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CHARACTERIZING COMPLETELY MULTIPLICATIVE FUNCTIONS BY GENERALIZED MÖBIUS FUNCTIONS

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Using the generalized Möbius functions, μ_α, first introduced by Hsu (1995), two characterizations of completely multiplicative functions are given; save a minor condition they read $(\mu_\alpha f)^{-1} = \mu_{-\alpha} f$ and $f^\alpha = \mu_{-\alpha} f$.

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1. Introduction. Hsu [6], see also Brown et al. [3], introduced a very interesting arithmetic function

$$\mu_\alpha(n) = \prod_{p\mid n} \binom{\alpha}{\nu_p(n)} (-1)^{\nu_p(n)}, \tag{1.1}$$

where $\alpha \in \mathbb{R}$, and $n = \prod_{p \text{ prime}} p^{\nu_p(n)}$ denotes the prime factorization of n.

This function is called the generalized Möbius function because $\mu_1 = \mu$, the well-known Möbius function. Note that $\mu_0 = I$, the identity function with respect to Dirichlet convolution, $\mu_{-1} = \zeta$, the arithmetic zeta function and $\mu_{\alpha+\beta} = \mu_\alpha * \mu_\beta$; α, β being real numbers. Recall that an arithmetic function f is said to be completely multiplicative if $f(1) \neq 0$ and $f(mn) = f(m)f(n)$ for all m and n. As a tool to characterize completely multiplicative functions, Apostol [1] or Apostol [2, Problem 28(b), page 49], it is known that for a multiplicative function f, f is completely multiplicative if and only if

$$(\mu f)^{-1} = \mu^{-1} f = \mu_{-1} f. \tag{1.2}$$

Our first objective is to extend this result to μ_α.

Theorem 1.1. *Let f be a nonzero multiplicative function and α a nonzero real number. Then f is completely multiplicative if and only if*

$$(\mu_\alpha f)^{-1} = \mu_{-\alpha} f. \tag{1.3}$$

In another direction, Haukkanen [5] proved that if f is a completely multiplicative function and α a real number, then $f^\alpha = \mu_{-\alpha} f$. Here and throughout, all powers refer to Dirichlet convolution; namely, for positive integral α, define $f^\alpha := f * \cdots * f$ (α times) and for real α, define $f^\alpha = \operatorname{Exp}(\alpha \operatorname{Log} f)$, where Exp and Log are Rearick's operators [9]. Our second objective is to establish the converse of this result. There is an additional hypothesis, referred to as condition (NE) which appears frequently. By condition (NE), we refer to the condition that: *if α is a negative even integer, then assume that $f(p^{-\alpha-1}) = f(p)^{-\alpha-1}$ for each prime p.*

Theorem 1.2. *Let f be a nonzero multiplicative function and $\alpha \in \mathbb{R} - \{0,1\}$. Assuming condition (NE), if $f^\alpha = \mu_{-\alpha} f$, then f is completely multiplicative.*

Because of the different nature of the methods, the proof of Theorem 1.2 is divided into two cases, namely, $\alpha \in \mathbb{Z}$ and $\alpha \notin \mathbb{Z}$. As applications of Theorem 1.2, we deduce an extension of Corollary 3.2 in [11] and a modified extension of [7, Theorem 4.1(i)].

2. Proof of Theorem 1.1. If f is completely multiplicative, then $(\mu_\alpha f)^{-1} = \mu_{-\alpha} f$ follows easily from Haukkanen's theorem [5]. To prove the other implication, it suffices to show that $f(p^k) = f(p)^k$ for each prime p and nonnegative integer k. This is trivial for $k = 0,1$. Assuming $f(p^j) = f(p)^j$ for $j = 0,1,\ldots,k-1$, we proceed by induction to settle the case $j = k > 1$. From hypothesis, we get

$$\mu_\alpha f * \mu_{-\alpha} f = I. \tag{2.1}$$

Thus

$$\begin{aligned} 0 = I(p^k) &= \sum_{i+j=k} \mu_{-\alpha}(p^i) f(p^i) \mu_\alpha(p^j) f(p^j) \\ &= (-1)^k \sum_{i+j=k} \binom{-\alpha}{i} \binom{\alpha}{j} f(p^i) f(p^j). \end{aligned} \tag{2.2}$$

Simplifying and using induction hypothesis, we get

$$-\left[\binom{\alpha+k-1}{k} + (-1)^k \binom{\alpha}{k}\right] f(p^k) = \sum_{j=1}^{k-1} \left[(-1)^j \binom{\alpha}{j} \binom{\alpha+k-j-1}{k-j}\right] f(p)^k. \tag{2.3}$$

From Riordan [10, identity (5), page 8], the coefficient of $f(p)^k$ on the right-hand side is equal to

$$\begin{aligned} &0 - \left[(-1)^0 \binom{\alpha}{0} \binom{\alpha+k-1}{k} + (-1)^k \binom{\alpha}{k} \binom{\alpha+k-k-1}{k-k}\right] \\ &= -\left[\binom{\alpha+k-1}{k} + (-1)^k \binom{\alpha}{k}\right] \neq 0 \end{aligned} \tag{2.4}$$

and the desired result follows.

Remark 2.1. (1) To prove the "only if" part of Theorem 1.1, instead of using Haukkanen's result, a direct proof based on [1, Theorem 4(a)] can be done as follows: if f is completely multiplicative, then $(\mu_\alpha f) * (\mu_{-\alpha} f) = (\mu_\alpha * \mu_{-\alpha}) f = \mu_0 f = If = I$.

(2) To prove the "if" part of Theorem 1.1, instead of using [10, identity (5)], a self-contained proof can be done as follows: from $(1+z)^{\alpha} \cdot (1+z)^{-\alpha} = 1$ we infer that, for $k > 1$,

$$\sum_{i+j=k} \binom{-\alpha}{i} \binom{\alpha}{j} = 0 \tag{2.5}$$

which implies

$$\binom{-\alpha}{k} + \binom{\alpha}{k} = -\left[\sum_{i=1}^{k-1} \binom{-\alpha}{i} \binom{\alpha}{k-i}\right]. \tag{2.6}$$

Thus,

$$\begin{aligned} 0 &= \sum_{i+j=k} \binom{-\alpha}{i} \binom{\alpha}{j} f(p^i) f(p^j) \\ &= \left[\binom{-\alpha}{k} + \binom{\alpha}{k}\right] f(p^k) + \left[\sum_{i=1}^{k-1} \binom{-\alpha}{i} \binom{\alpha}{k-i}\right] f(p)^k \end{aligned} \tag{2.7}$$

implies $f(p^k) = f(p)^k$.

3. Proof of Theorem 1.2.

The proof of Theorem 1.2 is much more involved and we treat the integral and nonintegral cases separately. This is because the former can be settled using only elementary binomial identities, while the proof of the latter, which is also valid for integral α, makes use of Rearick logarithmic operator, which deems nonelementary to us.

Proposition 3.1. *Let f be a nonzero multiplicative function and r a positive integer ≥ 2. If $f^r = \mu_{-r} f$, then f is completely multiplicative.*

Proof. Since f is multiplicative, it is enough to show that

$$f(p^k) = f(p)^k, \tag{3.1}$$

where p is a prime and k a nonnegative integer. This clearly holds for $k = 0, 1$. As an induction hypothesis, assume this holds for $0, 1, \ldots, k-1$ (≥ 1).

From

$$(\mu_{-r} f)(p^k) = f^r(p^k), \tag{3.2}$$

we get, using induction hypothesis,

$$\begin{aligned} \binom{-r}{k} (-1)^k f(p^k) &= \sum_{j_1+\cdots+j_r=k} f(p^{j_1}) f(p^{j_2}) \cdots f(p^{j_r}) \\ &= r f(p^k) + f(p)^k \sum_{\substack{j_1+\cdots+j_r=k \\ \text{all } j_i \neq k}} 1. \end{aligned} \tag{3.3}$$

Simplifying, we arrive at

$$\left[\binom{k+r-1}{r-1}-r\right]\left(f(p^k)-f(p)^k\right)=0. \tag{3.4}$$

Since $r\geq 2$, then $\binom{k+r-1}{r-1}-r\neq 0$, and we have the result. $\square$

Remark 3.2. The case $r=1$ is excluded for $\mu_{-1}f=\zeta f$ is always equal to f, and so the assumption is empty. The case $r=0$ is excluded because $I=f^0=\mu_0 f=If$ holds for any arithmetic function f with $f(1)=1$.

Proposition 3.3. *Let f be a nonzero multiplicative function and $-\alpha=r$ a positive integer. Assuming condition (NE), if $f^{-r}=\mu_r f$, then f is completely multiplicative.*

Proof. As in Proposition 3.1, we show by induction that $f(p^k)=f(p)^k$ for prime p and nonnegative integer k, noting that it holds trivially for $k=0,1$. The main assumption of the theorem gives

$$\mu_r f * f^r = I. \tag{3.5}$$

We have, for $k\geq 2$,

$$0=I(p^k)=\sum_{i+j_1+\cdots+j_r=k}(\mu_r f)(p^i)f(p^{j_1})\cdots f(p^{j_r}). \tag{3.6}$$

Using induction hypothesis and [10, identity (5)], the right-hand expression is

$$\begin{aligned}
&\sum_{j_1+\cdots+j_r=k} f(p^{j_1})\cdots f(p^{j_r})+f(p)^k\sum_{i=1}^{k-1}(-1)^i\binom{r}{i}\sum_{j_1+\cdots+j_r=k-i}1+(\mu_r f)(p^k)\\
&\quad = rf(p^k)+f(p)^k\left[\binom{k+r-1}{r-1}-r\right]+f(p)^k\sum_{i=1}^{k-1}(-1)^i\binom{r}{i}\binom{k-i+r-1}{r-1}\\
&\qquad +(-1)^k\binom{r}{k}f(p^k)=\left[r+(-1)^k\binom{r}{k}\right]\left(f(p^k)-f(p)^k\right).
\end{aligned} \tag{3.7}$$

For positive integers r and k (≥ 2), observe that $r+(-1)^k\binom{r}{k}=0$ if and only if $k=r-1$ and k is odd. The conclusion hence follows. $\square$

Remark 3.4. In the case of r being a positive even integer, without an additional assumption on $f(p^{r-1})$, Proposition 3.3 fails to hold as seen from the following example.

Take $r=4$. For each prime p, set

$$f(1)=f(p)=f(p^2)=1,\qquad f(p^3)=0 \tag{3.8}$$

and for $k\geq 4$, define $f(p^k)$ by the relation $(\mu_4 f * f^4)(p^k)=I(p^k)=0$.

Define other values of f by multiplicativity, namely,

$$f(p_1^{a_1}\cdots p_k^{a_k})=f(p_1^{a_1})\cdots f(p_k^{a_k}); \tag{3.9}$$

p_i prime, a_i nonnegative integer. This particular function satisfies $\mu_4 f = f^{-4}$ and is multiplicative, but not completely multiplicative.

Now for the case of nonintegral index, we need one more auxiliary result. For more details about the Rearick logarithm, see [8, 9].

Lemma 3.5. *Let f be an arithmetic function, p a prime, k a positive integer, and let* Log *denote the Rearick logarithmic operator defined by*

$$\operatorname{Log} f(1) = \log f(1),$$
$$\operatorname{Log} f(n) = \frac{1}{\log n} \sum_{d|n} f(d) f^{-1}\left(\frac{n}{d}\right) \log d \quad (n > 1). \tag{3.10}$$

If $f(1) = 1$, $f(p^i) = f(p)^i$ $(i = 1, 2, \ldots, k-1)$, then

$$(\operatorname{Log} f)(p^i) = \frac{f(p)^i}{i} \quad (i = 1, 2, \ldots, k-1). \tag{3.11}$$

Proof. From hypothesis, we have

$$f(1) = f^{-1}(1) = 1, \qquad f^{-1}(p) = -f(p), \tag{3.12}$$

and so

$$\operatorname{Log} f(1) = 0, \qquad \operatorname{Log} f(p) = f(p). \tag{3.13}$$

Next,

$$\operatorname{Log} f(p^2) = \frac{1}{\log p^2} \left[f(p^2) \log p^2 + f(p) f^{-1}(p) \log p \right] = \frac{1}{2} f(p)^2. \tag{3.14}$$

Now proceed by induction noting, as in the lemma of Carroll [4], that $f(1) = 1$ and $f(p^i) = f(p)^i$ $(i = 1, \ldots, k-1)$ imply $f^{-1}(p^i) = 0$ $(i = 2, 3, \ldots, k-1)$. We thus have

$$\begin{aligned} \operatorname{Log} f(p^i) &= \frac{1}{i} \sum_{s+t=i} s f(p^s) f^{-1}(p^t) \\ &= \frac{1}{i} \left(i f(p^i) - (i-1) f(p^{i-1}) f(p) \right) \\ &= \frac{1}{i} f(p)^i. \end{aligned} \tag{3.15}$$

□

Now for the final case, we prove the following proposition.

Proposition 3.6. *Let f be multiplicative and $\alpha \in \mathbb{R} - \mathbb{Z}$. If $f^\alpha = \mu_{-\alpha} f$, then f is completely multiplicative.*

Proof. As before, we proceed by induction on nonnegative integer k to show that $f(p^k) = f(p)^k$ the result being trivial for $k = 0, 1$.

Let D be the log-derivation on the ring of arithmetic functions (cf. [7, 8, 9]). Since

$$D(f^\alpha) = \alpha f^{\alpha-1} * Df = \alpha f^\alpha * D(\operatorname{Log} f) = \alpha \mu_{-\alpha} f * D(\operatorname{Log} f), \tag{3.16}$$

where Log denotes the Rearick logarithmic operator mentioned in Lemma 3.5, then taking derivation D on both sides of $\mu_{-\alpha}f = f^{\alpha}$ and evaluating at p^k, we get

$$\left(\mu_{-\alpha}f(p^k)\right)\log p^k = \alpha\sum_{i+j=k}\left(\mu_{-\alpha}f(p^i)\right)(\operatorname{Log} f)(p^j)\log p^j, \tag{3.17}$$

that is,

$$\begin{aligned}(-1)^k k\binom{-\alpha}{k}f(p^k) = \alpha\bigg[&\binom{-\alpha}{0}k(\operatorname{Log} f)(p^k)\\ &+\cdots+(-1)^{k-1}\binom{-\alpha}{k-1}f(p^{k-1})(\operatorname{Log} f)(p)\bigg].\end{aligned} \tag{3.18}$$

Using induction hypothesis and the lemma, we have

$$\begin{aligned}(-1)^k k\binom{-\alpha}{k}f(p^k) = {}&\alpha\binom{-\alpha}{0}k\left(-\frac{k-1}{k}f(p)^k+f(p^k)\right)\\ &+\cdots+\alpha(-1)^{k-1}\binom{-\alpha}{k-1}f(p^{k-1})f(p)\end{aligned} \tag{3.19}$$

and so with the aid of [10, identity (5)], we get

$$\left[(-1)^k k\binom{-\alpha}{k}-\alpha k\right]f(p^k) = \left[\alpha\binom{\alpha+k-1}{k-1}-\alpha k\right]f(p)^k. \tag{3.20}$$

Since $\alpha\in\mathbb{R}-\mathbb{Z}$, then the coefficients on both sides are the same nonzero real number, which immediately yields the desired conclusion. □

The following corollaries are immediate consequences of Theorem 1.2 and the main theorem in [5].

Corollary 3.7 (cf. [11, Corollary 3.2]). *Let $\alpha\in\mathbb{R}-\{0,1\}$, $k\in\mathbb{R}$, and f a nonzero multiplicative function. Define*

$$E^k(n) = n^k\quad(n\in\mathbb{N}),\qquad \tau=\mu_{-\alpha}f,\qquad \phi_\tau^{(k)} = E^k*\tau. \tag{3.21}$$

If f is completely multiplicative, then $\phi_\tau^{(k)} = E^k * f^{\alpha}$, and the converse is true provided condition (NE) holds.

Corollary 3.8 (cf. [7, Theorem 4.1(i)]). *Let $\alpha\in\mathbb{R}-\{0,1\}$ and f a nonzero multiplicative function. If f is completely multiplicative, then*

$$f*\operatorname{Log}\mu_{-\alpha}f = \alpha(f*\operatorname{Log} f) \tag{3.22}$$

and the converse is true provided that condition (NE) holds.

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