IJMMS 29:12 (2002) 687–699
PII. S0161171202013212
http://ijmms.hindawi.com
© Hindawi Publishing Corp.

A GEOMETRICAL ANALYSIS OF THE FIELD EQUATIONS IN FIELD THEORY

A. ECHEVERRÍA-ENRÍQUEZ, M. C. MUÑOZ-LECANDA, and N. ROMÁN-ROY

Received 14 May 2001 and in revised form 6 September 2001

We give a geometric formulation of the field equations in the Lagrangian and Hamiltonian formalisms of classical field theories (of first order) in terms of multivector fields. This formulation enables us to discuss the existence and nonuniqueness of solutions of these equations, as well as their integrability.

2000 Mathematics Subject Classification: 53C05, 53C80, 55R10, 55R99, 58A20, 70S05.

1. Introduction. In recent years, there have been new developments in the study of multisymplectic Hamiltonian systems [2] and, in particular, their application to describe field theories. In this study, *multivector fields* and their contraction with differential forms are used; this is an intrinsic formulation of the systems of partial differential equations locally describing the field. Thus, the integrability of such equations, that is, of multivector fields, is a matter of interest. Given a fiber bundle $\pi : E \to M$, certain integrable multivector fields in E are equivalent to integrable connections in $E \to M$ [8]. This result is applied in two particular situations:

- first, multivector fields in J^1E (the first-order jet bundle), in order to characterize integrable multivector fields whose integral manifolds are holonomic;
- second, the manifold $J^{1*}E \equiv \Lambda_1^m T^*E/\Lambda_0^m T^*E$ (where $\Lambda_1^m T^*E$ is the bundle of m-forms on E vanishing by the action of two π-vertical vector fields, and $\Lambda_0^m T^*E \equiv \pi^*\Lambda^m T^*M$), which is also a fiber bundle $J^{1*}E \to M$. Then, we will take multivector fields in $J^{1*}E$ in order to characterize those that are integrable.

From these results we can set the Lagrangian and Hamiltonian equations for *multisymplectic models* of first-order classical field theories in a geometrical way [3, 12, 14, 15, 18], in terms of multivector fields; which is equivalent to other formulations using *Ehresmann connections* in a jet bundle [4, 21], or their associated *jet fields* [7]. This formulation allows us to discuss several aspects of these equations, in particular, the existence and nonuniqueness of solutions. (In a recent work [19], an *extended Hamiltonian formalism* for field theories was proposed, but using multivector fields in $\Lambda_1^m T^*E$ instead of $J^{1*}E$. See also [16, 17], where multivector fields are used in another more specific context.)

In Section 2, we introduce the terminology and nomenclature concerning multivector fields in differentiable manifolds and fiber bundles. This is used in Section 3 for setting the field equations for Lagrangian field theories (of first-order) in terms of multivector fields, and for analyzing their characteristic features. Finally, the same study is carried out in Section 4 for Hamiltonian field theories.

Throughout this paper, $\pi : E \to M$ denotes a fiber bundle ($\dim M = m$, $\dim E = N+m$), where M is an oriented manifold with volume form $\omega \in \Omega^m(M)$. We denote by $\pi^1 : J^1E \to E$ the jet bundle of local sections of π, and $\bar{\pi}^1 = \pi \circ \pi^1 : J^1E \to M$ gives another fiber bundle structure. And (x^μ, y^A, v^A_μ) denote natural local systems of coordinates in J^1E, adapted to the bundle $E \to M$ ($\mu = 1, \dots, m$; $A = 1, \dots, N$), such that $\omega = dx^1 \wedge \cdots \wedge dx^m \equiv d^m x$. Manifolds are real, paracompact, connected, and C^∞. Maps are C^∞. The sum over crossed repeated indices is understood.

2. Multivector fields in differentiable manifolds. Let E be an n-dimensional differentiable manifold. Sections of $\Lambda^m(TE)$ are called *m-multivector fields* in E (they are contravariant skewsymmetric tensors of order m in E). Then, *contraction* with multivector fields is the usual one for tensor fields in $J^{1*}E$. We denote by $\mathfrak{X}^m(E)$ the set of m-multivector fields in E.

If $Y \in \mathfrak{X}^m(E)$, for every $p \in E$, there exists an open neighborhood $U_p \subset E$ and $Y_1, \dots, Y_r \in \mathfrak{X}(U_p)$ such that $Y \underset{U_p}{=} \sum_{1 \le i_1 < \cdots < i_m \le r} f^{i_1 \cdots i_m} Y_{i_1} \wedge \cdots \wedge Y_{i_m}$, with $f^{i_1 \cdots i_m} \in C^\infty(U_p)$ and $m \le r \le \dim E$. Then, $Y \in \mathfrak{X}^m(E)$ is said to be *locally decomposable* if, for every $p \in E$, there exists an open neighborhood $U_p \subset E$ and $Y_1, \dots, Y_m \in \mathfrak{X}(U_p)$ such that $Y \underset{U_p}{=} Y_1 \wedge \cdots \wedge Y_m$.

A nonvanishing m-multivector field $Y \in \mathfrak{X}^m(E)$ and an m-dimensional distribution $D \subset TE$ are *locally associated* if there exists a connected open set $U \subseteq E$ such that $Y|_U$ is a section of $\Lambda^m D|_U$. If $Y, Y' \in \mathfrak{X}^m(E)$ are nonvanishing multivector fields locally associated with the same distribution D, on the same connected open set U, then there exists a nonvanishing function $f \in C^\infty(U)$ such that $Y' \overset{U}{=} fY$. This fact defines an equivalence relation in the set of nonvanishing m-multivector fields in E, whose equivalence classes are denoted by $\{Y\}_U$. We have as a consequence the following theorem.

Theorem 2.1. *There is a one-to-one correspondence between the set of m-dimensional orientable distributions D in TE and the set of the equivalence classes $\{Y\}_E$ of nonvanishing, locally decomposable m-multivector fields in E.*

Proof. Let $\omega \in \Omega^m(E)$ be an orientation form for D. If $p \in E$, there exists an open neighborhood $U_p \subset E$ and $Y_1, \dots, Y_m \in \mathfrak{X}(U_p)$, with $i(Y_1 \wedge \cdots \wedge Y_m)\omega > 0$, such that $D|_{U_p} = \operatorname{span}\{Y_1, \dots, Y_m\}$. Then $Y_1 \wedge \cdots \wedge Y_m$ is a representative of a class of m-multivector fields associated with D in U_p. But the family $\{U_p;\ p \in E\}$ is a covering of E; let $\{U_\alpha;\ \alpha \in A\}$ be a locally finite refinement and $\{\rho_\alpha;\ \alpha \in A\}$ a subordinate partition of unity. If $Y^\alpha_1, \dots, Y^\alpha_m$ is a local basis of D in U_α, with $i(Y^\alpha_1 \wedge \cdots \wedge Y^\alpha_m)\omega > 0$, then $Y = \sum_\alpha \rho_\alpha Y^\alpha_1 \wedge \cdots \wedge Y^\alpha_m$ is a global representative of the class of nonvanishing m-multivector fields associated with D in E.

The converse is trivial because, if $Y|_U = Y^1_1 \wedge \cdots \wedge Y^1_m = Y^2_1 \wedge \cdots \wedge Y^2_m$, for different sets $\{Y^1_1, \dots, Y^1_m\}$, $\{Y^2_1, \dots, Y^2_m\}$, then $\operatorname{span}\{Y^1_1, \dots, Y^1_m\} = \operatorname{span}\{Y^2_1, \dots, Y^2_m\}$. □

If $Y \in \mathfrak{X}^m(E)$ is nonvanishing and locally decomposable and $U \subseteq E$ is a connected open set, then the distribution associated with the class $\{Y\}_U$ is denoted by $\mathcal{D}_U(Y)$. If $U = E$, we write $\mathcal{D}(Y)$.

A nonvanishing, locally decomposable multivector field $Y \in \mathfrak{X}^m(E)$ is said to be *integrable* (resp., *involutive*) if its associated distribution $\mathcal{D}_U(Y)$ is integrable (resp., involutive). Of course, if $Y \in \mathfrak{X}^m(E)$ is integrable (resp., involutive), then so is every other multivector field in its equivalence class $\{Y\}$, and all of them have the same integral manifolds. Moreover, Frobenius theorem allows us to say that a nonvanishing and locally decomposable multivector field is integrable if and only if it is involutive. Nevertheless, in many applications, we have locally decomposable multivector fields $Y \in \mathfrak{X}^m(E)$ which are not integrable in E, but integrable in a submanifold of E. An (local) algorithm for finding this submanifold has been developed [8].

The particular situation to which we will pay attention is the study of multivector fields in fiber bundles. If $\pi : E \to M$ is a fiber bundle, we will be interested in the case where the integral manifolds of integrable multivector fields in E are sections of π. Thus, $Y \in \mathfrak{X}^m(E)$ is said to be *π-transverse* if, at every point $y \in E$, $(i(Y)(\pi^*\omega))_y \neq 0$, for every $\omega \in \mathbf{\Omega}^m(M)$ with $\omega(\pi(y)) \neq 0$. If $Y \in \mathfrak{X}^m(E)$ is integrable, it is π-transverse if and only if its integral manifolds are local sections of $\pi : E \to M$. In this case, if $\phi : U \subset M \to E$ is a local section with $\phi(x) = y$ and $\phi(U)$ is the integral manifold of Y through y, then $T_y(\operatorname{Im}\phi)$ is $\mathcal{D}_y(Y)$.

3. Lagrangian equations in classical field theories. A *classical field theory* is described by its *configuration bundle* $\pi : E \to M$ and a *Lagrangian density* which is a $\bar{\pi}^1$-semibasic m-form on J^1E. A Lagrangian density is usually written as $\mathcal{L} = \pounds(\bar{\pi}^{1*}\omega)$, where $\pounds \in C^\infty(J^1E)$ is the *Lagrangian function* associated with $\mathcal{L}$ and ω.

The *Poincaré-Cartan m* and *$(m+1)$-forms* associated with the Lagrangian density $\mathcal{L}$ are defined using the *vertical endomorphism* $\mathcal{V}$ of the bundle J^1E:

$$\begin{aligned} \Theta_{\mathcal{L}} &:= i(\mathcal{V})\mathcal{L} + \mathcal{L} \in \mathbf{\Omega}^m(J^1E); \\ \Omega_{\mathcal{L}} &:= -d\Theta_{\mathcal{L}} \in \mathbf{\Omega}^{m+1}(J^1E). \end{aligned} \tag{3.1}$$

Then a *Lagrangian system* is a couple $(J^1E, \Omega_{\mathcal{L}})$. The Lagrangian system is *regular* if $\Omega_{\mathcal{L}}$ is 1-nondegenerate. In a natural chart in J^1E we have

$$\begin{aligned} \Omega_{\mathcal{L}} = &-\frac{\partial^2 \pounds}{\partial v_\nu^B \partial v_\mu^A} dv_\nu^B \wedge dy^A \wedge d^{m-1}x_\mu - \frac{\partial^2 \pounds}{\partial y^B \partial v_\mu^A} dy^B \wedge dy^A \wedge d^{m-1}x_\mu \\ &+ \frac{\partial^2 \pounds}{\partial v_\nu^B \partial v_\mu^A} v_\mu^A dv_\nu^B \wedge d^m x + \left(\frac{\partial^2 \pounds}{\partial y^B \partial v_\mu^A} v_\mu^A - \frac{\partial \pounds}{\partial y^B} + \frac{\partial^2 \pounds}{\partial x^\mu \partial v_\mu^B} \right) dy^B \wedge d^m x, \end{aligned} \tag{3.2}$$

where $d^{m-1}x_\mu \equiv i(\partial/\partial x^\mu) d^m x$; and the regularity condition is equivalent to $\det((\partial^2 \pounds / \partial v_\mu^A \partial v_\nu^B)(\bar{y})) \neq 0$, for every $\bar{y} \in J^1E$.

A variational problem can be stated for $(J^1E, \Omega_{\mathcal{L}})$ (*Hamilton principle*): the states of the field are the sections of π (denoted by $\Gamma(M,E)$) which are critical for the functional $\mathbf{L} : \Gamma(M,E) \to \mathbb{R}$ defined by $\mathbf{L}(\phi) := \int_M (j^1\phi)^*\mathcal{L}$, for every $\phi \in \Gamma(M,E)$. These critical sections can be characterized by the condition

$$(j^1\phi)^* i(X)\Omega_{\mathcal{L}} = 0, \quad \forall X \in \mathfrak{X}(J^1E). \tag{3.3}$$

In natural coordinates, if $\phi = (x^\mu, y^A(x))$, this condition is equivalent to demanding that the components of ϕ satisfy the Euler-Lagrange equations,

$$\frac{\partial \pounds}{\partial y^A}\bigg|_{j^1\phi} - \frac{\partial}{\partial x^\mu}\frac{\partial \pounds}{\partial v^A_\mu}\bigg|_{j^1\phi} = 0, \quad (\text{for } A = 1,\dots,N). \tag{3.4}$$

(For a more detailed description of all these concepts cf. [1, 3, 7, 11, 12, 13, 20, 21]).

The problem of finding these critical sections can be formulated equivalently as follows: to find a distribution D of $T(J^1E)$ satisfying the conditions:

- D is integrable (i.e., *involutive*);
- D is m-dimensional;
- D is $\bar{\pi}^1$-transverse;
- the integral manifolds of D are the critical sections of the Hamilton principle.

From the first and second conditions, there exist $X_1,\dots,X_m \in \mathfrak{X}(J^1E)$ (in involution), which locally span D. Therefore $X = X_1 \wedge \cdots \wedge X_m$ defines a section of $\Lambda^m T(J^1E)$, that is, a nonvanishing, locally decomposable multivector field in J^1E, whose local expression in natural coordinates is

$$X = \bigwedge_{\mu=1}^{m} f\left(\frac{\partial}{\partial x^\mu} + F^A_\mu \frac{\partial}{\partial y^A} + G^A_{\mu\rho}\frac{\partial}{\partial v^A_\rho}\right), \tag{3.5}$$

where f is a nonvanishing function. A representative of the class $\{X\}$ can be selected by the condition $i(X)(\bar{\pi}^{1*}\omega) = 1$ which leads to $f = 1$. Furthermore, the third and fourth conditions impose that X is $\bar{\pi}^1$-transverse, integrable and its integral manifolds are holonomic sections of $\bar{\pi}^1$.

Bearing this in mind, we want to characterize the integrable multivector fields in J^1E whose integral manifolds are canonical prolongations of the sections of π. So, consider the vector bundle projection $\kappa : TJ^1E \to TE$ defined by

$$\kappa(\bar{y},\bar{u}) := T_{\bar{\pi}^1(\bar{y})}\phi\left(T_{\bar{y}}\bar{\pi}^1(\bar{u})\right), \quad (\bar{y},\bar{u}) \in TJ^1E,\ \phi \in \bar{y}. \tag{3.6}$$

This projection is extended in a natural way to $\Lambda^m\kappa : \Lambda^m TJ^1E \to \Lambda^m TE$. Then, a $\bar{\pi}^1$-transverse multivector field $X \in \mathfrak{X}^m(J^1E)$ is said to be *semiholonomic*, or a *second-order partial differential equation*, if $\Lambda^m\kappa \circ X = \Lambda^m T\pi^1 \circ X$. In a natural chart in J^1E, the local expression of X is

$$X \equiv \bigwedge_{\mu=1}^{m} f\left(\frac{\partial}{\partial x^\mu} + v^A_\mu \frac{\partial}{\partial y^A} + G^A_{\mu\rho}\frac{\partial}{\partial v^A_\rho}\right), \tag{3.7}$$

where $f \in C^\infty(J^1E)$ is an arbitrary nonvanishing function. On the other hand, $X \in \mathfrak{X}^m(J^1E)$ is said to be *holonomic* if it is integrable, $\bar{\pi}^1$-transverse and its integral sections $\psi : M \to J^1E$ are holonomic. Then, it can be proved [8] that a multivector field $X \in \mathfrak{X}^m(J^1E)$ is holonomic if and only if it is integrable and semiholonomic.

Of course, if $X \in \mathfrak{X}^m(J^1E)$ is a semiholonomic (resp., holonomic) multivector field, then all those in the class $\{X\} \subset \mathfrak{X}^m(J^1E)$ are semiholonomic (resp., holonomic) too. As a local expression of a representative we can take

$$X \equiv \bigwedge_{\mu=1}^{m} \left(\frac{\partial}{\partial x^\mu} + v^A_\mu \frac{\partial}{\partial y^A} + G^A_{\mu\rho}\frac{\partial}{\partial v^A_\rho}\right). \tag{3.8}$$

Given a section $\phi = (x^\mu, f^A)$, if $j^1\phi = (x^\mu, f^A, \partial f^A/\partial x^\rho)$ is an integral section of this semiholonomic multivector field, then $v^A_\mu = \partial f^A/\partial x^\mu$ and the components of ϕ are a solution to the system of partial differential equations,

$$G^A_{\nu\rho}\left(x^\mu, f^A, \frac{\partial f^A}{\partial x^\mu}\right) = \frac{\partial^2 f^A}{\partial x^\rho \partial x^\nu}. \tag{3.9}$$

On the other hand, it can be proved [8] that classes of locally decomposable and $\bar{\pi}^1$-transverse multivector fields are in one-to-one correspondence with orientable connections in the bundle $\pi : J^1E \to M$ (this correspondence is characterized by the fact that $\mathfrak{D}(X)$ is the *horizontal subbundle* of the connection). For the multivector field (3.8), the associated Ehresmann connection has the local expression

$$\nabla = dx^\mu \otimes \left(\frac{\partial}{\partial x^\mu} + v^A_\mu \frac{\partial}{\partial y^A} + G^A_{\mu\rho}\frac{\partial}{\partial v^A_\rho}\right). \tag{3.10}$$

Then $X \in \mathfrak{X}^m(J^1E)$ is integrable if and only if the connection ∇ associated with the class $\{X\}$ is *flat*, that is, the curvature of ∇ vanishes everywhere. Thus, system (3.9) has a solution if and only if the following additional system of equations holds (for every B, μ, ρ, η)

$$\begin{gathered} 0 = G^B_{\eta\mu} - G^B_{\mu\eta}, \\ 0 = \frac{\partial G^B_{\eta\rho}}{\partial x^\mu} + v^A_\mu \frac{\partial G^B_{\eta\rho}}{\partial y^A} + G^A_{\mu\gamma}\frac{\partial G^B_{\eta\rho}}{\partial v^A_\gamma} - \frac{\partial G^B_{\mu\rho}}{\partial x^\eta} - v^A_\eta \frac{\partial G^B_{\mu\rho}}{\partial y^A} - G^A_{\eta\gamma}\frac{\partial G^B_{\mu\rho}}{\partial v^A_\gamma}. \end{gathered} \tag{3.11}$$

Now, the problem posed by the Hamilton principle can be stated in the following way:

Theorem 3.1. *Let $(J^1E, \Omega_{\mathcal{L}})$ be a Lagrangian system. The critical sections of the Lagrangian variational problem are the integral sections of a class of holonomic multivector fields $\{X_{\mathcal{L}}\} \subset \mathfrak{X}^m(J^1E)$, such that*

$$i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0 \quad \forall X_{\mathcal{L}} \in \{X_{\mathcal{L}}\}. \tag{3.12}$$

Proof. The critical sections must be the integral sections of a class of holonomic multivector fields $\{X_{\mathcal{L}}\} \subset \mathfrak{X}^m(J^1E)$, as a consequence of the above discussion.

Now, using the local expression (3.2) of $\Omega_{\mathcal{L}}$, and taking (3.8) as the representative of the class of semiholonomic multivector fields $\{X_{\mathcal{L}}\}$, from the relation $i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$ we have that the coefficients on dv^A_μ, dy^A, and dx^μ must vanish. But for the coefficients on dv^A_μ, we obtain the identities

$$0 = \left(v^B_\mu - v^B_\mu\right)\frac{\partial^2 \pounds}{\partial v^A_\nu \partial v^B_\mu} \quad \forall A, \nu; \tag{3.13}$$

meanwhile the condition for the coefficients on dy^A leads to the system of equations

$$\frac{\partial^2 \pounds}{\partial v^B_\nu \partial v^A_\mu} G^B_{\nu\mu} = \frac{\partial \pounds}{\partial y^A} - \frac{\partial^2 \pounds}{\partial x^\mu \partial v^A_\mu} - \frac{\partial^2 \pounds}{\partial y^B \partial v^A_\mu} v^B_\mu \quad A = 1, \dots, N. \tag{3.14}$$

Therefore if $j^1\phi = (x^\mu, f^A, \partial f^A/\partial x^\nu)$ must be an integral section of $X_{\mathcal{L}}$, then $v^A_\mu = \partial f^A/\partial x^\mu$, and hence the coefficients $G^B_{\nu\mu}$ must satisfy (3.9). As a consequence, system (3.14) is equivalent to the Euler-Lagrange equations for the section ϕ. Note that, from the above conditions, the coefficients on dx^μ vanish identically. □

So, in Lagrangian field theories, we search for (classes of) nonvanishing and locally decomposable multivector fields $X_{\mathcal{L}} \in \mathfrak{X}^m(J^1E)$ such that

(1) the equation $i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$ holds;
(2) $X_{\mathcal{L}}$ are semiholonomic;
(3) $X_{\mathcal{L}}$ are integrable.

Then we introduce the following nomenclature:

Definition 3.2. $X_{\mathcal{L}} \in \mathfrak{X}^m(J^1E)$ is said to be an *Euler-Lagrange multivector field* for $\mathcal{L}$ if it is semiholonomic and is a solution to the equation $i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$.

Observe that neither the compatibility of system (3.14) nor the integrability of (3.9) are assured. Thus, the existence of Euler-Lagrange multivector fields is not guaranteed in general, and if they exist, they are not necessarily integrable.

Theorem 3.3 (existence and local multiplicity of Euler-Lagrange multivector fields). *Let $(J^1E, \Omega_{\mathcal{L}})$ be a regular Lagrangian system. Then*

(1) *there exist classes of Euler-Lagrange multivector fields for $\mathcal{L}$;*
(2) *in a local system, these multivector fields depend on $N(m^2-1)$ arbitrary functions.*

Proof. (1) First we analyze the local existence of solutions and then their global extension.

In a chart of natural coordinates in J^1E, using the local expression (3.2) of $\Omega_{\mathcal{L}}$ and taking the multivector field given in (3.5) (with $f = 1$) as the representative of the class $\{X_{\mathcal{L}}\}$, from the relation $i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$, we have that the coefficients on dv^A_μ, dy^A, and dx^μ must vanish.

Thus, for the coefficients on dv^A_μ, we obtain that

$$0 = \left(F^B_\mu - v^B_\mu\right)\frac{\partial^2 \pounds}{\partial v^A_\nu \partial v^B_\mu} \quad \forall A, \nu. \tag{3.15}$$

But if $\mathcal{L}$ is regular, the matrix $(\partial^2 \pounds/\partial v^A_\nu \partial v^B_\mu)$ is regular. Therefore $F^B_\mu = v^B_\mu$ (for every B, μ) which proves that if $X_{\mathcal{L}}$ exists it is semiholonomic.

Subsequently, from the condition for the coefficients on dy^A, and taking into account that we have obtained $F^B_\mu = v^B_\mu$, we obtain the set of (3.14), which is a system of N linear equations on the functions $G^B_{\nu\mu}$. This is a compatible system as a consequence of the regularity of $\mathcal{L}$, since the matrix of the coefficients has a (constant) rank equal to N (observe that the matrix of this system is obtained as a rearrangement of rows of the Hessian matrix).

From the above results, we obtain that the coefficients on dx^μ vanish identically. From these results, we are able to assure the local existence of (classes of) multivector fields satisfying the desired conditions. The corresponding global solutions are then obtained using a partition of unity subordinated to a covering of J^1E made of local natural charts.

(2) The expression of a semiholonomic multivector field $X_{\mathcal{L}} \in \{X_{\mathcal{L}}\}$ is given by (3.8). So, it is determined by the Nm^2 coefficients $G^B_{\nu\mu}$, which are related by the N independent equations (3.14). Therefore, there are $N(m^2-1)$ arbitrary functions. □

Now the problem is to find a class of integrable Euler-Lagrange multivector fields, if indeed it exists. Thus, we can choose, from the solutions to this system, those such that $X_{\mathcal{L}}$ verify the integrability condition, that is, the associated connection $\nabla_{\mathcal{L}}$ is flat (3.11). If (3.14) and the first group of (3.11) allow us to isolate $N+(1/2)Nm(m-1)$ coefficients $G^A_{\mu\nu}$ as functions on the remaining ones; and the set of $(1/2)Nm^2(m-1)$ partial differential equations (the second group of (3.11)) on these remaining coefficients satisfies the conditions on Cauchy-Kowalewska's theorem [6], then the existence of integrable Euler-Lagrange multivector fields is assured.

Remark 3.4 (*singular Lagrangian systems*). For singular Lagrangian systems, the existence of Euler-Lagrange multivector fields is not assured except perhaps on some submanifold $S \hookrightarrow J^1E$. Furthermore, locally decomposable and $\bar{\pi}^1$-transverse multivector fields, which are solutions of the field equations, can exist (in general, on some submanifold of J^1E), but none of them is semiholonomic (at any point of this submanifold). As in the regular case, although Euler-Lagrange multivector fields exist on some submanifold S, their integrability is not assured except perhaps on another smaller submanifold $I \hookrightarrow S$; such that the integral sections are contained in I. This condition implies that $\bar{\pi}^1|_I : I \to M$ must be onto on M.

The local treatment of the singular case is as follows: starting from (3.5), and taking the representative obtained by making $f_\mu = 1$, for every μ, we can impose the semiholonomic condition by making $F^A_\mu = v^A_\mu$, for every A, μ. Therefore, we have system (3.14) for the coefficients $G^A_{\mu\nu}$; but this system is not compatible in general except perhaps in a set of points $S_1 \subset J^1E$, which is assumed to be a nonempty closed submanifold. Then, there are Euler-Lagrange multivector fields on S_1, but the number of arbitrary functions on which they depend is not the same as in the regular case, since this number depends on the dimension of S_1 and the rank of the Hessian matrix of £. Next, the tangency condition must be analyzed; and finally the question of integrability must be considered as above, but for a submanifold of S_1.

4. Hamiltonian equations in classical field theories. For the Hamiltonian formalism of field theories, the choice of a multimomentum phase space or multimomentum bundle is not unique (see [10]). In this work we take

$$J^{1*}E \equiv \frac{\Lambda^m_1 T^*E}{\Lambda^m_0 T^*E}, \tag{4.1}$$

where $\Lambda^m_1 T^*E$ is the bundle of m-forms on E vanishing by the action of two π-vertical vector fields, and $\Lambda^m_0 T^*E \equiv \pi^*\Lambda^m T^*M$. We have the natural projections

$$\begin{aligned} \tau^1 &: J^{1*}E \longrightarrow E, \\ \bar{\tau}^1 = \pi \circ \tau^1 &: J^{1*}E \longrightarrow M \end{aligned} \tag{4.2}$$

and we denote by (x^μ, y^A, p^μ_A) the natural local systems of coordinates in $J^{1*}E$ adapted to these bundle structures ($\mu = 1,\dots,m$; $A = 1,\dots,N$).

For constructing Hamiltonian systems, $J^{1*}E$ must be endowed with a geometric structure. There are different ways for doing this, namely, using Hamiltonian sections [3], or Hamiltonian densities [3, 10, 12]. So we construct the Hamilton-Cartan m and $(m+1)$ forms $\Theta_h \in \mathbf{\Omega}^m(J^{1*}E)$ and $\Omega_h = -d\Theta_h \in \mathbf{\Omega}^{m+1}(J^{1*}E)$, which have the local expressions (in an open set $U \subset J^{1*}E$):

$$\begin{aligned}\Theta_h &= p_A^\mu dy^A \wedge d^{m-1}x_\mu - H d^m x,\\ \Omega_h &= -dp_A^\mu \wedge dy^A \wedge d^{m-1}x_\mu + dH \wedge d^m x.\end{aligned} \tag{4.3}$$

$H \in C^\infty(U)$ is a *local Hamiltonian function*, which is given by a Hamiltonian section $h : J^{1*}E \to \Lambda_1^m T^*E$ as follows: if (x^μ, y^A, p_A^μ, p) denotes a natural system of adapted coordinates in $\Lambda_1^m T^*E$, then $h(x^\alpha, y^A, p_A^\alpha) = (x^\alpha, y^A, p_A^\alpha, -H)$. A couple $(J^{1*}E, \Omega_h)$ is said to be a *Hamiltonian system*.

We can state a variational problem for $(J^{1*}E,\ \Omega_h)$ (*Hamilton-Jacobi principle*): the states of the field are the sections of $\bar{\tau}^1$ which are critical for the functional $\mathbf{H}(\psi) := \int_M \psi^*\Theta_h$, for every $\psi \in \Gamma(M, J^{1*}E)$. They are characterized by the condition [3, 10]

$$\psi^* i(X)\Omega_h = 0 \quad \forall X \in \mathfrak{X}(J^{1*}E). \tag{4.4}$$

In natural coordinates, if $\psi(x) = (x^\mu, y^A(x), p_A^\mu(x))$, this condition leads to the system

$$\left.\frac{\partial y^A}{\partial x^\mu}\right|_\psi = \left.\frac{\partial H}{\partial p_A^\mu}\right|_\psi, \qquad \left.\frac{\partial p_A^\mu}{\partial x^\mu}\right|_\psi = -\left.\frac{\partial H}{\partial y^A}\right|_\psi, \tag{4.5}$$

which is known as the *Hamilton-De Donder-Weyl equations*.

Let $(J^{1*}E, \Omega_h)$ be a Hamiltonian system. The problem of finding critical sections solutions of the Hamilton-Jacobi principle can be formulated equivalently as follows: to find a distribution D of $T(J^{1*}E)$ satisfying the conditions:

- D is integrable (i.e., *involutive*);
- D is m-dimensional,
- D is $\bar{\tau}^1$-transverse.
- The integral manifolds of D are the critical sections of the Hamilton-Jacobi principle.

Then, from the first and the second conditions, there exist $X_1, \dots, X_m \in \mathfrak{X}(J^{1*}E)$ (in involution), which locally span D. Therefore $X = X_1 \wedge \cdots \wedge X_m$ defines a section of $\Lambda^m T(J^{1*}E)$, that is, a nonvanishing, locally decomposable multivector field in $J^{1*}E$, whose local expression in natural coordinates is

$$X = \bigwedge_{\mu=1}^{m} f\left(\frac{\partial}{\partial x^\mu} + F_\mu^A \frac{\partial}{\partial y^A} + G_{A\mu}^\rho \frac{\partial}{\partial p_A^\rho}\right), \tag{4.6}$$

where $f \in C^\infty(J^{1*}E)$ is a nonvanishing function. A representative of the class $\{X\}$ can be selected by the condition $i(X)(\bar{\tau}^{1*}\omega) = 1$ which leads to $f = 1$.

Therefore, the problem posed by the Hamilton-Jacobi principle can be stated in the following way:

Theorem 4.1. *The critical sections of the Hamilton-Jacobi principle are the sections $\psi \in \Gamma_c(M, J^{1*}E)$ such that they are the integral sections of a class of integrable and $\bar{\tau}^1$-transverse multivector fields $\{X_{\mathcal{H}}\} \subset \mathfrak{X}^m(J^{1*}E)$ satisfying*

$$i(X_{\mathcal{H}})\Omega_h = 0 \quad \forall X_{\mathcal{H}} \in \{X_{\mathcal{H}}\}. \tag{4.7}$$

Proof. The critical sections must be the integral sections of a class of integrable and $\bar{\tau}^1$-transverse multivector fields $\{X_{\mathcal{H}}\} \subset \mathfrak{X}^m(J^{1*}E)$, as a consequence of the above discussion.

Now, using the local expression (4.3) of Ω_h and taking the multivector field (4.6) (with $f = 1$) as a representative of the class $\{X_{\mathcal{H}}\}$, from $i(X_{\mathcal{H}})\Omega_h = 0$ we obtain that the coefficients on dp_A^μ must vanish:

$$0 = F_\nu^A - \frac{\partial H}{\partial p_A^\nu} \quad \forall A, \nu; \tag{4.8}$$

and the same happens for the coefficients on dy^A:

$$0 = G_{A\mu}^\mu + \frac{\partial H}{\partial y^A} \quad A = 1, \dots, N. \tag{4.9}$$

(Using these results, the coefficients on dx^μ vanish identically.)

Now, if $\psi(x) = (x^\mu, y^A(x^\nu), p_A^\mu(x^\nu))$ has to be an integral section of $X_{\mathcal{H}}$ then

$$F_\mu^A \circ \psi = \frac{\partial y^A}{\partial x_\mu}, \qquad G_{A\mu}^\mu \circ \psi = -\frac{\partial p_A^\mu}{\partial x^\mu}; \tag{4.10}$$

and (4.8) and (4.9) are the Hamilton-De Donder-Weyl equations (4.5) for ψ. □

Thus, we search for (classes of) $\bar{\tau}^1$-transverse and locally decomposable multivector fields $X_{\mathcal{H}} \in \mathfrak{X}^m(J^{1*}E)$ such that

(1) $i(X_{\mathcal{H}})\Omega_h = 0$ holds;
(2) $X_{\mathcal{H}}$ are integrable.

Classes of locally decomposable and $\bar{\tau}^1$-transverse multivector fields are in one-to-one correspondence with connections in the bundle $\bar{\tau}^1 : J^{1*}E \to M$. Then $X_{\mathcal{H}}$ is integrable if and only if the curvature of the connection associated with this class vanishes everywhere.

Definition 4.2. A multivector field $X_{\mathcal{H}} \in \mathfrak{X}^m(J^{1*}E)$ will be called a *Hamilton-De Donder-Weyl (HDW) multivector field* for the system $(J^{1*}E, \Omega_h)$ if it is $\bar{\tau}^1$-transverse, locally decomposable and verifies the equation $i(X_{\mathcal{H}})\Omega_h = 0$.

For a Hamiltonian system, the existence of Hamilton-De Donder-Weyl multivector fields is guaranteed, although they are not necessarily integrable.

Theorem 4.3 (existence and local multiplicity of HDW-multivector fields)**.** *Let $(J^{1*}E, \Omega_h)$ be a Hamiltonian system. Then*

(1) *there exist classes of HDW-multivector fields $\{X_{\mathcal{H}}\}$;*
(2) *in a local system, the above solutions depend on $N(m^2 - 1)$ arbitrary functions.*

Proof. (1) Bearing in mind the proof of Theorem 4.1, we have that (4.8) makes a system of Nm linear equations which determines univocally the functions F_ν^A, while

(4.9) is a compatible system of N linear equations on the Nm^2 functions $G^{\mu}_{A\nu}$. These results assure the local existence. The global solutions are obtained using a partition of unity subordinated to a covering of $J^{1*}E$ made of natural charts.

(2) In natural coordinates in $J^{1*}E$, a representative of a class of HDW-multivector fields $X_{\mathcal{H}} \in \{X_{\mathcal{H}}\}$ is given by (4.6) (with $f = 1$). Therefore, it is determined by the Nm coefficients F^A_ν, which are obtained as the solution to (4.8), and by the Nm^2 coefficients $G^{\mu}_{A\nu}$, which are related by the N independent equations (4.9). Therefore, there are $N(m^2-1)$ arbitrary functions. □

In order to find a class of integrable HDW-multivector fields (if it exists) we must impose that $X_{\mathcal{H}}$ verify the integrability condition: the curvature of the associated connection $\nabla_{\mathcal{H}}$ vanishes everywhere, that is, the following system of equations holds (for $1 \le \mu < \eta \le m$)

$$\begin{aligned} 0 &= \frac{\partial F^B_\eta}{\partial x^\mu} + F^A_\mu \frac{\partial F^B_\eta}{\partial y^A} + G^{\gamma}_{A\mu}\frac{\partial F^B_\eta}{\partial p^\gamma_A} - \frac{\partial F^B_\mu}{\partial x^\eta} - F^A_\eta \frac{\partial F^B_\mu}{\partial y^A} - G^{\rho}_{A\eta}\frac{\partial F^B_\mu}{\partial p^\rho_A} \\ &= \frac{\partial^2 H}{\partial x^\mu \partial p^\eta_B} + \frac{\partial H}{\partial p^\mu_A}\frac{\partial^2 H}{\partial y^A \partial p^\eta_B} + G^{\gamma}_{A\mu}\frac{\partial^2 H}{\partial p^\gamma_A \partial p^\eta_B} - \frac{\partial^2 H}{\partial x^\eta \partial p^\eta_B} \\ &\quad - \frac{\partial H}{\partial p^\eta_A}\frac{\partial^2 H}{\partial y^A \partial p^\mu_B} - G^{\rho}_{A\eta}\frac{\partial^2 H}{\partial p^\rho_A \partial p^\mu_B}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} 0 &= \frac{\partial G^\rho_{B\eta}}{\partial x^\mu} + F^A_\mu \frac{\partial G^\rho_{B\eta}}{\partial y^A} + G^{\gamma}_{A\mu}\frac{\partial G^\rho_{B\eta}}{\partial p^\gamma_A} - \frac{\partial G^\rho_{B\mu}}{\partial x^\eta} - F^A_\eta \frac{\partial G^\rho_{B\mu}}{\partial y^A} - G^{\gamma}_{A\eta}\frac{\partial G^\rho_{B\mu}}{\partial p^\gamma_A} \\ &= \frac{\partial G^\rho_{B\eta}}{\partial x^\mu} + \frac{\partial H}{\partial p^\mu_A}\frac{\partial G^\rho_{B\eta}}{\partial y^A} + G^{\gamma}_{A\mu}\frac{\partial G^\rho_{B\eta}}{\partial p^\gamma_A} - \frac{\partial G^\rho_{B\mu}}{\partial x^\eta} - \frac{\partial H}{\partial p^\eta_A}\frac{\partial G^\rho_{B\mu}}{\partial y^A} - G^{\gamma}_{A\eta}\frac{\partial G^\rho_{B\mu}}{\partial p^\gamma_A}, \end{aligned} \tag{4.12}$$

(where use is made of the Hamiltonian equations). Hence the number of arbitrary functions will be in general less than $N(m^2-1)$.

As this is a system of partial differential equations with linear restrictions, there is no way of assuring the existence of an integrable solution. Considering the Hamiltonian equation (4.9) for the coefficients $G^{\mu}_{A\nu}$, together with the integrability conditions (4.11) and (4.12), we have $N + (1/2)Nm(m-1)$ linear equations and $(1/2)Nm^2(m-1)$ partial differential equations. Then, if the set of linear restrictions (4.9) and (4.11) allow us to isolate $N + (1/2)Nm(m-1)$ coefficients $G^{\mu}_{A\nu}$ as functions on the remaining ones; and the set of $(1/2)Nm^2(m-1)$ partial differential equations (4.12) on these remaining coefficients satisfies certain conditions, then the existence of integrable HDW-multivector fields (in $J^{1*}E$) is assured. If this is not the case, we can eventually select some particular HDW-multivector field solution, and apply an integrability algorithm in order to find a submanifold $\mathcal{I} \hookrightarrow J^{1*}E$ (if it exists), where this multivector field is integrable (and tangent to $\mathcal{I}$).

Remarks. • (*Restricted Hamiltonian systems*). There are many interesting cases in field theories where the Hamiltonian field equations are established not in $J^{1*}E$, but rather in a submanifold $\mathrm{j}_0 : P \hookrightarrow J^{1*}E$, such that P is a fiber bundle over E (and M), and the corresponding projections $\tau^1_0 : P \to E$ and $\bar{\tau}^1_0 : P \to M$ satisfy $\tau^1 \circ \mathrm{j}_0 = \tau^1_0$ and $\bar{\tau}^1 \circ \mathrm{j}_0 = \bar{\tau}^1_0$.

Now, the existence of HDW-multivector fields is not assured. However, an algorithmic procedure can be outlined with the aim of obtaining a submanifold S_f of P, where HDW-multivector fields exist, can be outlined. Of course the solution is not unique, in general, but the number of arbitrary functions is not the same as above (it depends on the dimension of S_f).

Finally, the question of integrability must be considered, and similar considerations to those above must be made for the submanifold S_f instead of $J^{1*}E$.

• (*Hamiltonian system associated with a hyper-regular Lagrangian system*). If the Hamiltonian system $(J^{1*}E, \Omega_h)$ is associated with a *hyper-regular Lagrangian system*, then there exists the so-called *Legendre map*, which is a diffeomorphism between J^1E and $J^{1*}E$ [3, 5, 10]. In this case, it can be proved [10] that, if $X_{\mathcal{L}} \in \mathfrak{X}^m(J^1E)$ and $X_{\mathcal{H}} \in \mathfrak{X}^m(J^{1*}E)$ are multivector fields solution of the Lagrangian and Hamiltonian field equations, respectively, then

$$\Lambda^m TF\mathcal{L} \circ X_{\mathcal{L}} = f X_{\mathcal{H}} \circ F\mathcal{L} \tag{4.13}$$

for some $f \in C^\infty(J^{1*}E)$. That is, we have the following (commutative) diagram:

$$\begin{tikzcd} \Lambda^m TJ^1E \arrow[r, "\Lambda^m TF\mathcal{L}"'] & \Lambda^m TJ^{1*}E \\ J^1E \arrow[u, "X_{\mathcal{L}}"] \arrow[r, "F\mathcal{L}"] & J^{1*}E \arrow[u, "X_{\mathcal{H}}"'] \end{tikzcd} \tag{4.14}$$

we say that the classes $\{X_{\mathcal{L}}\}$ and $\{X_{\mathcal{H}}\}$ are $F\mathcal{L}$-related.

5. Conclusions and outlook. We have used multivector fields in fiber bundles for setting and studying the Lagrangian and Hamiltonian field equations of first-order classical field theories. In particular, we have shown that:

• The field equations for first-order classical field theories in the Lagrangian formalism (*Euler-Lagrange equations*) can be written using multivector fields in J^1E. This description allows us to write the field equations for field theories in an analogous way to the dynamical equations for (time-dependent) Lagrangian mechanical systems.

• The Lagrangian equations can have no integrable solutions in J^1E, for neither regular nor singular Lagrangian systems.

In the regular case, *Euler-Lagrange multivector fields* (i.e., semiholonomic solutions to the equation $i(X_{\mathcal{L}})\Omega_{\mathcal{L}} = 0$) always exist; but they are not necessarily integrable. In the singular case, not even the existence of such an Euler-Lagrange multivector field is assured. In both cases, the multivector field solution (if it exists) is not unique.

• The Hamiltonian field equations can be written using multivector fields in $J^{1*}E$ (the multimomentum bundle of the Hamiltonian formalism) in an analogous way to the dynamical equations for (time-dependent) Hamiltonian mechanical systems.

• The field equations $i(X_{\mathcal{H}})\Omega_h = 0$, with $X_{\mathcal{H}} \in \mathfrak{X}^m(J^{1*}E)$ locally decomposable and $\bar{\tau}^1$-transverse, have solution everywhere in $J^{1*}E$, which is not unique; that is, there are classes of *Hamilton-De Donder-Weyl multivector fields* which are solutions to these equations. Nevertheless, these multivector fields are not necessarily integrable everywhere in $J^{1*}E$.

• This multivector field formulation is especially useful for characterizing *symmetries*, both in the Lagrangian and Hamiltonian formalisms of field theories. First attempts at this characterization have been already carried out [9], but new developments in this area are expected in the future.

Acknowledgments. We are grateful for the financial support of the Comisión Interministerial de Ciencia y Tecnología (CICYT) PB98-0920. We thank Mr. Jeff Palmer for his assistance in preparing the English version of the manuscript.

References

[1] E. Binz, J. Śniatycki, and H. Fischer, *Geometry of Classical Fields*, North-Holland Mathematics Studies, vol. 154, North-Holland, Amsterdam, 1988.

[2] F. Cantrijn, L. A. Ibort, and M. de León, *Hamiltonian structures on multisymplectic manifolds*, Rend. Sem. Mat. Univ. Politec. Torino **54** (1996), no. 3, 225–236.

[3] J. F. Cariñena, M. Crampin, and L. A. Ibort, *On the multisymplectic formalism for first order field theories*, Differential Geom. Appl. **1** (1991), no. 4, 345–374.

[4] M. de León, J. Marín-Solano, and J. C. Marrero, *Ehresmann connections in classical field theories*, Differential Geometry and Its Applications (Granada, 1994), An. Fis. Monogr., vol. 2, CIEMAT, Madrid, 1995, pp. 73–89.

[5] ———, *A geometrical approach to classical field theories: a constraint algorithm for singular theories*, New Developments in Differential Geometry (Debrecen, 1994) (L. Tamássy and J. Szenthe, eds.), Math. Appl., vol. 350, Kluwer Academic Publishers, Dordrecht, 1996, pp. 291–312.

[6] J. Dieudonné, *Éléments d'Analyse. Tome IV. Chapitres XVIII à XX*, Cahiers Scientifiques, Fasc. 34., Gauthier-Villars, Paris, 1977 (French).

[7] A. Echeverría-Enríquez, M. C. Muñoz-Lecanda, and N. Román-Roy, *Geometry of Lagrangian first-order classical field theories*, Fortschr. Phys. **44** (1996), 235–280.

[8] ———, *Multivector fields and connections: setting Lagrangian equations in field theories*, J. Math. Phys. **39** (1998), no. 9, 4578–4603.

[9] ———, *Multivector field formulation of Hamiltonian field theories: equations and symmetries*, J. Phys. A **32** (1999), no. 48, 8461–8484.

[10] ———, *Geometry of multisymplectic Hamiltonian first-order field theories*, J. Math. Phys. **41** (2000), no. 11, 7402–7444.

[11] P. L. García, *The Poincaré-Cartan invariant in the calculus of variations*, Symposia Mathematica, Vol. 14 (Convegno di Geometria Simplettica e Fisica Matematica, INDAM, Rome, 1973), Academic Press, London, 1974, pp. 219–246.

[12] G. Giachetta, L. Mangiarotti, and G. Sardanashvily, *New Lagrangian and Hamiltonian Methods in Field Theory*, World Scientific Publishing, Singapore, 1997.

[13] H. Goldschmidt and S. Sternberg, *The Hamilton-Cartan formalism in the calculus of variations*, Ann. Inst. Fourier (Grenoble) **23** (1973), no. 1, 203–267.

[14] M. J. Gotay, *A multisymplectic framework for classical field theory and the calculus of variations. I. Covariant Hamiltonian formalism*, Mechanics, Analysis and Geometry: 200 years after Lagrange (M. Francaviglia, ed.), North-Holland Delta Ser., North-Holland, Amsterdam, 1991, pp. 203–235.

[15] F. Hélein and J. Kouneiher, *Finite dimensional Hamiltonian formalism for gauge and field theories*, http://arxiv.org/abs/math-ph/0010036.com, 2000.

[16] I. V. Kanatchikov, *Novel algebraic structures from the polysymplectic form in field theory*, GROUP21, Physical Applications and Mathematical Aspects of Geometry, Groups and Algebras (H. A. Doebner, W. Scherer, and C. Schulte, eds.), vol. 2, World Scientific Publishing, Singapore, 1997.

[17] ———, *Canonical structure of classical field theory in the polymomentum phase space*, Rep. Math. Phys. **41** (1998), no. 1, 49–90.

[18] J. E. Marsden and S. Shkoller, *Multisymplectic geometry, covariant Hamiltonians, and water waves*, Math. Proc. Cambridge Philos. Soc. **125** (1999), no. 3, 553–575.

[19] C. Paufler and H. Romer, *Geometry of Hamiltonean n-vectors fields in multisymplectic field theory*, http://arxiv.org/abs/math-ph/0102008.com, 2001.

[20] G. Sardanashvily, *Generalized Hamiltonian Formalism for Field Theory. Constraint Systems*, World Scientific Publishing, New Jersey, 1995.

[21] D. J. Saunders, *The Geometry of Jet Bundles*, London Mathematical Society Lecture Note Series, vol. 142, Cambridge University Press, Cambridge, 1989.

A. Echeverría-Enríquez: Departamento de Matemática Aplicada IV., Edificio C-3, Campus Norte UPC., C/ Jordi Girona 1. E-08034 Barcelona, Spain

M. C. Muñoz-Lecanda: Departamento de Matemática Aplicada IV., Edificio C-3, Campus Norte UPC., C/ Jordi Girona 1. E-08034 Barcelona, Spain
E-mail address: matmcml@mat.upc.es

N. Román-Roy: Departamento de Matemática Aplicada IV., Edificio C-3, Campus Norte UPC., C/ Jordi Girona 1. E-08034 Barcelona, Spain
E-mail address: matnrr@mat.upc.es