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ON AN INFINITE SERIES FOR $(1+1/x)^x$ AND ITS APPLICATION

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An infinite series for $(1+1/x)^x$ is deduced. As an application, a refinement of Carleman's inequality is achieved.

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The well-known Carleman's inequality states that if $a_n \geq 0$, $n = 1,2,\dots$, and $0 < \sum_{n=1}^{\infty} a_n < \infty$, then

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} a_n. \tag{1}$$

Recently, Yang and Debnath [4] improved (1) to

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \frac{1}{2(n+1)}\right) a_n. \tag{2}$$

In [3], a further refinement of (2) is presented as follows:

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 + \frac{1}{n+1/5}\right)^{-1/2} a_n. \tag{3}$$

The key step in the establishment of inequalities (2) and (3) is aimed at estimates of $(1+1/x)^x$. In this note, we derive an equality for $(1+1/x)^x$ in terms of an infinite series. As an application, we further strengthen inequality (3). The main results of this note are presented as follows.

Theorem 1. *For any $x > 0$,*

$$\left(1 + \frac{1}{x}\right)^x = e\left(1 - \sum_{n=1}^{\infty} \frac{b_n}{(1+x)^n}\right), \tag{4}$$

where $b_n > 0$ and satisfies the recurrence relation

$$b_1 = \frac{1}{2}, \qquad b_{n+1} = \frac{1}{(n+1)(n+2)} - \frac{1}{n+1} \sum_{i=1}^{n} \frac{b_i}{n-i+2}. \tag{5}$$

Carleman's inequality (1) is correspondingly refined as follows.

THEOREM 2. *If $a_n \geq 0$, $n = 1, 2, \ldots$, and $0 < \sum_{n=1}^{\infty} a_n < \infty$, then*

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} < e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{m} \frac{b_k}{(1+n)^k}\right) a_n, \tag{6}$$

where m is any positive integer and $b_k > 0$ is given by (5).

To prove Theorem 1, we now introduce three lemmas.

LEMMA 3. *For $x > 0$, $t = 1/(1+x)$,*

$$\left(1 + \frac{1}{x}\right)^x = e \exp\left(-\sum_{n=1}^{\infty} \frac{t^n}{n(n+1)}\right). \tag{7}$$

PROOF. For $x > 0$, $0 < t = 1/(1+x) < 1$, we have

$$\left(1 + \frac{1}{x}\right)^x = \left(\frac{1}{1-t}\right)^{(1-t)/t} = \exp\left(-\frac{1-t}{t} \ln(1-t)\right). \tag{8}$$

Using the power series

$$\ln(1-t) = -\sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1}, \tag{9}$$

which converges for $0 < t < 1$, we have

$$\begin{aligned} \left(1 + \frac{1}{x}\right)^x &= \exp\left((1-t) \sum_{n=0}^{\infty} \frac{t^n}{n+1}\right) \\ &= \exp\left(1 - \sum_{n=1}^{\infty} \frac{t^n}{n(n+1)}\right) \\ &= e \exp\left(-\sum_{n=1}^{\infty} \frac{t^n}{n(n+1)}\right). \end{aligned} \tag{10}$$

This proves (7) as desired. □

LEMMA 4. *For $0 < t < 1$,*

$$\exp\left(-\sum_{n=1}^{\infty} \frac{t^n}{n(n+1)}\right) = 1 - \sum_{n=1}^{\infty} b_n t^n, \tag{11}$$

where b_n satisfies the recurrence relation (5).

PROOF. Set

$$\begin{aligned} p(t) &= -\sum_{n=1}^{\infty} \frac{t^n}{n(n+1)}, \\ f(t) &= \exp\left(-\sum_{n=1}^{\infty} \frac{t^n}{n(n+1)}\right) = \exp(p(t)). \end{aligned} \tag{12}$$

It is clear that the power series of $p(t)$ converges uniformly for $0 < t < 1$ and $f(0) = \exp(p(0)) = 1$. Therefore, we can expand $f(t)$ as a power series in the form of (11). To show that the recurrence relation (5) holds, by the chain rule, we have

$$b_1 = -f'(0) = -f(0)p'(0) = \frac{1}{2}. \tag{13}$$

Next we have, using the Leibniz rule,

$$f^{(k+1)}(x) = \left(f(x)p'(x)\right)^{(k)} = \sum_{i=0}^{k} \binom{k}{i} f^{(i)}(x)p^{(k-i+1)}(x), \tag{14}$$

where $f^{(i)}$ indicates the ith derivative of $f(x)$ for $i \geq 1$ and $f^{(0)} = f$. By virtue of the facts

$$b_{k+1} = -\frac{f^{(k+1)}(0)}{(k+1)!}, \qquad p^{(i)}(0) = -\frac{i!}{i(i+1)}, \qquad \binom{k}{i} = \frac{k!}{i!(k-i)!}, \tag{15}$$

separating the first term in (14) from the summation, we get

$$b_{k+1} = \frac{1}{(k+1)(k+2)} - \frac{1}{k+1} \sum_{i=1}^{k} \frac{b_i}{k-i+2}, \tag{16}$$

from which the recurrence relation (5) follows. This proves Lemma 4. □

To find b_n in (11), starting with $b_1 = 1/2$, and applying the recurrence relation (5) repeatedly, we obtain

$$\begin{aligned} b_2 &= \frac{1}{6} - \frac{1}{4}b_1 = \frac{1}{24}, \\ b_3 &= \frac{1}{12} - \frac{1}{9}b_1 - \frac{1}{6}b_2 = \frac{1}{48}, \\ b_4 &= \frac{1}{20} - \frac{1}{16}b_1 - \frac{1}{12}b_1 - \frac{1}{8}b_3 = \frac{73}{5760}. \end{aligned} \tag{17}$$

For $n \geq 5$, the computation of b_n is considerably longer and complicated. Implementing the recurrence relation (5) with Maple, we easily find the next six coefficients as follows:

$$\begin{aligned} & b_5 = \frac{11}{1280}, \qquad b_6 = \frac{3625}{580608}, \qquad b_7 = \frac{5525}{1161216}, \\ & b_8 = \frac{5233001}{1393459200}, \qquad b_9 = \frac{1212281}{398131200}, \qquad b_{10} = \frac{927777937}{367873228800}. \end{aligned} \tag{18}$$

Those calculations suggest the following lemma.

Lemma 5. *If b_n satisfies the recurrence relation (5), then $b_n > 0$ for all $n \geq 1$.*

Proof. In view of the recurrence relation (5), we see that $b_{n+1} > 0$ is equivalent to

$$\sum_{i=1}^{n} \frac{b_i}{n-i+2} < \frac{1}{n+2}. \tag{19}$$

We make the inductive hypothesis that (19) is true for all positive integers n. This hypothesis is true for $n = 1$ as $b_1 = 1/2$ and

$$\frac{b_1}{2} = \frac{1}{4} < \frac{1}{3}. \tag{20}$$

Now, by the recurrence relation (5), we have

$$\begin{aligned}
&\frac{1}{k+3} - \sum_{i=1}^{k+1} \frac{b_i}{k-i+3} \\
&= \frac{1}{k+3} - \sum_{i=1}^{k} \frac{b_i}{k-i+3} - \frac{b_{k+1}}{2} \\
&= \frac{1}{k+3} - \sum_{i=1}^{k} \frac{b_i}{k-i+3} - \frac{1}{2(k+1)} \left(\frac{1}{k+2} - \sum_{i=1}^{k} \frac{b_i}{k-i+2} \right) \\
&= \frac{2(k+1)(k+2)-(k+3)}{2(k+1)(k+2)(k+3)} - \sum_{i=1}^{k} \frac{2(k+1)(k-i+2)-(k-i+3)}{2(k+1)(k-i+3)} \frac{b_i}{k-i+2} \\
&= \frac{2k^2+5k+1}{2(k+1)(k+3)} \left\{ \frac{1}{k+2} - \sum_{i=1}^{k} \frac{[2(k+1)(k-i+2)-(k-i+3)](k+3)}{(k-i+3)[2(k+1)(k+2)-(k+3)]} \frac{b_i}{k-i+2} \right\} \\
&> \frac{2k^2+5k+1}{2(k+1)(k+3)} \left\{ \frac{1}{k+2} - \sum_{i=1}^{k} \frac{b_i}{k-i+2} \right\} \\
&> 0,
\end{aligned} \tag{21}$$

from which (19) holds for $n = k+1$. Here we have used the fact

$$\begin{aligned}
&\frac{[2(k+1)(k-i+2)-(k-i+3)](k+3)}{(k-i+3)[2(k+1)(k+2)-(k+3)]} \\
&\qquad = \frac{2(k+1)\big((k-i+2)/(k-i+3)\big)-1}{2(k+1)\big((k+2)/(k+3)\big)-1} < 1, \quad \text{for } 1 \le i \le k
\end{aligned} \tag{22}$$

and the inductive hypothesis for $n = k$. Therefore, the lemma now follows by the principle of mathematical induction. □

Now, we turn to the proof of Theorem 1.

Proof of Theorem 1. By virtue of (7) and (11), taking $t = 1/(1+x)$, we have

$$\left(1 + \frac{1}{x}\right)^x = e\left(1 - \sum_{n=1}^{\infty} \frac{b_n}{(1+x)^n}\right). \tag{23}$$

By Lemmas 4 and 5, we have that $b_n > 0$ and satisfies the recurrence relation (5). This proves Theorem 1. □

Remark 6. As an added bonus, taking $x = n$ in (23), we have

$$\left(1+\frac{1}{n}\right)^n = e\left(1-\sum_{k=1}^{\infty}\frac{b_k}{(1+n)^k}\right). \tag{24}$$

Thus, for any positive integer $m \geq 1$, we obtain

$$\left(1+\frac{1}{n}\right)^n < e\left(1-\sum_{k=1}^{m}\frac{b_k}{(1+n)^k}\right). \tag{25}$$

On the other hand, noticing that $b_k \leq 1/k(k+1)$ from (5), we have

$$\left(1+\frac{1}{n}\right)^n > e\left(1-\sum_{k=1}^{\infty}\frac{1}{k(k+1)(1+n)^k}\right). \tag{26}$$

Combining inequalities (24) and (26), we deduce that

$$e\left(1-\sum_{k=1}^{\infty}\frac{1}{k(k+1)(1+n)^k}\right) < \left(1+\frac{1}{n}\right)^n < e\left(1-\sum_{k=1}^{m}\frac{b_k}{(1+n)^k}\right). \tag{27}$$

This improves Kloosterman's inequality [2, pages 324–325] and [4, inequality (2.7)].

Next, we prove Theorem 2 by modifying the approach used to prove Hardy's inequality [1].

Proof of Theorem 2. For any positive sequence $\{c_n\}$, using the arithmetic-geometric average inequality, we have

$$\left(\prod_{k=1}^{n} c_k a_k\right)^{1/n} \leq \frac{1}{n}\sum_{k=1}^{n} c_k a_k. \tag{28}$$

So that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(a_1 a_2 \cdots a_n\right)^{1/n} &= \sum_{n=1}^{\infty}\left(\frac{\prod_{k=1}^{n} c_k a_k}{\prod_{k=1}^{n} c_k}\right)^{1/n} \\ &\leq \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n} c_k\right)^{-1/n}\left(\frac{1}{n}\sum_{k=1}^{n} c_k a_k\right).\end{aligned} \tag{29}$$

Exchanging the order of the summation in the last inequality, we have

$$\sum_{n=1}^{\infty}\left(a_1 a_2 \cdots a_n\right)^{1/n} \leq \sum_{k=1}^{\infty} c_k a_k \sum_{n=k}^{\infty}\frac{1}{n}\left(\prod_{k=1}^{n} c_k\right)^{-1/n}. \tag{30}$$

Set

$$c_k = \left(1+\frac{1}{k}\right)^k k, \quad k = 1,2,\ldots, \tag{31}$$

we have

$$\prod_{k=1}^{n} c_k = (1+n)^n, \tag{32}$$

and hence

$$\sum_{n=k}^{\infty} \frac{1}{n}\left(\prod_{k=1}^{n} c_k\right)^{-1/n} = \sum_{n=k}^{\infty} \frac{1}{n(n+1)} = \frac{1}{k}. \tag{33}$$

Thus, by virtue of (30), we deduce that

$$\sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} \le \sum_{k=1}^{\infty} \frac{1}{k} c_k a_k = \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n a_n. \tag{34}$$

Taking $x = n$ in Theorem 1, we have refined Carleman's inequality (1) as

$$\begin{aligned} \sum_{n=1}^{\infty} (a_1 a_2 \cdots a_n)^{1/n} &\le e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{\infty} \frac{b_k}{(1+n)^k}\right) a_n \\ &< e \sum_{n=1}^{\infty} \left(1 - \sum_{k=1}^{m} \frac{b_k}{(1+n)^k}\right) a_n, \end{aligned} \tag{35}$$

where m is any positive integer. This proves Theorem 2 as required. □

Remark 7. It is clear that (2) is the special case of (35) at $m = 1$. Furthermore, by the binomial series, we have

$$\left(1+\frac{1}{n+1/5}\right)^{-1/2} > 1 - \frac{1}{2(n+1)} - \frac{1}{24(n+1)^2}, \quad \text{for } n = 1,2,\ldots. \tag{36}$$

Therefore, when $m = 2$, (35) strengthens (3).

References

[1] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge University Press, Cambridge, 1952.

[2] D. S. Mitrinović, *Analytic Inequalities*, Die Grundlehren der mathematischen Wisenschaften, vol. 1965, Springer-Verlag, New York, 1970.

[3] P. Yan and G. Sun, *A strengthened Carleman's inequality*, J. Math. Anal. Appl. **240** (1999), no. 1, 290–293.

[4] B. Yang and L. Debnath, *Some inequalities involving the constant e, and an application to Carleman's inequality*, J. Math. Anal. Appl. **223** (1998), no. 1, 347–353.

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