i\theta}|^r d\theta\right\}^{1/r} \max_{|z|=1} |p'(z)|. \tag{1.3}$$

The result is sharp and equality in (1.3) holds for $p(z) = (z+1)^n$.

If we let $r \to \infty$ in (1.3) we get (1.1). Aziz and Ahemad [1] generalized (1.3) in the sense that $\max_{|z|=1} |p'(z)|$ on $|z| = 1$ on the right-hand side of (1.3) is replaced by a factor involving the integral mean of $|p'(z)|$ on $|z| = 1$ and proved the following result.

Theorem 1.2. *If $p(z)$ is a polynomial of degree n having all its zeros in $|z| \le k \le 1$, then for $r > 0$, $p > 1$, $q > 1$ with $1/p + 1/q = 1$,*

$$n\left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{1/r} \le \left\{\int_0^{2\pi} |1+ke^{i\theta}|^{qr} d\theta\right\}^{1/qr} \left\{\int_0^{2\pi} |p'(e^{i\theta})|^{pr} d\theta\right\}^{1/pr}. \tag{1.4}$$

If we let $r \to \infty$ and $p \to \infty$ (so that $q \to 1$) in (1.4) we get (1.2).

In this paper, we will first extend Theorem 1.2 to the class of polynomials $p(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$, $1 \le \mu \le n$, having all the zeros in $|z| \le k$; $k \le 1$, and thereby obtain a generalization of it. More precisely, we prove the following result.

Theorem 1.3. *If $p(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$, $1 \le \mu \le n$ is a polynomial of degree n having all its zeros in $|z| \le k$; $k \le 1$, then for each $r > 0$, $p > 1$, $q > 1$ with $1/p + 1/q = 1$,*

$$n\left\{\int_0^{2\pi} |p(e^{i\theta})|^r d\theta\right\}^{1/r} \le \left\{\int_0^{2\pi} |1+k^\mu e^{i\theta}|^{pr} d\theta\right\}^{1/pr} \left\{\int_0^{2\pi} |p'(e^{i\theta})|^{qr} d\theta\right\}^{1/qr}. \tag{1.5}$$

Remark 1.4. If we let $r \to \infty$ and $q \to \infty$ (so that $p \to 1$) in (1.5) we get (1.2) for $\mu = 1$.

Our next result is an improvement of Theorem 1.3 which in turn gives a generalization as well as a refinement of Theorem 1.2.

Theorem 1.5. *If $p(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$, $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$; $k \le 1$ and $m = \min_{|z|=k} |p(z)|$, then for every real or complex number β with $|\beta| \le 1$, $r > 0$, $p > 1$, $q > 1$ with $1/p + 1/q = 1$,*

$$\begin{aligned} n&\left\{\int_0^{2\pi} \left|p(e^{i\theta}) + \frac{\beta m e^{i(n-1)\theta}}{k^{n-\mu}}\right|^r d\theta\right\}^{1/r} \\ &\le \left\{\int_0^{2\pi} |1+k^\mu e^{i\theta}|^{pr} d\theta\right\}^{1/pr} \left\{\int_0^{2\pi} |p'(e^{i\theta})|^{qr} d\theta\right\}^{1/qr}. \end{aligned} \tag{1.6}$$

Remark 1.6. Letting $r \to \infty$ and $q \to \infty$ (so that $p \to 1$) in (1.6) and choosing the argument of β suitably with $|\beta| = 1$, it follows that, if $p(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$, $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k$; $k \le 1$, then

$$\max_{|z|=1} |p'(z)| \ge \frac{n}{1+k^\mu}\left\{\max_{|z|=1} |p(z)| + \frac{1}{k^{n-\mu}} \min_{|z|=1} |p(z)|\right\}. \tag{1.7}$$

Inequality (1.7) was recently proved by Aziz and Shah [2].

2. Lemmas.

For the proof of Theorem 1.5 we will make use of the following lemmas.

Lemma 2.1 (see Aziz and Shah [2, Lemma 2]). *If $p(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$ is a polynomial of degree n having all its zeros in $|z| \le k \le 1$, then*

$$|q'(z)| \le k^\mu |p'(z)| \quad \text{for } |z| = 1,\ 1 \le \mu \le n, \tag{2.1}$$

where here and throughout $q(z) = z^n \overline{p(1/\bar{z})}$.

Lemma 2.2 (see Rather [6]). *If $p(z) = a_n z^n + \sum_{v=\mu}^{n} a_{n-v} z^{n-v}$, $1 \le \mu \le n$, is a polynomial of degree n having all its zeros in $|z| \le k \le 1$, and $m = \min_{|z|=k} |p(z)|$, then*

$$k^{\mu}\left|p'(z)\right| \ge \left|q'(z)\right| + \frac{mn}{k^{n-\mu}} \quad \text{for } |z| = 1. \tag{2.2}$$

3. Proof of theorems

Proof of Theorem 1.3. Suppose that $p(z)$ has all its zeros in $|z| \le k \le 1$, therefore, by Lemma 2.1 we have

$$k^{\mu}\left|p'(z)\right| \ge \left|q'(z)\right| \quad \text{for } |z| = 1. \tag{3.1}$$

Also $q(z) = z^n \overline{p(1/\bar{z})}$ so that $p(z) = z^n \overline{q(1/\bar{z})}$, we have

$$p'(z) = nz^{n-1}\overline{q\left(\frac{1}{\bar{z}}\right)} - z^{n-2}\overline{q'\left(\frac{1}{\bar{z}}\right)}. \tag{3.2}$$

Equivalently,

$$zp'(z) = nz^{n}\overline{q\left(\frac{1}{\bar{z}}\right)} - z^{n-1}\overline{q'\left(\frac{1}{\bar{z}}\right)} \tag{3.3}$$

which implies

$$\left|p'(z)\right| = \left|nq(z) - zq'(z)\right| \quad \text{for } |z| = 1. \tag{3.4}$$

Using (3.1) in (3.4) we get

$$\left|q'(z)\right| \le k^{\mu}\left|nq(z) - zq'(z)\right| \quad \text{for } |z| = 1;\ 1 \le \mu \le n. \tag{3.5}$$

Since $p(z)$ has all its zeros in $|z| \le k \le 1$, by the Gauss-Lucas theorem all the zeros of $p'(z)$ also lie in $|z| \le 1$. This implies that the polynomial

$$z^{n-1}\overline{p'\left(\frac{1}{\bar{z}}\right)} = nq(z) - zq'(z) \tag{3.6}$$

has all its zeros in $|z| \ge 1/k \ge 1$.

Therefore, it follows from (3.5) that the function

$$w(z) = \frac{zq'(z)}{k^{\mu}(nq(z) - zq'(z))} \tag{3.7}$$

is analytic for $|z| \le 1$ and $|w(z)| \le 1$ for $|z| \le 1$. Furthermore $w(0) = 0$. Thus the function $1 + k^{\mu}w(z)$ is subordinate to the function $1 + k^{\mu}z$ in $|z| \le 1$. Hence by a well-known property of subordination [3] we have for $r > 0$ and for $0 \le \theta < 2\pi$,

$$\int_0^{2\pi} \left|1 + k^{\mu}w(e^{i\theta})\right|^r d\theta \le \int_0^{2\pi} \left|1 + k^{\mu}e^{i\theta}\right|^r d\theta. \tag{3.8}$$

Also from (3.7), we have

$$1+k^{\mu}w(z)=\frac{nq(z)}{nq(z)-zq'(z)} \tag{3.9}$$

or

$$|nq(z)|=|1+k^{\mu}w(z)||nq(z)-zq'(z)|. \tag{3.10}$$

Using (3.4) and also $|p(z)|=|q(z)|$ in (3.10), we have

$$n|p(z)|=|1+k^{\mu}w(z)||p'(z)| \quad \text{for } |z|=1. \tag{3.11}$$

Combining (3.8) and (3.11) we get

$$n^r\int_0^{2\pi}|p(e^{i\theta})|^r d\theta \le \int_0^{2\pi}|1+k^{\mu}e^{i\theta}|^r|p'(e^{i\theta})|^r d\theta \quad \text{for } r>0. \tag{3.12}$$

Now applying Hölder's inequality for $p>1$, $q>1$ with $1/p+1/q=1$ to (3.12), we get

$$\begin{aligned} n^r&\int_0^{2\pi}|p(e^{i\theta})|^r d\theta \\ &\le \left\{\int_0^{2\pi}|1+k^{\mu}e^{i\theta}|^{pr}d\theta\right\}^{1/p}\left\{\int_0^{2\pi}|p'(e^{i\theta})|^{qr}d\theta\right\}^{1/q} \quad \text{for } r>0 \end{aligned} \tag{3.13}$$

which is equivalent to

$$\begin{aligned} n&\left\{\int_0^{2\pi}|p(e^{i\theta})|^r d\theta\right\}^{1/r} \\ &\le \left\{\int_0^{2\pi}|1+k^{\mu}e^{i\theta}|^{pr}d\theta\right\}^{1/pr}\left\{\int_0^{2\pi}|p'(e^{i\theta})|^{qr}d\theta\right\}^{1/qr} \quad \text{for } r>0 \end{aligned} \tag{3.14}$$

which proves the desired result. □

Proof of Theorem 1.5. Since $p(z)$ has all its zeros in $|z|\le k\le 1$, therefore, by Lemma 2.2 we get

$$k^{\mu}|p'(z)| \ge |q'(z)|+\frac{mn}{k^{n-\mu}} \quad \text{for } |z|=1,\ 1\le\mu\le n. \tag{3.15}$$

Also by (3.4) for $|z|=1$, we have

$$|p'(z)|=|nq(z)-zq'(z)|. \tag{3.16}$$

Now using (3.15) for every complex β with $|\beta|\le 1$, we get

$$\begin{aligned} \left|q'(z)+\bar{\beta}\frac{mn}{k^{n-\mu}}\right| &\le |q'(z)|+\frac{mn}{k^{n-\mu}} \\ &\le k^{\mu}|p'(z)| \\ &= k^{\mu}|nq(z)-zq'(z)| \quad \text{for } |z|=1. \end{aligned} \tag{3.17}$$

Since $p(z)$ has all its zeros in $|z| \le k \le 1$, the result follows on the same lines as that of Theorem 1.3. Hence we omit the proof. □

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