

RELATIONSHIP BETWEEN IDEALS OF BCI-ALGEBRAS AND ORDER IDEALS OF ITS ADJOINT SEMIGROUP

MICHIRO KONDO

(Received 4 October 2000 and in revised form 6 February 2001)

ABSTRACT. We consider the relationship between ideals of a BCI-algebra and order ideals of its adjoint semigroup. We show that (1) if I is an ideal, then $I = M^{-1}(M(I))$, (2) $M(M^{-1}(J))$ is the order ideal generated by $J \cap R(X)$, (3) if X is a BCK-algebra, then $J = M(M^{-1}(J))$ for any order ideal J of X , thus, for each BCK-algebra X there is a one-to-one correspondence between the set $\mathcal{I}(X)$ of all ideals of X and the set $\mathcal{O}(X)$ of all order ideals of it, and (4) the order $M(M^{-1}(J))$ is an order ideal if and only if $M^{-1}(J)$ is an ideal. These results are the generalization of those denoted by Huang and Wang (1995) and Li (1999). We can answer the open problem of Li affirmatively.

2000 Mathematics Subject Classification. 06F35, 03G25.

1. Introduction. In [1, 2, 4], the relationship between *filters* of a BCK-algebra and *order ideals* of its *adjoint semigroup* is considered. There is a gap in the proofs of the results obtained there. For example, [2, Theorem 2.8] and [4, Theorem 2.3], where it is proved for a map $I \rightarrow M(I)$ to be onto, is not correct. They only show that if I is a filter then $M(I)$ is an order ideal and that if J is an order ideal then $M^{-1}(J)$ is a filter. In order to fill the gap of the proof and to develop the theory of adjoint semigroups of BCI-algebras, we consider *ideals* instead of filters and show more results about order ideals. As a particular case, we give an affirmative solution to the problem left open in [4].

First of all, we define *BCI-algebras*.

Let $\langle X; *, 0 \rangle$ be an algebraic structure of type $\langle 2, 0 \rangle$. We call it a *BCI-algebra* when it satisfies the conditions: for any $x, y, z \in X$,

- (1) $(x * y) * (x * z) \leq z * y$,
- (2) $x * (x * y) \leq y$,
- (3) $x * x = 0$,
- (4) if $x * y = y * x = 0$ then $x = y$.

The relation " \leq " is defined as follows:

$$x \leq y \Leftrightarrow x * y = 0. \tag{1.1}$$

It is easy to show that $\langle X; \leq \rangle$ is a partially-ordered set and the following proposition holds.

PROPOSITION 1.1. *Let X be a BCI-algebra. For any $x, y, z \in X$,*

- (a) $x * 0 = x$,
- (b) $0 * (x * y) = (0 * x) * (0 * y)$,

- (c) $(x * y) * z = (x * z) * y,$
- (d) $(x * z) * (y * z) \leq x * y,$
- (e) $x \leq y$ implies $x * z \leq y * z,$
- (f) $x \leq y$ implies $z * x \leq z * y.$

Moreover, a BCI-algebra is called a BCK-algebra if it satisfies the condition

$$(5) 0 * x = 0.$$

Any subset I of X is said to be an *ideal* if

- (I1) $0 \in I,$
- (I2) $x * y, y \in I \Rightarrow x \in I.$

Let X be any BCI-algebra. For an element $a \in X,$ we define a map a^{-1} from X into itself by

$$(x)a^{-1} = x * a. \tag{1.2}$$

The map a^{-1} is called a *right map* in [3]. We also define $M(X)$ and $R(X)$ for any BCI-algebra X as follows:

$$M(X) = \{a_1^{-1}a_2^{-1} \cdots a_n^{-1} \mid a_1, a_2, \dots, a_n \in X\}, \quad R(X) = \{a^{-1} \mid a \in X\}, \tag{1.3}$$

where $a_1^{-1}a_2^{-1} \cdots a_n^{-1}$ is a map defined by

$$(x)a_1^{-1}a_2^{-1} \cdots a_n^{-1} = (\cdots ((x * a_1) * a_2) \cdots) * a_n. \tag{1.4}$$

We call $M(X)$ an *adjoint semigroup* of $X.$ It follows from the properties of BCI-algebras that $M(X)$ is a *commutative monoid* with unit $0^{-1}.$ We introduce a relation \sqsubseteq on $M(X):$

$$b_1^{-1}b_2^{-1} \cdots b_m^{-1} \sqsubseteq a_1^{-1}a_2^{-1} \cdots a_n^{-1} \tag{1.5}$$

$$\Downarrow$$

$$(\cdots ((u * a_1) * a_2) \cdots) * a_n \leq (\cdots ((u * b_1) * b_2) \cdots) * b_m \quad (u \in X).$$

It is obvious that $\langle M(X), \sqsubseteq \rangle$ is a partially ordered set. We note that for any element $a, b \in X,$

$$a^{-1} \sqsubseteq b^{-1} \iff a \leq b. \tag{1.6}$$

In fact, if $a^{-1} \sqsubseteq b^{-1}$ then $u * b \leq u * a$ for every $u \in X.$ If we take $u = a,$ then $a * b = 0$ and thus $a \leq b.$ Conversely, if we suppose $a \leq b$ then, since $(u * b) * (u * a) \leq a * b = 0$ for any $u \in X,$ we have $u * b \leq u * a.$ This means that $a^{-1} \sqsubseteq b^{-1}.$

For any subset S of $X,$ we define

$$M(S) = \{a_1^{-1}a_2^{-1} \cdots a_n^{-1} \mid a_i \in S \ (1 \leq i \leq n)\}. \tag{1.7}$$

A subsemigroup J of $M(X)$ is called an *order ideal* if it satisfies the condition

$$\forall x \in J \text{ and } y \in R(X), \text{ if } y \sqsubseteq x \text{ then } y \in J. \tag{1.8}$$

PROPOSITION 1.2. *Let X be a BCI-algebra. Then the following statements are equivalent:*

- (a) X : BCK-algebra.

(b) 0^{-1} is the smallest element in $M(X)$.

(c) For any element $x^{-1}, y^{-1} \in R(X)$, $x^{-1} \sqsubseteq x^{-1}y^{-1}$.

PROOF. (a) \Rightarrow (b). For every element $a_1^{-1}a_2^{-1} \cdots a_n^{-1} \in M(X)$ ($a_i \in X$) and $u \in X$, since X is a BCK-algebra, we have $((\cdots (u * a_1) * \cdots) * a_n) * (u * 0) = ((\cdots (0 * a_1) * \cdots) * a_n) = 0$. This implies $(\cdots (u * a_1) * \cdots) * a_n \leq u * 0$ for any element $u \in X$. Thus we get $0^{-1} \sqsubseteq a_1^{-1} \cdots a_n^{-1}$ and 0^{-1} is the smallest element of $M(X)$.

(b) \Rightarrow (c). Suppose that 0^{-1} is the smallest element of $M(X)$. Since $0^{-1} \sqsubseteq y^{-1}$ and the operation of $M(X)$ preserves the order \sqsubseteq , we have $x^{-1} = x^{-1}0^{-1} \sqsubseteq x^{-1}y^{-1}$.

(c) \Rightarrow (a). If we take $x^{-1} = 0^{-1}$ then we have $0^{-1} \sqsubseteq y^{-1}$ and hence $0 \leq y$ for any $y \in X$. This means that X is a BCK-algebra. \square

2. Basic properties of ideals and order ideals. In this section, we consider the basic properties of ideals and order ideals. Moreover, we investigate relations between ideals and order ideals. First of all we have the following result.

PROPOSITION 2.1. *If I is an order ideal, then $M(I)$ is an order ideal.*

PROOF. It is obvious that $M(I)$ is a subsemigroup of $M(X)$. Suppose $x \in M(I)$, $y \in R(X)$, and $y \sqsubseteq x$. There are elements $a_1, \dots, a_n \in I$ and $b \in X$ such that $x = a_1^{-1} \cdots a_n^{-1}$ and $y = b^{-1}$. Since $y \sqsubseteq x$, we have, for any $u \in X$,

$$(\cdots (u * a_1) * \cdots) * a_n \leq u * b. \tag{2.1}$$

Especially, if we take $u = b$, then

$$(\cdots (b * a_1) * \cdots) * a_n = 0 \in I. \tag{2.2}$$

Since I is the ideal and $a_i \in I$, we have $b \in I$ and hence $y = b^{-1} \in M(I)$. \square

PROPOSITION 2.2. *For any subsets $S, S' \subseteq X$, if $S \neq S'$ then $M(S) \neq M(S')$, that is, M is an injection.*

PROOF. Suppose that $S \neq S'$ for subsets S, S' of X . There exists an element $a \in X$ such that $a \in S$ but $a \notin S'$. Since $a^{-1} \in M(S)$ and $a^{-1} \notin M(S')$, we get $M(S) \neq M(S')$. Thus the map M is an injection. \square

We define a map M^{-1} by

$$M^{-1}(T) = \{x \mid x^{-1} \in T\} \tag{2.3}$$

for every $T \subseteq M(X)$. It is clear that $M^{-1}(T) = M^{-1}(T \cap R(X))$.

PROPOSITION 2.3. *If J is an order ideal, then $M^{-1}(J)$ is an ideal.*

PROOF. Since J is an order ideal, it is of course a subsemigroup of $M(X)$ and hence $0^{-1} \in J$, that is, $0 \in M^{-1}(J)$.

We assume that $x, y * x \in M^{-1}(J)$ for $x, y \in X$. It follows from the definition of M that $x^{-1}(y * x)^{-1} \in J$. On the other hand, we have $y^{-1} \sqsubseteq x^{-1}(y * x)^{-1}$ by the property of BCI-algebras. This yields that $y^{-1} \in J$ and hence that $y \in M^{-1}(J)$. \square

From the above we have the following relations about ideals and order ideals:

$$\begin{aligned}
 I : \text{ideal} &\implies M(I) : \text{order ideal} \implies M^{-1}(M(I)) : \text{ideal}, \\
 J : \text{order ideal} &\implies M^{-1}(J) : \text{ideal} \implies M(M^{-1}(J)) : \text{order ideal}.
 \end{aligned}
 \tag{2.4}$$

It is natural to ask whether I is identical with $M^{-1}(M(I))$ or so is J with $M(M^{-1}(J))$. By simple calculation, we see $I \subseteq M^{-1}(M(I))$ and $M(M^{-1}(J)) \subseteq J$. We answer the question for the case of ideals.

In general, we have the following result.

THEOREM 2.4. *If S is a subset of X with 0 , then $S = M^{-1}(M(S))$.*

PROOF. Let x be any element of S . It follows from the definition of M that $x^{-1} \in M(S)$ and that $x \in M^{-1}(M(S))$, which implies that $S \subseteq M^{-1}(M(S))$. Thus we have $M(S) \subseteq M(M^{-1}(M(S)))$.

Suppose that S is a subset of X with 0 . It follows that $M(S)$ is a subsemigroup. In fact, we get $0^{-1} \in M(S)$ by $0 \in S$. For any element $\alpha = a_1^{-1} \cdots a_n^{-1}$ and $\beta = b_1^{-1} \cdots b_m^{-1}$ ($a_i, b_j \in S$ ($1 \leq i \leq n, 1 \leq j \leq m$))), we have $\alpha \cdot \beta = a_1^{-1} \cdots a_n^{-1} \cdot b_1^{-1} \cdots b_m^{-1} \in M(S)$. This means that $M(S)$ is the subsemigroup. If $x \in M(M^{-1}(M(S)))$, then there exist $a_i \in M^{-1}(M(S))$ ($1 \leq i \leq n$) such that $x = a_1^{-1} \cdots a_n^{-1}$. Since $a_i^{-1} \in M(S)$ and $M(S)$ is the subsemigroup, it follows from $x = a_1^{-1} \cdots a_n^{-1} \in M(S)$ that $M(M^{-1}(M(S))) \subseteq M(S)$. These imply that $M(S) = M(M^{-1}(M(S)))$ and $S = M^{-1}(M(S))$ by [Proposition 2.2](#). □

COROLLARY 2.5. *If I is an ideal, then $I = M^{-1}(M(I))$.*

3. Relations between order ideals J and $M(M^{-1}(J))$. We proceed to investigate relations between order ideals J and $M(M^{-1}(J))$. In general, we have $M(M^{-1}(J)) \subseteq J$ for any order ideal J . Now the following question arises:

Are always $J = M(M^{-1}(J))$? or Under what condition do we get $J = M(M^{-1}(J))$?

THEOREM 3.1. *For any order ideal J , $M(M^{-1}(J))$ is the order ideal generated by $J \cap R(X)$.*

PROOF. It is clear that $M(M^{-1}(J))$ is an order ideal and $J \cap R(X) \subseteq M(M^{-1}(J))$.

To establish a proof of the theorem, we only have to show that $M(M^{-1}(J)) \subseteq K$ for any order ideal K containing $J \cap R(X)$. If $x \in M(M^{-1}(J))$ then there are elements $a_i \in M^{-1}(J)$ such that $x = a_1^{-1} \cdots a_n^{-1}$. From $a_i^{-1} \in J$, we have $a_i^{-1} \in J \cap R(X) \subseteq K$. Since K is an order ideal, we also have $x = a_1^{-1} \cdots a_n^{-1} \in K$ and hence $M(M^{-1}(J)) \subseteq K$. □

We denote by $[T]$ the order ideal generated by $T \subseteq M(X)$.

COROLLARY 3.2. *For any BCI-algebra X , $M(X)$ is the order ideal generated by $R(X)$, that is, $M(X) = (R(X))$.*

To ask whether $J = M(M^{-1}(J))$ is equivalent to do $J = (J \cap R(X))$ for every order ideal J . As the next example shows, we cannot in general conclude $J = M(M^{-1}(J))$.

EXAMPLE 3.3. Let $X = \{0, 1, 2, \dots, a\}$ and let $*$ be an operation defined as in the following table:

0	0	1	2	3	...	a
0	0	0	0	0	...	a
1	1	0	0	0	...	a
2	2	1	0	0	...	a
3	3	2	1	0	...	a
⋮	⋮	⋮	⋮	⋮	⋮	⋮
a	a	a	a	a	...	0

It turns out that X is a BCI-algebra and both $J = \{0^{-1}, 1^{-1}, 2^{-1}, \dots, a^{-1}a^{-1}\}$ and $J \cap R(X) = \{0^{-1}, 1^{-1}, 2^{-1}, \dots\}$ are order ideals. This means that the order ideal $M(M^{-1}(J))$ generated by $J \cap R(X)$ is identical with $J \cap R(X)$ itself. Hence we have $J \neq M(M^{-1}(J))$.

In spite of the above, in case of BCK-algebras, we can show $J = M(M^{-1}(J))$. It is proved in [4] that M^{-1} is an injection in case of X being a finite BCK-algebra. We can prove the same result for any BCK-algebra without the assumption being finite.

THEOREM 3.4. *Let X be a BCK-algebra and J an order ideal. Then $J = M(M^{-1}(J))$.*

PROOF. It suffices to prove that $J \subseteq M(M^{-1}(J))$. Suppose that $x \in J (\subseteq M(X))$. There exist $a_i \in X$ such that $x = a_1^{-1} \cdots a_n^{-1}$. Since X is a BCK-algebra, we have $a_i^{-1} \subseteq a_1^{-1} \cdots a_n^{-1} = x \in J$ for each i . From $a_i^{-1} \in J$ and $a_i \in M^{-1}(J)$, we conclude that $x = a_i^{-1} \cdots a_n^{-1} \in M(M^{-1}(J))$. This implies that $J \subseteq M(M^{-1}(J))$, that is, $J = M(M^{-1}(J))$. □

COROLLARY 3.5. *For any BCK-algebra X , I is an ideal if and only if $M(I)$ is an order ideal.*

PROOF. It follows from Proposition 2.2 that if I is an ideal then $M(I)$ is an order ideal. Conversely, let $M(I)$ be any order ideal. Since X is a BCK-algebra, we have $M(I) = M(M^{-1}(M(I)))$ from Theorem 3.4. This implies $I = M^{-1}(M(I))$ because the operator M is injective. It follows that I is the ideal of X . □

THEOREM 3.6. *Let X be a BCI-algebra. Then $M(X) = R(X)$ implies $J = M(M^{-1}(J))$ for any order ideal J .*

PROOF. We only show $J \subseteq M(M^{-1}(J))$. If $x \in J$ then there exist $a_i \in X$ such that $x = a_1^{-1} \cdots a_n^{-1}$. It follows from the assumption that $x = a_1^{-1} \cdots a_n^{-1} = b^{-1}$ for some $b \in X$. Since $x \in J$, we have $b^{-1} \in J$ and hence $x = b^{-1} \in M(M^{-1}(J))$ by $b \in M^{-1}(J)$. This means that $J \subseteq M(M^{-1}(J))$. □

REMARK 3.7. The converses of Theorems 3.4 and 3.6 do not hold as in the next example.

EXAMPLE 3.8. For $X = \{0, a, b, c\}$, we define an operation $*$ as in the following table:

*	0	a	b	c
0	0	0	0	c
a	a	0	a	c
b	b	b	0	c
c	c	c	c	0

It is clear that X is not a BCK-algebra but a BCI-algebra. In this case we also have $M(X) = \{0^{-1}, a^{-1}, b^{-1}, c^{-1}, c^{-1}c^{-1}\}$ and $R(X) = \{0^{-1}, a^{-1}, b^{-1}, c^{-1}\}$, that is, $M(X) \neq R(X)$. But it turns out that $J = M(M^{-1}(J))$ for every order ideal J of X .

4. Relations between I and $M(I)$, J and $M^{-1}(J)$. Two maps M and M^{-1} are both monotone. While $I = M^{-1}(M(I))$ for any ideal I , we have in general $J \neq M(M^{-1}(J))$. This means that M and M^{-1} are not converse maps to each other. We also see that a map M whose domain is restricted to the set of all ideals is an injection, but we do not know about M^{-1} until now. Moreover there is a following problem left open in [4]: let X be a BCI-algebra and $M(X)$ be an adjoint semigroup.

(a) Is a map $I \rightarrow M(I)$ an injection?

(b) Is there a one-to-one correspondence between the set $\mathcal{I}(X)$ of all ideals and the set $\mathcal{O}(X)$ of all order ideals?

In the following, we give affirmative answers to the questions above.

PROPOSITION 4.1. *For every $S \subseteq X$, if $M^{-1}(M(S))$ is an ideal then $M^{-1}(M(S))$ is the ideal generated by S .*

PROOF. We only show that $M^{-1}(M(S)) \subseteq I$ for any ideal I such that $S \subseteq I$. Suppose that $x \in M^{-1}(M(S))$. Since $x^{-1} \in M(S)$, there exist $s_i \in S$ such that $x^{-1} = s_1^{-1} \cdots s_n^{-1}$. It follows that $(\cdots (x * s_1) * \cdots) * s_n = 0 \in I$ and hence $x \in I$ from $s_i \in S \subseteq I$. This yields the desired result. \square

PROPOSITION 4.2. *If I is an ideal then $M(I)$ is an order ideal generated by $\{a^{-1} \mid a \in I\}$.*

PROOF. It is clear from Proposition 2.2 that $M(I)$ is an order ideal and $\{a^{-1} \mid a \in I\} \subseteq M(I)$. Let J be an order ideal of $M(X)$ and $\{a^{-1} \mid a \in I\} \subseteq J$. If $x \in M(I)$ then there exist $a_i \in I$ such that $x = a_1^{-1} \cdots a_n^{-1}$. It follows from $a_i \in I$ that $a_i^{-1} \in \{a^{-1} \mid a \in I\} \subseteq J$. From supposition, J is the order ideal and of course it is a subsemigroup. Since $a_i^{-1} \in J$, we get that $x = a_1^{-1} \cdots a_n^{-1} \in J$. This implies $M(I) \subseteq J$ and hence $M(I)$ is the order ideal generated by $\{a^{-1} \mid a \in I\}$. \square

PROPOSITION 4.3. *For any subset $S \subseteq X$, $M^{-1}(M(S))$ is an ideal if and only if $M(S)$ is an order ideal.*

PROOF. Suppose that $M^{-1}(M(S))$ is an ideal. It is obvious from the definition that $M(M^{-1}(M(S))) \subseteq M(S)$. Since $M^{-1}(M(S))$ is an ideal, it is generated by S from the above, so we have $S \subseteq M^{-1}(M(S))$. This means $M(S) \subseteq M(M^{-1}(M(S)))$ and consequently $M(S) = M(M^{-1}(M(S)))$, that is, $M(S)$ is the order ideal by Proposition 2.2.

If $M(S)$ is an order ideal, then $M^{-1}(M(S))$ is an ideal from Proposition 2.3. \square

THEOREM 4.4. *If J is a subsemigroup of $M(X)$, then*

- (1) $M^{-1}(M(M^{-1}(J))) = M^{-1}(J)$,
- (2) $M(M^{-1}(J))$ is an order ideal if and only if $M^{-1}(J)$ is an ideal.

PROOF. (1) It is clear from assumption of the theorem that $M(M^{-1}(J)) \subseteq J$ and hence $M^{-1}(M(M^{-1}(J))) \subseteq M^{-1}(J)$. On the other hand, it follows from $M^{-1}(J) \subseteq X$ that $x \in M^{-1}(J)$ implies $x^{-1} \in M(M^{-1}(J))$ and $x \in M^{-1}(M(M^{-1}(J)))$. Thus we have $M^{-1}(J) \subseteq M^{-1}(M(M^{-1}(J)))$ and $M^{-1}(M(M^{-1}(J))) = M^{-1}(J)$.

(2) In case of $M(M^{-1}(J))$ being an order ideal, since $M^{-1}(M(M^{-1}(J)))$ is an ideal, $M^{-1}(J)$ is the ideal by the above. The converse is clear. \square

PROPOSITION 4.5. *If I is an ideal, then $x \in I$ if and only if $x^{-1} \in M(I)$.*

PROOF. Suppose that $x^{-1} \in M(I)$. There exist $a_i \in I$ such that $x^{-1} = a_1^{-1} \cdots a_n^{-1}$. Since $(\cdots (x * a_1) * \cdots) * a_n = 0 \in I$, we get $x \in I$. The reverse case is clear. \square

A subset S of X is called *closed* if $S = M^{-1}(M(S))$. It is clear from the definition of closedness that S is a closed set if and only if the condition holds: $x^{-1} \in M(S)$ if and only if $x \in S$.

From the above we have a relationship between an ideal I and an order ideal $M(I)$.

PROPOSITION 4.6. *For any subset I , I is an ideal if and only if I is a closed set and $M(I)$ is an order ideal.*

By [Proposition 2.2](#), M is an injection. For M^{-1} , we have the following proposition.

PROPOSITION 4.7. *For any map M , M^{-1} is an injection if and only if $J = M(M^{-1}(J))$ for any order ideal J .*

PROOF. From [Theorem 4.4](#), we have $M^{-1}(M(M^{-1}(J))) = M^{-1}(J)$ for any order ideal J . Since M^{-1} is injective, we have $J = M(M^{-1}(J))$.

Suppose that $M^{-1}(J) = M^{-1}(K)$ for two order ideals J, K . We obtain $J = M(M^{-1}(J)) = M(M^{-1}(K)) = K$ and hence M^{-1} is injective. \square

Thus, from [Theorem 3.4](#), we have

$$X : \text{BCK-algebra} \implies M^{-1} : \text{injection.} \tag{4.1}$$

This generalizes the following result in [\[2\]](#) (which unfortunately has a gap in its proof):

$$X : \text{finite BCK-algebra} \implies M^{-1} : \text{injection.} \tag{4.2}$$

Also we have from [Proposition 2.2](#) and [Theorem 3.6](#) that for X being a BCI-algebra and each ideal I, I' ,

$$\begin{aligned} I \neq I' &\implies M(I) \neq M(I'), \\ M(X) = R(X) &\implies M^{-1} : \text{injection.} \end{aligned} \tag{4.3}$$

This gives affirmative answers to the Open Problem in [\[4\]](#) which says that: let X be a BCI-algebra and $M(X)$ an adjoint semigroup of it. Then

- (a) Is a map $I \rightarrow M(I)$ injective?
- (b) If $M(X) = R(X)$, then is there a one-to-one correspondence from the set of all ideals of X to the set of all order ideals of $M(X)$?

Because as to (b) if $M(X) = R(X)$ then M^{-1} is injective and it follows from [Theorem 3.6](#) that $J = M(M^{-1}(J))$ for every order ideal J . This implies that M and M^{-1} are converse maps to each other and hence there is a one-to-one correspondence from $\mathcal{I}(X)$ to $\mathcal{O}(X)$.

5. Characterization of some BCI-algebras by $R(X)$ and $M(X)$

PROPOSITION 5.1. *For any BCI-algebra X , $R(X)$ satisfies the condition that for any $a^{-1} \in R(X)$ there is $b^{-1} \in R(X)$ such that $a^{-1}b^{-1} = 0^{-1}$ if and only if X is p -semisimple, that is, $0 * (0 * a) = a$ for every $a \in X$.*

PROOF. Suppose that $R(X)$ satisfies the condition that for for each a^{-1} there exists $b^{-1} \in R(X)$ ($a^{-1}b^{-1} = 0^{-1}$). This implies $(u * a) * b = 0$ for every $u \in X$ and hence $0 * a = b$ by taking $u = b$. We thus have $(u * a) * (0 * a) = u$ and $0 * (0 * a) = a$ by taking $u = a$. This means that X is a p -semisimple BCI-algebra.

It is easy to prove the converse by taking $b = 0 * a$. For, if we take $b = 0 * a$ then it suffices to show that $(x * a) * b = x$ for every $x \in X$. It is clear from the following

$$\begin{aligned}
 (x * a) * (0 * a) &= 0 * (0 * ((x * a) * (0 * a))) \\
 &= 0 * ((0 * (x * a)) * (0 * (0 * a))) \\
 &= 0 * (((0 * x) * (0 * a)) * a) \\
 &= 0 * (((0 * (0 * a)) * x) * a) \\
 &= 0 * ((a * x) * a) \\
 &= 0 * ((a * a) * x) \\
 &= 0 * (0 * x) \\
 &= x.
 \end{aligned}
 \tag{5.1}$$

□

If $R(X)$ satisfies the condition that for any $a^{-1} \in R(X)$ there exists $b^{-1} \in R(X)$ such that $a^{-1}b^{-1} = 0^{-1}$, then $M(X)$ becomes an *abelian group*.

COROLLARY 5.2. *For any BCI-algebra X , if X is p -semisimple then $M(X)$ is an abelian group.*

In [2], *positive implicative BCK-algebras* are characterized by $R(X)$. We also give a characterization of those as follows.

PROPOSITION 5.3. *The following conditions are equivalent to each other:*

- (1) *for every BCI-algebra X , $R(X)$ is idempotent (i.e, for all $a^{-1} \in R(X)$ $a^{-1}a^{-1} = a^{-1}$),*
- (2) *every BCI-algebra X is a positive implicative BCK-algebra (i.e, a BCK-algebra satisfying the condition: for all $u, a \in X$ $(u * a) * a = u * a$),*
- (3) *$\{u \in M(X) \mid u \sqsubseteq a^{-1}\}$ is an order ideal for any $a \in X$.*

PROOF. It is straightforward from the definition of positive implicativeness that (1) \Leftrightarrow (2).

(1) \Rightarrow (3). Let $\Gamma_a = \{u \in M(X) \mid u \sqsubseteq a^{-1}\}$ and $u = x_1^{-1} \cdots x_n^{-1}, v = y_1^{-1} \cdots y_m^{-1}$ be in Γ_a . Since $uv = x_1^{-1} \cdots x_n^{-1} \cdot y_1^{-1} \cdots y_m^{-1} \sqsubseteq a^{-1}a^{-1} = a^{-1}$, Γ_a is a subsemigroup of $M(X)$. If $u = x_1^{-1} \cdots x_n^{-1} \in \Gamma_a$ and $b^{-1} \sqsubseteq u$, then we have $b^{-1} \sqsubseteq a^{-1}$ from $b^{-1} \sqsubseteq u \sqsubseteq a^{-1}$. It follows from $b^{-1} \in \Gamma_a$ that Γ_a is an order ideal.

(3) \Rightarrow (1). If Γ_a is an order ideal, then it is also a subsemigroup. Thus, we have $a^{-1}a^{-1} \in \Gamma_a$ from $a^{-1} \in \Gamma_a$, that is, $a^{-1}a^{-1} \sqsubseteq a^{-1}$. This yields $u * a \leq (u * a) * a$ for any $u \in X$. If we take $u = a$, then we have $0 \leq 0 * a$. It follows from $0 * (0 * a) \leq a$ that $0 \leq a$.

In this case we also have $((u * a) * a) * (u * a) = 0 * a = 0$, $(u * a) * a \leq u * a$ and $(u * a) * a = u * a$. This means $a^{-1}a^{-1} = a^{-1}$ and hence $R(X)$ is idempotent. \square

REFERENCES

- [1] W. Huang, *On the semigroup theory of BCI-algebras*, Pure Appl. Math. **9** (1993), 28-34.
- [2] W. Huang and D. Wang, *Adjoint semigroups of BCI-algebras*, Southeast Asian Bull. Math. **19** (1995), no. 3, 95-98. [MR 96h:06039](#). [Zbl 0859.06016](#).
- [3] M. Kondo, *Some properties of left maps in BCK-algebras*, Math. Japon. **36** (1991), no. 1, 173-174. [CMP 1 093 369](#). [Zbl 0717.06011](#).
- [4] J. Li, *Relationship between filters of BCK-algebra and order ideals of its adjoint semigroup*, Math. Japon. **49** (1999), no. 3, 447-450. [MR 2000d:06028](#). [Zbl 0932.06016](#).

MICHIRO KONDO: DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, SHIMANE UNIVERSITY, MATSUE, 690-8504, JAPAN
E-mail address: kondo@cis.shimane-u.ac.jp