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FUZZY PROPERTIES IN FUZZY CONVERGENCE SPACES

GUNTHER JÄGER

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Based on the concept of limit of prefilters and residual implication, several notions in fuzzy topology are fuzzyfied in the sense that, for each notion, the degree to which it is fulfilled is considered. We establish therefore theories of degrees of compactness and relative compactness, of closedness, and of continuity. The resulting theory generalizes the corresponding "crisp" theory in the realm of fuzzy convergence spaces and fuzzy topology.

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1. Introduction. In most papers and contributions to the theory of $[0,1]$-topological spaces, the considered properties (like compactness) are viewed in a crisp way, that is, the properties either hold or fail. In [16], R. Lowen suggested that also the properties should be considered fuzzy, that is, one should be able to measure a degree to which a property holds. There are some papers dealing with such approaches. E. Lowen and R. Lowen [11] consider compactness degrees, and in [19], measures of separation in $[0,1]$-topological spaces are investigated. In [17], Šostak developed a theory of compactness degrees and connectedness degrees in $[0,1]$-fuzzy topological spaces, and developed, in [18], a theory of degrees of precompactness and completeness in the so-called Hutton fuzzy uniform spaces. These latter theories are related to the present work as they are explicitly based on a generalized inclusion (which is, however, not resulting from a residual implication).

In this paper, we follow these ideas in a systematic way. Starting from the notion of limit of a prefilter as defined in [15], we consider a semigroup operation $*$ on $[0,1]$ which is finitely distributive over arbitrary joins and, therefore, has a right adjoint $\to$, that is, a residual implication operator. In this way, a natural way of obtaining truly fuzzy extensions of properties in fuzzy convergence spaces [12, 13] is to replace subsethood, $a \le b$ of two fuzzy sets $a,b \in [0,1]^X$ by degrees of subsethood, $\text{subset}(a,b) = \bigwedge_{x\in X}(a(x) \to b(x))$ [1]. Exploiting this idea leads to the theory considered in this paper. We extend some results of an earlier paper [10], where degrees of closedness and degrees of compactness were studied and a theory of degrees of continuity and of degrees of relative compactness is established. Note that stratified $[0,1]$-topological spaces [5, 14] as well as Choquet convergence spaces [3] are fuzzy convergence spaces [13], that is, our approach works also in this more special context.

2. Preliminaries. Throughout the paper, we consider fuzzy subsets with membership values in the real unit interval $[0,1]$, that is, $a,b,c,\ldots \in [0,1]^X$. We assume that the reader is familiar with the usual definitions and notations in fuzzy set theory and fuzzy topology. We especially denote the pointwise extensions of the lattice operations $\wedge$, $\vee$, and the order relation $\le$ from $[0,1]$ to $[0,1]^X$ again by $\wedge$, $\vee$, $\le$, respectively. Moreover, we write $\varnothing$ for the constant function 0. Further we want to consider an additional operation $*:[0,1]\times[0,1]\to[0,1]$ with the following properties:

(I) $\alpha*(\beta*\gamma)=(\alpha*\beta)*\gamma$;
(II) $\alpha*\beta=\beta*\alpha$;
(III) $\alpha*1=\alpha$;
(IV) $\beta\le\gamma\Rightarrow\alpha*\beta\le\alpha*\gamma$;
(V) $\alpha*\bigvee_{\lambda\in\Lambda}\beta_\lambda=\bigvee_{\lambda\in\Lambda}(\alpha*\beta_\lambda)$;

that is, $*$ is a t*-norm* on $[0,1]$ which is distributive over arbitrary joins. Standard examples are the minimum $\alpha*\beta=\alpha\wedge\beta$, the product $\alpha*\beta=\alpha\cdot\beta$, or the *Lukasiewicz* t*-norm* $\alpha*\beta=\alpha T_m\beta=(\alpha+\beta-1)\vee 0$. Property (V) allows the definition of a *residual implication* $\to$ defined by $\alpha\to\beta:=\bigvee\{\lambda\mid\alpha*\lambda\le\beta\}$.

Lemma 2.1. *Let X be a nonempty set and let $a,b\in[0,1]^X$. Then*

$$\bigwedge_{x\in X}(a(x)\longrightarrow b(x))=\bigvee\{\alpha\in[0,1]\mid a(x)*\alpha\le b(x)\ \forall x\in X\}. \tag{2.1}$$

The proof is straightforward and therefore left to the reader.

Lemma 2.2. *The residual implication has the following properties* $(\alpha,\beta,\gamma,\alpha_\lambda,\beta_\lambda\in[0,1];\ (\lambda\in\Lambda))$:

(i) $\gamma\le\alpha\to\beta$ *if and only if* $\gamma*\alpha\le\beta$;
(ii) $\alpha\to\beta=1$ *if and only if* $\alpha\le\beta$;
(iii) *if* $\beta\le\gamma$ *then* $\alpha\to\beta\le\alpha\to\gamma$ *and* $\gamma\to\alpha\le\beta\to\alpha$;
(iv) $(\alpha\wedge\gamma)\to(\beta\wedge\gamma)\ge\alpha\to\beta$;
(v) $0\to\alpha=1$;
(vi) $\bigwedge_\lambda(\alpha_\lambda\to\beta)=(\bigvee_\lambda\alpha_\lambda)\to\beta$;
(vii) $\bigwedge_\lambda(\alpha\to\beta_\lambda)=\alpha\to(\bigwedge_\lambda\beta_\lambda)$;
(viii) $\alpha*(\alpha\to\beta)\le\beta$;
(ix) $(\alpha\to\beta)*(\beta\to\gamma)\le\alpha\to\gamma$;
(x) $\bigwedge_\lambda(\alpha_\lambda\to\beta_\lambda)\le(\bigvee_\lambda\alpha_\lambda)\to(\bigvee_\lambda\beta_\lambda)$.

Proof. Many of the assertions are easy consequences of (i) and can be found, for example, in [4]. We only prove (ix): we have $\alpha*((\alpha\to\beta)*(\beta\to\gamma))=(\alpha*(\alpha\to\beta))*(\beta\to\gamma)\le\beta*(\beta\to\gamma)\le\gamma$, by (viii). From (i) the claim follows. $\square$

3. Fuzzy convergence spaces. A *prefilter* (see [15]) $\mathbb{F}$ on $a\in[0,1]^X$ is a filter in the lattice $F_X(a):=\{b\in[0,1]^X\mid b\le a\}$, that is, $\varnothing\notin\mathbb{F}\ne\varnothing$; $f,g\in\mathbb{F}\Rightarrow f\wedge g\in\mathbb{F}$ and $F_X(a)\ni g\ge f\in\mathbb{F}\Rightarrow g\in\mathbb{F}$. We denote the set of all prefilters on a by $\mathbb{F}(a)$ and order this set by set inclusion. For $\mathbb{F}\in\mathbb{F}(a)$ we denote by $c(\mathbb{F})=\bigwedge_{f\in\mathbb{F}}\bigvee_{x\in X}f(x)$ its *characteristic value*. A prefilter is called *prime* if whenever $f\vee g\in\mathbb{F}$ then $f\in\mathbb{F}$ or $g\in\mathbb{F}$ [15]. For example, the *point prefilters* $[\alpha\mathbf{1}_x]=\{f\in F_X(a)\mid f(x)\ge\alpha\}$ are prime prefilters. We denote the set of all prime prefilters on a by $\mathbb{F}_p(a)$. It is shown in [15] that the set $\mathbb{P}(\mathbb{F}):=\{\mathbb{G}\in\mathbb{F}_p(a)\mid\mathbb{G}\ge\mathbb{F}\}$ contains minimal elements and we denote $\mathbb{P}_m(\mathbb{F}):=\{\mathbb{G}\in\mathbb{P}(\mathbb{F})\mid\mathbb{G}\text{ minimal}\}$. We call $\mathbb{B}\subset F_X(a)$ a *prefilterbase* if and only if $\varnothing\notin\mathbb{B}\ne\varnothing$ and $b,c\in\mathbb{B}\Rightarrow\exists d\in\mathbb{B}$, such that, $d\le b\wedge c$. For a prefilterbase, we denote by $[\mathbb{B}]_a = [\mathbb{B}] = \{f \in F_X(a) \mid \exists b \in \mathbb{B} : b \le f\}$, the generated prefilter. For $b \le a$ and $\mathbb{F} \in \mathbb{F}(a)$ with $f \wedge b \ne \emptyset$ for all $f \in \mathbb{F}$, we put $\mathbb{F}_b := \{f \wedge b \mid f \in \mathbb{F}\} \in \mathbb{F}(b)$. For further results concerning prefilters we refer the reader to [15].

A *fuzzy convergence space* [6, 12, 13] $(a, \lim)$ is a fuzzy set $a \in [0,1]^X$ together with a mapping $\lim : \mathbb{F}(a) \to F_X(a)$ subject to the conditions:

(PST) for all $\mathbb{F} \in \mathbb{F}(a) : \lim \mathbb{F} = \bigwedge_{\mathbb{G} \in \mathbb{P}_m(\mathbb{F})} \lim \mathbb{G}$,

(F1p) for all $\mathbb{F} \in \mathbb{F}_p(a) : \lim \mathbb{F} \le c(\mathbb{F})$,

(F2p) for all $\mathbb{F}, \mathbb{G} \in \mathbb{F}_p(a) : \mathbb{F} \le \mathbb{G} \Rightarrow \lim \mathbb{G} \le \lim \mathbb{F}$,

(C1) for all $x \in a_0$, $0 < \alpha \le a(x) : \alpha \mathbf{1}_x \le \lim[\alpha \mathbf{1}_x]$.

By reason of (PST), it is sufficient to define the mapping lim only for prime prefilters. Standard examples for fuzzy convergence spaces are *fuzzy topological spaces* (X, Δ) in the sense of R. Lowen [14] (i.e., *stratified* $[0,1]$*-topological spaces* in the notation of [5]) and *Choquet limit spaces* (X, τ) [3].

For a fuzzy convergence space $(a, \lim)$ and $b \le a$, we denote $\bar{b} := \bigvee_{b \in \mathbb{F} \in \mathbb{F}_p(a)} \lim \mathbb{F}$ its *lim-closure* [7].

If $a \in [0,1]^X$ and $b \in [0,1]^Y$ we put $\mathrm{mor}(a,b) = \{\varphi : X \to Y \mid \varphi(a) \le b\}$ and we write $\varphi : a \to b$ for $\varphi \in \mathrm{mor}(a,b)$. For $c \le b$, the *inverse image* $\varphi^{\leftarrow}(c)$ of c under $\varphi \in \mathrm{mor}(a,b)$ is defined by $\varphi^{\leftarrow}(c)(x) = c(\varphi(x)) \wedge a(x) = \varphi^{-1}(c) \wedge a(x)$, $x \in X$. For $\mathbb{F} \in \mathbb{F}(a)$, we define $\varphi(\mathbb{F})$ as the prefilter on b generated by the prefilterbase $\{\varphi(f) \mid f \in \mathbb{F}\}$. If $(a, \lim^a)$, $(b, \lim^b)$ are fuzzy convergence spaces then we call $\varphi : a \to b$ *continuous* if and only if $\varphi(\lim^a \mathbb{F}) \le \lim^b \varphi(\mathbb{F})$ for all $\mathbb{F} \in \mathbb{F}_p(a)$. The category with fuzzy convergence spaces as objects and continuous mappings as morphisms is denoted by FCS.

Let now $(a, \lim) \in |\mathrm{FCS}|$ and let $b \le a$. We define on b the fuzzy convergence $\lim|_b$ induced by $(a, \lim)$,

$$\lim|_b \mathbb{F} = \lim[\mathbb{F}] \wedge b, \tag{3.1}$$

and call $(b, \lim|_b)$ a *subspace* of $(a, \lim)$ (cf. [6]).

If $a^\lambda \in [0,1]^{X_\lambda}$, $(\lambda \in \Lambda)$ and $(a^\lambda, \lim_\lambda) \in |\mathrm{FCS}|$ for all $\lambda \in \Lambda$, then we define the *product space* $(\prod a^\lambda, \pi - \lim)$ putting for $\mathbb{F} \in \mathbb{F}(\prod a^\lambda)$

$$\pi - \lim \mathbb{F} = \prod_{\lambda \in \Lambda} \lim_\lambda \mathrm{pr}_\lambda(\mathbb{F}). \tag{3.2}$$

For more details we refer to [6].

4. The degree of closedness of a fuzzy set.

The definitions and results of this section were already established in [10]. However, we propose the proofs of the propositions in a more systematical way making use of Lemma 2.2.

Deffinition 4.1. Let $(a, \lim) \in |\mathrm{FCS}|$ and let $b \le a$. We call

$$\mathrm{cl}\,(b, (a, \lim)) = \mathrm{cl}(b) := \bigwedge_{\mathbb{F} \in \mathbb{F}_p(a) : b \in \mathbb{F}, x \in X} (\lim \mathbb{F}(x) \longrightarrow b(x)), \tag{4.1}$$

the *degree of closedness* of b in $(a, \lim)$.

In [6], we called a fuzzy subset $b \le a$ of a fuzzy convergence space $(a, \lim)$ *lim-closed*, if $b \in \mathbb{F} \in \mathbb{F}_p(a)$ implies $\lim \mathbb{F} \le b$. From Lemma 2.2(ii), it is immediately evident that $\mathrm{cl}(b) = 1$ if and only if b is lim-closed.

Proposition 4.2. *Let* $(a, \lim) \in |\mathrm{FCS}|$ *and let* $b, b^\lambda, c \le a$, $(\lambda \in \Lambda)$. *The following holds:*

(i) $\mathrm{cl}(b \vee c) \ge \mathrm{cl}(b) \wedge \mathrm{cl}(c)$;
(ii) $\mathrm{cl}(\bigwedge_{\lambda\in\Lambda} b^\lambda) \ge \bigwedge_{\lambda\in\Lambda} \mathrm{cl}(b^\lambda)$;
(iii) $\mathrm{cl}(a \wedge \alpha) = 1$ *for all* $\alpha \in [0,1]$.

Proof. (i) Put $\gamma := \mathrm{cl}(b) \wedge \mathrm{cl}(c)$. Then $\gamma \le \lim \mathbb{F}(x) \to b(x)$ for all $b \in \mathbb{F} \in \mathbb{F}_p(a)$, $x \in X$, and $\gamma \le \lim \mathbb{F}(x) \to c(x)$ for all $c \in \mathbb{F} \in \mathbb{F}_p(a)$, $x \in X$. If $b \vee c \in \mathbb{F} \in \mathbb{F}_p(a)$ then without of generality $b \in \mathbb{F}$; and hence, with Lemma 2.2(v), $\gamma \le \lim \mathbb{F}(x) \to (b \vee c)(x)$ for every $x \in X$. The arbitrariness of $\mathbb{F} \in \mathbb{F}_p(a)$ finally yields $\gamma \le \mathrm{cl}(b \vee c)$.

(ii) Put $\gamma := \bigwedge_{\lambda\in\Lambda} \mathrm{cl}(b^\lambda)$. Then for every $\mathbb{F} \in \mathbb{F}_p(a)$ such that $b^\lambda \in \mathbb{F}$, for every $x \in X$, and for every $\lambda \in \Lambda$, we have $\gamma \le \lim \mathbb{F}(x) \to b^\lambda(x)$. If now $\bigwedge_{\lambda\in\Lambda} b^\lambda \in \mathbb{F} \in \mathbb{F}_p(a)$ then $b^\lambda \in \mathbb{F}$ for every $\lambda \in \Lambda$; by Lemma 2.2(vii), hence

$$\gamma \le \bigwedge_{\lambda\in\Lambda} \left(\lim \mathbb{F}(x) \longrightarrow b^\lambda(x)\right) = \lim \mathbb{F}(x) \longrightarrow \bigwedge_{\lambda\in\Lambda} b^\lambda(x). \tag{4.2}$$

From this the claim follows.

(iii) Its proof follows with condition (F1p) Section 3, as $c(\mathbb{F}) \le \alpha$ for $a \wedge \alpha \in \mathbb{F} \in \mathbb{F}_p(a)$ and by Lemma 2.2(ii). □

Proposition 4.2 allows for a fuzzy convergence space $(a, \lim)$, via

$$o(b) := \mathrm{cl}(\mathbf{C}b) \tag{4.3}$$

(with $\mathbf{C}b(x) := a(x) - b(x)$, $x \in X$, the *pseudocomplement* of b with respect to a, cf. [8]), the definition of a fuzzy $[0,1]$-topology in the sense of [5, 17].

Corollary 4.3. *Let* $(a, \lim) \in |\mathrm{FCS}|$. *The following holds:*

(i) *the union of two lim-closed fuzzy sets is lim-closed;*
(ii) *the intersection of a family of lim-closed fuzzy sets is lim-closed;*
(iii) $a \wedge \alpha$ *is lim-closed for every* $\alpha \in [0,1]$.

Proposition 4.4. *Let* $(a, \lim) \in |\mathrm{FCS}|$ *and let* $c \le b \le a$. *Then*

$$\mathrm{cl}\left(c, (b, \lim|_b)\right) \ge \mathrm{cl}\left(c, (a, \lim)\right). \tag{4.4}$$

Proof. Put $\gamma := \mathrm{cl}(c, (a, \lim))$ and let $\mathbb{F} \in \mathbb{F}_p(b)$ such that $c \in \mathbb{F}$. Then $c \in [\mathbb{F}] \in \mathbb{F}_p(a)$, thus for every $x \in X$, we conclude with Lemma 2.2(iii) that

$$\gamma \le \lim[\mathbb{F}](x) \longrightarrow c(x) \le \left(\lim[\mathbb{F}](x) \wedge b(x)\right) \longrightarrow c(x) = \lim|_b \mathbb{F}(x) \longrightarrow c(x). \tag{4.5}$$

Hence by arbitrariness of $\mathbb{F}$, we get $\gamma \le \mathrm{cl}(c, (b, \lim|_b))$. □

Corollary 4.5. *Let* $(a, \lim) \in |\mathrm{FCS}|$ *and let* $c \le b \le a$. *If* c *is* lim*-closed then* c *is* $\lim|_b$*-closed.*

We end this section describing the degree of closedness for special operations $*$. For $* = \wedge$ we obtain, with Lemma 2.1,

$$\mathrm{cl}(b) = \bigvee \{\alpha \mid \lim \mathbb{F} \wedge \alpha \le b \ \forall b \in \mathbb{F} \in \mathbb{F}_p(a)\}; \tag{4.6}$$

and for $* = T_m$, the Lukasiewicz t-norm, we deduce from Lemma 2.1 that

$$\mathrm{cl}(b) = 1 - \bigwedge \{\alpha \mid \lim \mathbb{F} \le b + \alpha \ \forall b \in \mathbb{F} \in \mathbb{F}_p(a)\}. \tag{4.7}$$

5. The degree of continuity of a mapping. We extend the notion of continuity of a mapping $\varphi : a \to b$.

Deffinition 5.1. Let $(a, \lim^a), (b, \lim^b) \in |\mathrm{FCS}|$ and let $\varphi : a \to b$ be a mapping. Then

$$\begin{aligned} &\mathrm{cont}\,(\varphi, (a, \lim^a), (b, \lim^b)) \\ &\quad = \mathrm{cont}(\varphi) := \bigwedge_{\mathbb{F} \in \mathbb{F}_p(a), x \in X} (\lim^a \mathbb{F}(x) \longrightarrow \lim^b \varphi(\mathbb{F})(\varphi(x))) \end{aligned} \tag{5.1}$$

is called the *continuity degree* of φ.

Obviously again it holds that φ is continuous if and only if $\mathrm{cont}(\varphi) = 1$.

Proposition 5.2. *Let $(a, \lim^a), (b, \lim^b), (c, \lim^c) \in |\mathrm{FCS}|$ and let $\varphi : a \to b$ and $\psi : b \to c$. Then*

$$\mathrm{cont}(\psi \circ \varphi) \ge \mathrm{cont}(\varphi) * \mathrm{cont}(\psi). \tag{5.2}$$

Proof. Put $\delta := \mathrm{cont}(\varphi)$, $\epsilon := \mathrm{cont}(\psi)$. If $\mathbb{F} \in \mathbb{F}_p(a)$ and $x \in X$ then

$$\begin{aligned} \delta &\le \lim^a \mathbb{F}(x) \longrightarrow \lim^b \varphi(\mathbb{F})(\varphi(x)), \\ \epsilon &\le \lim^b \varphi(\mathbb{F})(\varphi(x)) \longrightarrow \lim^c \psi \circ \varphi(\mathbb{F})(\psi \circ \varphi(x)) \end{aligned} \tag{5.3}$$

(with $\psi \circ \varphi(\mathbb{F}) = \psi(\varphi(\mathbb{F}))$). It follows with Lemma 2.2(ix) that

$$\delta * \epsilon \le \lim^a \mathbb{F}(x) \longrightarrow \lim^c \psi \circ \varphi(\mathbb{F})(\psi \circ \varphi(x)), \tag{5.4}$$

from which the claim follows. □

Corollary 5.3. *Let $(a, \lim^a), (b, \lim^b), (c, \lim^c) \in |\mathrm{FCS}|$ and let $\varphi : a \to b$ and $\psi : b \to c$. If φ and ψ are continuous then so is $\psi \circ \varphi$.*

Corollary 5.4. *Let $(a, \lim^a), (b, \lim^b) \in |\mathrm{FCS}|$; $\lim^* \ge \lim^a$; $\lim' \le \lim^b$ and let $\varphi : a \to b$. Then*

$$\mathrm{cont}\,(\varphi, (a, \lim^a), (b, \lim^b)) \le \mathrm{cont}\,(\varphi, (a, \lim^*), (b, \lim')). \tag{5.5}$$

Proof. This follows from the continuity of the identity mappings. □

Proposition 5.5. *Let $(a, \lim^a), (b, \lim^b) \in |\mathrm{FCS}|$; $c \le a$ and let $\varphi : a \to b$. Then*

$$\mathrm{cont}\,(\varphi|_c, (c, \lim^a|_c), (\varphi(c), \lim^b|_{\varphi(c)})) \ge \mathrm{cont}\,(\varphi, (a, \lim^a), (b, \lim^b)). \tag{5.6}$$

Proof. We have

$$\begin{aligned}\operatorname{cont}(\varphi|_c) &= \bigwedge_{\mathbb{F}\in\mathbb{F}_p(c),x\in X}\left(\lim{}^a|_c\mathbb{F}(x)\longrightarrow \lim{}^b|_{\varphi(c)}\varphi(\mathbb{F})(\varphi(x))\right)\\ &= \bigwedge_{\mathbb{F}\in\mathbb{F}_p(c),x\in X}\left(\lim{}^a[\mathbb{F}](x)\wedge c(x)\longrightarrow \lim{}^b[\varphi(\mathbb{F})](\varphi(x))\wedge\varphi(c)(\varphi(x))\right)\\ &\geq \bigwedge_{\mathbb{F}\in\mathbb{F}_p(a),x\in X}\left(\lim{}^a\mathbb{F}(x)\wedge c(x)\longrightarrow \lim{}^b\varphi(\mathbb{F})(\varphi(x))\wedge\varphi(c)(\varphi(x))\right).\end{aligned} \tag{5.7}$$

By reason of Lemma 2.2(iii) and as $\varphi(c)(\varphi(x))\geq c(x)$ the latter is

$$\operatorname{cont}(\varphi|_c)\geq \bigwedge_{\mathbb{F}\in\mathbb{F}_p(a),x\in X}\left(\left(\lim{}^a\mathbb{F}(x)\wedge c(x)\right)\longrightarrow\left(\lim{}^b\varphi(\mathbb{F})(\varphi(x))\wedge c(x)\right)\right). \tag{5.8}$$

From Lemma 2.2(iv) we finally deduce

$$\operatorname{cont}(\varphi|_c)\geq \bigwedge_{\mathbb{F}\in\mathbb{F}_p(a),x\in X}\left(\lim{}^a\mathbb{F}(x)\longrightarrow \lim{}^b\varphi(\mathbb{F})(\varphi(x))\right), \tag{5.9}$$

from which the claim follows. □

Corollary 5.6. *Let $(a,\lim^a),(b,\lim^b)\in|\mathrm{FCS}|$; $c\leq a$ and let $\varphi: a\to b$ be continuous. Then also $\varphi|_c:(c,\lim^a|_c)\to(\varphi(c),\lim^b|_{\varphi(c)})$ is continuous.*

Proposition 5.7. *Let $(a,\lim^a),(b,\lim^b)\in|\mathrm{FCS}|$; $e\leq b$ and $\varphi: a\to b$ be a mapping. Then*

$$\operatorname{cl}(\varphi^{\leftarrow}(e))\geq \operatorname{cl}(e)*\operatorname{cont}(\varphi). \tag{5.10}$$

Proof. Let $\delta:=\operatorname{cl}(e)$ and $\epsilon:=\operatorname{cont}(\varphi)$. If $\mathbb{F}\in\mathbb{F}_p(a)$, $\varphi^{\leftarrow}(e)\in\mathbb{F}$, $x\in X$ then $\varphi(\varphi^{\leftarrow}(e))\in\varphi(\mathbb{F})$ and hence also $e\in\varphi(\mathbb{F})$. Thus

$$\begin{aligned}\delta&\leq\lim{}^b\varphi(\mathbb{F})(\varphi(x))\longrightarrow e(\varphi(x)),\\ \epsilon&\leq\lim{}^a\mathbb{F}(x)\longrightarrow\lim{}^b\varphi(\mathbb{F})(\varphi(x)).\end{aligned} \tag{5.11}$$

Finally by Lemma 2.2(ix), (vii) and as $\lim^a\mathbb{F}(x)\leq a(x)$, we deduce that

$$\begin{aligned}\epsilon*\delta&\leq\lim{}^a\mathbb{F}(x)\longrightarrow e(\varphi(x))\\ &=\lim{}^a\mathbb{F}(x)\longrightarrow\left(e(\varphi(x))\wedge a(x)\right)\\ &=\lim{}^a\mathbb{F}(x)\longrightarrow\varphi^{\leftarrow}(e)(x).\end{aligned} \tag{5.12}$$

From this the claim follows. □

Corollary 5.8. *Let $(a,\lim^a),(b,\lim^b)\in|\mathrm{FCS}|$; $e\leq b$ and $\varphi: a\to b$ be a mapping.*
(i) *If φ is continuous then $\operatorname{cl}(\varphi^{\leftarrow}(e))\geq\operatorname{cl}(e)$.*
(ii) *If φ is continuous and e is $\lim^b$-closed, then $\varphi^{\leftarrow}(e)$ is $\lim^a$-closed.*

We again end this section describing the continuity degrees for special operations $*$. For $* = \wedge$ we obtain, with Lemma 2.1,

$$\operatorname{cont}(\varphi) = \bigvee \{\alpha \in [0,1] \mid \varphi(\lim^a \mathbb{F}) \wedge \alpha \le \lim^b \varphi(\mathbb{F})\ \forall \mathbb{F} \in \mathbb{F}_p(a)\}; \tag{5.13}$$

and for $* = T_m$ we obtain, again with the help of Lemma 2.1,

$$\operatorname{cont}(\varphi) = 1 - \bigwedge \{\beta \in [0,1] \mid \varphi(\lim^a \mathbb{F}) \le \lim^b \varphi(\mathbb{F}) + \beta\ \forall \mathbb{F} \in \mathbb{F}_p(a)\}. \tag{5.14}$$

6. Degrees of compactness and of relative compactness. In this section, we extend the theory of relative compact subsets established in [2, 9] and repeat, sketching new proofs, the theory of compactness degrees developed in [10] (which extends the theory of compactness in fuzzy convergence spaces [6] and the theory of measures of compactness in $[0,1]$-topological spaces [11]). Some additional results concerning compactness degrees are included.

Deffinition 6.1. Let $(a,\lim) \in |\mathrm{FCS}|$ and let $b \le a$. We call

(i) $\mathbf{c}(a) = \mathbf{c}(a,\lim) = \bigwedge_{\mathbb{F}\in\mathbb{F}_p(a)} (c(\mathbb{F}) \to \bigvee_{x\in X} \lim \mathbb{F}(x))$ the *compactness degree* of $(a,\lim)$,

(ii) $\mathbf{c}(b) = \mathbf{c}(b,\lim|_b)$ the *compactness degree* of b, and

(iii) $\mathbf{rc}(b) = \mathbf{rc}(b,(a,\lim)) = \bigwedge_{\mathbb{F}\in\mathbb{F}_p(a): b\in\mathbb{F}} (c(\mathbb{F}) \to \bigvee_{x\in X} \lim \mathbb{F}(x))$ the *degree of relative compactness* of b in $(a,\lim)$.

Clearly, $(a,\lim)$ is compact [6] if and only if $\mathbf{c}(a) = 1$; and b is relatively compact in $(a,\lim)$ [9] if and only if $\mathbf{rc}(b) = 1$.

Proposition 6.2. *Let $(a,\lim) \in |\mathrm{FCS}|$ and let $b \le a$. Then $\mathbf{c}(b) \le \mathbf{rc}(b)$.*

Proof. Let $\mathbb{F} \in \mathbb{F}_p(a)$ such that $b \in \mathbb{F}$. Then $\mathbb{F}_b \in \mathbb{F}_p(b)$, $c(\mathbb{F}_b) = c(\mathbb{F})$, and $[\mathbb{F}_b] = \mathbb{F}$. Hence we conclude that

$$\begin{aligned}\mathbf{c}(b) &= \bigwedge_{\mathbb{F}\in\mathbb{F}_p(b)} \Big(c(\mathbb{F}) \longrightarrow \bigvee_x \lim|_b \mathbb{F}(x)\Big) \\ &\le \bigwedge_{\mathbb{F}\in\mathbb{F}_p(a): b\in\mathbb{F}} \Big(c(\mathbb{F}_b) \longrightarrow \bigvee_x \lim|_b \mathbb{F}_b(x)\Big)\end{aligned} \tag{6.1}$$

and by Lemma 2.2(iii),

$$\mathbf{c}(b) \le \bigwedge_{\mathbb{F}\in\mathbb{F}_p(a): b\in\mathbb{F}} \Big(c(\mathbb{F}) \longrightarrow \bigvee_x \lim \mathbb{F}(x)\Big) = \mathbf{rc}(b). \tag{6.2}$$

□

Corollary 6.3. *A compact fuzzy subset of a fuzzy convergence space is relatively compact.*

Proposition 6.4. *Let $(a,\lim) \in |\mathrm{FCS}|$ and let $c \le b \le a$. Then $\mathbf{rc}(b) \le \mathbf{rc}(c)$.*

The proof is obvious.

Corollary 6.5. *A fuzzy subset of a relatively compact fuzzy set is relatively compact.*

Proposition 6.6. *Let* $(a,\lim) \in |\mathsf{FCS}|$ *and let* $b \le a$*. Then* $\mathbf{c}(\bar{b}) \le \mathbf{rc}(b)$.

Proof. Combination of Propositions 6.2 and 6.4. □

Corollary 6.7. *A fuzzy subset, whose* lim*-closure is compact, is relatively compact.*

Proposition 6.8. *Let* $(a,\lim) \in |\mathsf{FCS}|$ *and let* $\bar{c} \le b \le a$*. Then* $\mathbf{rc}(c,(a,\lim)) \le \mathbf{rc}(c,(b,\lim|_b))$.

Proof. Let $\mathbb{F} \in \mathbb{F}_p(b)$ such that $c \in \mathbb{F}$. Then $c \in [\mathbb{F}]_a \in \mathbb{F}_p(a)$. From $\bar{c} \le b$, we deduce that $\lim[\mathbb{F}]_a \le \bar{c} \le b$, hence $\lim|_b\mathbb{F} = b \wedge \lim[\mathbb{F}]_a = \lim[\mathbb{F}]_a$, from which it follows that

$$\begin{aligned} \mathbf{rc}(c,(b,\lim|_b)) &= \bigwedge_{c\in\mathbb{F}\in\mathbb{F}_p(b)} \Big(c([\mathbb{F}]_a) \longrightarrow \bigvee_x \lim[\mathbb{F}]_a(x) \Big) \\ &\ge \bigwedge_{c\in\mathbb{F}\in\mathbb{F}_p(a)} \Big(c(\mathbb{F}) \longrightarrow \bigvee_x \lim\mathbb{F}(x) \Big) = \mathbf{rc}(c,(a,\lim)). \end{aligned} \tag{6.3}$$

□

Corollary 6.9. *Let* $(a,\lim) \in |\mathsf{FCS}|$ *and let* $\bar{c} \le b \le a$*. If* c *is relatively compact in* $(a,\lim)$ *then* c *is relatively compact in* $(b,\lim|_b)$.

Proposition 6.10. *Let* $(a,\lim) \in |\mathsf{FCS}|$ *and let* $b,c \le a$*. Then*

(i) $\mathbf{c}(b) \wedge \mathbf{c}(c) \le \mathbf{c}(b \vee c)$*; and,*

(ii) $\mathbf{rc}(b) \wedge \mathbf{rc}(c) \le \mathbf{rc}(b \vee c)$.

Proof. The proof of (i) was already shown in [10] and can be deduced similarly to (ii). We prove (ii). Let $\gamma := \mathbf{rc}(b) \wedge \mathbf{rc}(c)$. Let further $\mathbb{F} \in \mathbb{F}_p(a)$ such that $b \vee c \in \mathbb{F}$. Then, without loss of generality, $b \in \mathbb{F}$; hence by definition of γ and of $\mathbf{rc}(b)$

$$\gamma \le c(\mathbb{F}) \longrightarrow \bigvee_x \lim\mathbb{F}(x), \tag{6.4}$$

from which the claim follows. □

Corollary 6.11. *The union of two compact (resp., relatively compact) fuzzy sets is compact (resp., relatively compact).*

For the next proposition see also the related Proposition 3.4 in [10].

Proposition 6.12. *Let* $(a,\lim) \in |\mathsf{FCS}|$ *and let* $b \le a$*. Then* $\mathbf{c}(b) \ge \mathbf{c}(a) * \mathrm{cl}(b)$.

Proof. Let $\delta := \mathbf{c}(a)$ and $\eta := \mathrm{cl}(b)$. If $\mathbb{F} \in \mathbb{F}_p(b)$ then $b \in [\mathbb{F}] \in \mathbb{F}_p(a)$ and $c([\mathbb{F}]) = c(\mathbb{F})$. Hence

$$\delta \le c(\mathbb{F}) \longrightarrow \bigvee_x \lim[\mathbb{F}](x), \tag{6.5}$$

and for all $x \in X$

$$\eta \le \lim[\mathbb{F}](x) \longrightarrow b(x). \tag{6.6}$$

Hence, by Lemma 2.2(i), $\delta * c(\mathbb{F}) \leq \bigvee_x \lim[\mathbb{F}](x)$; and for all $x \in X$ we have $\eta * \lim[\mathbb{F}](x) \leq b(x)$. Thus it follows that

$$\begin{aligned}\bigvee_x \lim|_b\mathbb{F}(x) &= \bigvee_x \big(b(x) \wedge \lim[\mathbb{F}](x)\big)\\ &\geq \bigvee_x \big((\eta * \lim[\mathbb{F}](x)) \wedge \lim\mathbb{F}(x)\big)\\ &= \bigvee_x \big(\eta * \lim[\mathbb{F}](x)\big)\\ &= \eta * \bigvee_x \lim[\mathbb{F}](x) \geq \eta * \delta * c(\mathbb{F}).\end{aligned} \tag{6.7}$$

Hence $\eta * \delta \leq c(\mathbb{F}) \to \bigvee_x \lim|_b\mathbb{F}(x)$. From this the claim follows. □

Corollary 6.13. (i) *The compactness degree of a* lim*-closed fuzzy subset of a fuzzy convergence space is at least as high as the compactness degree of the whole space.*

(ii) *A* lim*-closed fuzzy subset of a compact fuzzy convergence space is compact.*

The following result generalizes [10, Proposition 3.6].

Proposition 6.14. *Let* $(a, \lim^a), (b, \lim^b) \in |\mathrm{FCS}|$ *and let* $\varphi : a \to b$ *and* $c \leq a$. *Then*

(i) $\mathbf{c}(\varphi(c)) \geq \mathbf{c}(c) * \mathrm{cont}(\varphi)$;

(ii) $\mathbf{rc}(\varphi(c)) \geq \mathbf{rc}(c) * \mathrm{cont}(\varphi)$.

Proof. (i) Let first $c = a$ and $\varphi(c) = b$. For $\mathbb{F} \in \mathbb{F}_p(b)$ with $c(\mathbb{F}) > 0$ we have (cf. [6]) $\varphi^{\leftarrow}(\mathbb{F}) \in \mathbb{F}(a)$ and $c(\varphi^{\leftarrow}(\mathbb{F})) = c(\mathbb{F})$. Hence there exists $\mathbb{G} \in \mathbb{F}_p(a)$, $\mathbb{G} \geq \varphi^{\leftarrow}(\mathbb{F})$ and $c(\mathbb{G}) = c(\varphi^{\leftarrow}(\mathbb{F})) = c(\mathbb{F})$ [6]. Clearly $\varphi(\mathbb{G}) \geq \mathbb{F}$. We conclude from Lemma 2.2(x) and (ix) together with condition (F2p) Section 3 that

$$\begin{aligned}\mathbf{c}(c) * \mathrm{cont}(\varphi) &\leq \Big(c(\mathbb{F}) \longrightarrow \bigvee_x \lim^a\mathbb{G}(x)\Big) * \Big(\bigwedge_x \big(\lim^a\mathbb{G}(x) \longrightarrow \lim^b\varphi(\mathbb{G})(\varphi(x))\big)\Big)\\ &\leq \Big(c(\mathbb{F}) \longrightarrow \bigvee_x \lim^a\mathbb{G}(x)\Big) * \Big(\bigvee_x \lim^a\mathbb{G}(x) \longrightarrow \bigvee_x \lim^b\varphi(\mathbb{G})(\varphi(x))\Big)\\ &\leq c(\mathbb{F}) \longrightarrow \bigvee_x \lim^b\varphi(\mathbb{G})(\varphi(x))\\ &\leq c(\mathbb{F}) \longrightarrow \bigvee_x \lim^b\mathbb{F}(\varphi(x))\\ &\leq c(\mathbb{F}) \longrightarrow \bigvee_y \lim^b\mathbb{F}(y).\end{aligned} \tag{6.8}$$

Hence $\mathbf{c}(c) * \mathrm{cont}(\varphi) \leq \mathbf{c}(\varphi(c))$. The general case follows from this with Proposition 5.5.

(ii) Its proof goes analogously to (i). □

Corollary 6.15. (i) *The compactness degree of the image of a fuzzy set under a continuous mapping is not smaller than the compactness degree of the fuzzy set.*

(ii) *The continuous image of a compact fuzzy set is compact.*

(iii) *The degree of relative compactness of the image of a fuzzy set under a continuous mapping is not smaller than the degree of relative compactness of the fuzzy set.*
(iv) *The continuous image of a relatively compact fuzzy set is relatively compact.*

Proposition 6.16 (Tychonoff). *Let $a^\lambda \in [0,1]^{X_\lambda}$, $(a^\lambda, \lim_\lambda) \in |\mathrm{FCS}|$ $(\lambda \in \Lambda)$ and let $b \le \prod_{\lambda\in\Lambda} a^\lambda$ and $b^\lambda \le a^\lambda$ $(\lambda \in \Lambda)$. The following holds.*
(i) $\mathbf{c}(\prod_{\lambda\in\Lambda} a^\lambda) \ge \bigwedge_{\lambda\in\Lambda} \mathbf{c}(a^\lambda)$.
If $\bigvee_{x_\lambda\in X_\lambda} a^\lambda(x_\lambda) = \alpha_0 > 0$ for all $\lambda \in \Lambda$ then equality holds.
(ii) $\mathbf{rc}(b, (\prod_{\lambda\in\Lambda} a^\lambda, \pi - \lim)) = \bigwedge_{\lambda\in\Lambda} \mathbf{rc}(\mathrm{pr}_\lambda(b), (a^\lambda, \lim_\lambda))$.
(iii) $\mathbf{rc}(\prod_{\lambda\in\Lambda} b^\lambda, (\prod_{\lambda\in\Lambda} a^\lambda, \pi - \lim)) \ge \bigwedge_{\lambda\in\Lambda} \mathbf{rc}(b^\lambda, (a^\lambda, \lim_\lambda))$.
If $\bigvee_{x_\lambda\in X_\lambda} b^\lambda(x_\lambda) = \beta_0 > 0$ for all $\lambda \in \Lambda$ then equality holds.

Proof. (i) was proved in [10]. We prove (ii). The inequality $\mathbf{rc}(b) \le \bigwedge_\lambda \mathbf{rc}(\mathrm{pr}_\lambda(b))$ follows at once with the continuity of the pr_λ, by Proposition 6.14. On the other hand, let $\mathbb{F} \in \mathbb{F}_p(\prod a^\lambda)$, $b \in \mathbb{F}$. Then $\mathrm{pr}_\lambda(b) \in \mathrm{pr}_\lambda(\mathbb{F}) \in \mathbb{F}_p(a^\lambda)$ for all λ. Put $\gamma := \bigwedge_\lambda \mathbf{rc}(\mathrm{pr}_\lambda(b))$. Then, for every $\lambda \in \Lambda$,

$$\begin{aligned}\gamma &\le c(\mathrm{pr}_\lambda(\mathbb{F})) \longrightarrow \bigvee_{x_\lambda\in X_\lambda} \lim_\lambda \mathrm{pr}_\lambda(\mathbb{F})(x_\lambda)\\ &= c(\mathbb{F}) \longrightarrow \bigvee_{x_\lambda\in X_\lambda} \lim_\lambda \mathrm{pr}_\lambda(\mathbb{F})(x_\lambda)\end{aligned} \tag{6.9}$$

and hence

$$\gamma \le \bigwedge_{\lambda\in\Lambda}\left(c(\mathbb{F}) \longrightarrow \bigvee_{x_\lambda\in X_\lambda} \lim_\lambda \mathrm{pr}_\lambda(\mathbb{F})(x_\lambda)\right) \tag{6.10}$$

by Lemma 2.2(vii)

$$\gamma \le c(\mathbb{F}) \longrightarrow \bigwedge_{\lambda\in\Lambda}\bigvee_{x_\lambda\in X_\lambda} \lim_\lambda \mathrm{pr}_\lambda(\mathbb{F})(x_\lambda). \tag{6.11}$$

As $[0,1]$ is completely distributive we conclude

$$\bigwedge_{\lambda\in\Lambda}\bigvee_{x_\lambda\in X_\lambda} \lim_\lambda \mathrm{pr}_\lambda(\mathbb{F})(x_\lambda) = \bigvee_{(x_\lambda)\in\prod X_\lambda}\bigwedge_{\lambda\in\Lambda} \lim_\lambda \mathrm{pr}_\lambda(\mathbb{F})(x_\lambda) \tag{6.12}$$

and hence

$$\gamma \le c(\mathbb{F}) \longrightarrow \bigvee_{(x_\lambda)\in\prod X_\lambda} \pi - \lim \mathbb{F}((x_\lambda)). \tag{6.13}$$

As $\mathbb{F} \in \mathbb{F}_p(\prod a^\lambda)$, $b \in \mathbb{F}$ was arbitrarily chosen, the claim follows.

The first part of (iii) follows from Proposition 6.4 and (ii) as $\mathrm{pr}_\mu(\prod b^\lambda) \le b^\mu$. Under the assumptions of the second part it even holds that $\mathrm{pr}_\mu(\prod b^\lambda) = b^\mu$ and hence we have equality. □

Corollary 6.17. *Let $a^\lambda \in [0,1]^{X_\lambda}$, $(a^\lambda, \lim_\lambda) \in |\mathrm{FCS}|$ $(\lambda \in \Lambda)$ and let $b \le \prod_{\lambda\in\Lambda} a^\lambda$ and $b^\lambda \le a^\lambda$ $(\lambda \in \Lambda)$. The following holds:*

(i) *if every a^λ is compact then so is their product. If $\bigvee_{x_\lambda\in X_\lambda} a^\lambda(x_\lambda) = \alpha_0 > 0$ for all $\lambda \in \Lambda$; then from the compactness of the product, the compactness of each factor follows;*

(ii) *a fuzzy set* b *is relatively compact in* $(\prod a^\lambda, \pi - \lim)$ *if and only if* $\mathrm{pr}_\lambda(b)$ *is relatively compact in* $(a^\lambda, \lim_\lambda)$ *for every* $\lambda \in \Lambda$*;*

(iii) *if* b^λ *is relatively compact in* $(a^\lambda, \lim_\lambda)$ *for every* $\lambda \in \Lambda$*, then* $\prod b^\lambda$ *is relatively compact in* $(\prod a^\lambda, \pi - \lim)$*. If* $\bigvee_{x_\lambda \in X_\lambda} b^\lambda(x_\lambda) = \beta_0 > 0$ *for all* $\lambda \in \Lambda$*, then from the relative compactness of* $\prod b^\lambda$ *in* $(\prod a^\lambda, \pi - \lim)$ *the relative compactness of each* b^λ *in* $(a^\lambda, \lim_\lambda)$ *follows.*

We conclude this section giving the compactness degrees and the degrees of relative compactness for special operations $*$. In case $* = \wedge$ we get

$$\begin{aligned} \mathbf{c}(a) &= \bigvee \Big\{ \alpha \mid c(\mathbb{F}) \wedge \alpha \le \sup_{x \in X} \lim \mathbb{F}(x) \ \forall \mathbb{F} \in \mathbb{F}_p(a) \Big\}, \\ \mathbf{rc}(b) &= \bigvee \Big\{ \alpha \mid c(\mathbb{F}) \wedge \alpha \le \sup_{x \in X} \lim \mathbb{F}(x) \ \forall b \in \mathbb{F} \in \mathbb{F}_p(a) \Big\}; \end{aligned} \tag{6.14}$$

and in case $* = T_m$ we compute with Lemma 2.1

$$\begin{aligned} \mathbf{c}(a) &= 1 - \bigvee \Big\{ c(\mathbb{F}) - \sup_{x \in X} \lim \mathbb{F}(x) \mid \mathbb{F} \in \mathbb{F}_p(a) \Big\}, \\ \mathbf{rc}(b) &= 1 - \bigvee \Big\{ c(\mathbb{F}) - \sup_{x \in X} \lim \mathbb{F}(x) \mid b \in \mathbb{F} \in \mathbb{F}_p(a) \Big\}. \end{aligned} \tag{6.15}$$

We mention without proof that in the case of $a = 1_X$, (X, Δ) a fuzzy topological space, the compactness degree for $* = T_m$ is just the degree of compactness in E. Lowen and R. Lowen [11]. In this way, the compactness degrees here not only generalize the theory of compactness in FCS but also generalize the theory of compactness degrees in FTS, the category of fuzzy topological spaces.

7. Conclusions. The theory of “truly” fuzzy properties developed in this paper relies mainly on the notion of residual implication with respect to the operation $*$. Hence it can easily be extended to more general situations, where the real unit interval is, for example, replaced by a more general lattice L. We have only to make sure, that the operation $* : L \times L \to L$ then will still fulfill the properties (I), (II), (III), (IV), and (V) of Section 2 and that the residual implication $\to$ will fulfill the Lemma 2.2. For lattices that are suitable for this direction of research, we refer the reader to [5].

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Gunther Jäger: Department of Mathematics (Pure and Applied), Rhodes University, 6140 Grahamstown, South Africa

E-mail address: g.jaeger@ru.ac.za