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COMPACT HERMITIAN OPERATORS ON PROJECTIVE TENSOR PRODUCTS OF BANACH ALGEBRAS

T. K. DUTTA, H. K. NATH, and H. K. SARMAH

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Let U and V be, respectively, an infinite- and a finite-dimensional complex Banach algebras, and let $U \otimes_p V$ be their projective tensor product. We prove that (i) every compact Hermitian operator T_1 on U gives rise to a compact Hermitian operator T on $U \otimes_p V$ having the properties that $\|T_1\| = \|T\|$ and $\operatorname{sp}(T_1) = \operatorname{sp}(T)$; (ii) if U and V are separable and U has Hermitian approximation property (HAP), then $U \otimes_p V$ is also separable and has HAP; (iii) every compact analytic semigroup (CAS) on U induces the existence of a CAS on $U \otimes_p V$ having some nice properties. In addition, the converse of the above results are discussed and some open problems are posed.

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1. Introduction. We first introduce the terminologies which are initially needed for our main purpose.

1.1. The projective tensor norm $\|\cdot\|_p$ is defined on the algebraic tensor product $U \otimes V$ of two complex Banach algebras U and V by

$$\|a\|_p = \inf\left\{ \sum_{i=1}^{n} ||u_i|| \, ||v_i|| : a = \sum_{i=1}^{n} u_i \otimes v_i \right\}, \tag{1.1}$$

where the infimum is taken over all (finite) representations of $a \in U \otimes V$. The *projective tensor product* $U \otimes_p V$ is the completion of $U \otimes V$ with this norm. Furthermore, a norm $\|\cdot\|_\pi$ on $U \otimes V$ is said to be a *cross-norm* if $\|u \otimes v\|_\pi = \|u\| \|v\|$. (For detailed discussion of various tensor products, see [1, 3, 6, 7].)

1.2. A *state* on a unital Banach algebra U with the unit e is a continuous linear functional f such that $\|f\| = f(e) = 1$, and an element u in U is *Hermitian* if and only if its numerical range, that is, $N(u) = \{f(u) : f \text{ is a state on } U\}$, is contained in the real line. Equivalently, u is Hermitian if and only if $\lim_{\alpha \to 0+}(1/\alpha)[\|e + i\alpha u\| - 1] = 0$. If an operator T on U is such that it is a Hermitian as an element of the operator algebra, then T is called a *Hermitian operator* on U.

1.3. A linear transformation T mapping a normed linear space X into a normed linear space Y is said to be *compact* if, given any sequence $\{x_n\}$ in X such that $\{\|x_n\|\}$ is bounded, the sequence $\{Tx_n\}$ has a convergent subsequence. If T is both compact and Hermitian, it is then termed as a *compact Hermitian operator*. A compact Hermitian operator enjoys many powerful technical results (see [4]).

Throughout this paper, unless stated specifically, U denotes an infinite-dimensional complex Banach algebra with its dual U^*, V is a finite-dimensional complex Banach algebra with $\dim V = k$, $\{e_1, e_2, \dots, e_k\}$ a standardized basis for V with $\|e_j\| = 1$, for all $j = 1, 2, \dots, k$ and $U \otimes_p V$ their projective tensor product. Our primary aim is to study how a compact Hermitian operator on U, the Hermitian approximation property in U and a compact analytic semigroup on U gives rise to the significant existence of those on $U \otimes_p V$. The converse route from $U \otimes_p V$ to U with these properties is also investigated with some fruitful outcomes.

2. The main results.

We first prove a simple but illuminating lemma.

Lemma 2.1. *Every element a of $U \otimes_p V$ can be expressed in the form $\sum_{j=1}^{k} w_j \otimes e_j$.*

Proof. We know that $U \otimes_p V$ is a Banach algebra consisting of all elements of the form

$$a = \sum_{i=1}^{\infty} u_i \otimes v_i, \quad \text{where } \sum_{i=1}^{\infty} \|u_i\| \|v_i\| < \infty, \tag{2.1}$$

(see [5]). We can choose v_i such that $\|v_i\| = 1$ for all i. If $\|v_i\| \neq 1$, put $v_i' = v_i / \|v_i\|$ and replace v_i by $\|v_i\| v_i'$ and adjoin $\|v_i\|$ to u_i. Then $v_i = \sum_{j=1}^{k} \alpha_{ij} e_j$, where α_{ij}'s are scalars.

Now

$$\begin{aligned} a_n &= \sum_{i=1}^{n} u_i \otimes v_i = \sum_{i=1}^{n} u_i \otimes \sum_{j=1}^{k} \alpha_{ij} e_j \\ &= \sum_{j=1}^{k} \left(\sum_{i=1}^{n} \alpha_{ij} u_i \right) \otimes e_j = \sum_{j=1}^{k} w_{j,n} \otimes e_j, \quad \text{where } w_{j,n} = \sum_{i=1}^{n} \alpha_{ij} u_i \in U. \end{aligned} \tag{2.2}$$

If $v = \sum_{j=1}^{k} \beta_j e_j$ is an arbitrary element of V, then the map $\phi : V \to \ell_k^{\infty}$, defined by $\phi(v) = (\beta_1, \beta_2, \dots, \beta_k)$, is a topological isomorphism. Hence there is a positive constant M such that $\|\phi(v)\| \le M\|v\|$, for all $v \in V$. Substituting v_i in v, $\|\phi(v_i)\| \le M\|v_i\| = M$, for all i. This gives

$$\operatorname*{Max}_{1 \le j \le k} \{ |\alpha_{ij}| \} \le M. \tag{2.3}$$

Also,

$$\sum_{i=1}^{n} |\alpha_{ij}| \|u_i\| \le M \sum_{i=1}^{n} \|u_i\| \le M \sum_{i=1}^{\infty} \|u_i\| < \infty. \tag{2.4}$$

This shows that the partial sums of the series $\sum_{i=1}^{\infty} \alpha_{ij} u_i$ are absolutely uniformly bounded and hence the series $\sum_{i=1}^{\infty} \alpha_{ij} u_i$ is absolutely summable. Consequently, $\sum_{i=1}^{\infty} \alpha_{ij} u_i$ is summable to an element, say w_j in U. Thus, $\lim_{n \to \infty} w_{j,n} = w_j$ and so $a = \lim_{n \to \infty} a_n = \sum_{j=1}^{k} w_j \otimes e_j$. □

Next, our first key result is the following.

Theorem 2.2. *Every compact Hermitian operator T_1 on U gives rise to a compact Hermitian operator T on $U \otimes_p V$ having the properties that $\|T_1\| = \|T\|$ and $\operatorname{sp}(T_1) = \operatorname{sp}(T)$.*

Proof. We prove the result in different steps.

Step 1. One can define a map $T : U \otimes_p V \to U \otimes_p V$ by the rule

$$T\left(\sum_i u_i \otimes v_i\right) = \sum_i (T_1 u_i) \otimes v_i \quad \forall a = \sum_i u_i \otimes v_i \in U \otimes_p V. \tag{2.5}$$

It is easy to show that the map is well defined. Moreover, the linearity of T follows immediately from its definition.

Next we aim at proving the bound of T. For any arbitrary element $a \in U \otimes_p V$ and $\varepsilon > 0$, the definition of the projective norm provides a finite representation $\sum_{i=1}^n u_i \otimes v_i$ such that

$$\|a\|_p + \varepsilon > \sum_{i=1}^n ||u_i|| \, ||v_i||. \tag{2.6}$$

For this representation of a we obtain

$$\|Ta\|_p = \left\| \sum_{i=1}^n (T_1 u_i) \otimes v_i \right\|_p \le ||T_1|| \sum_{i=1}^n ||u_i|| \, ||v_i|| < ||T_1|| (\|a\|_p + \varepsilon). \tag{2.7}$$

Since ε is arbitrary, it follows that $\|T_a\|_p \le \|T_1\| \, \|a\|_p$ for each $a \in U \otimes_p V$.

Consequently, T is bounded. Furthermore, the compactness of T can be proved without much difficulty.

Step 2. Our next attempt is to show that T is Hermitian. T_1 is given to be Hermitian and therefore $\lim_{\alpha \to 0+} (1/\alpha)\{\|I_1 + i\alpha T_1\| - 1\} = 0$, where α is real and I_1 is the identity map on U. Let I be the identity map on $U \otimes_p V$, $\varepsilon > 0$, and let $\sum_{i=1}^n u_i \otimes v_i$ be a finite representation of a such that $\|a\|_p + \varepsilon > \sum_{i=1}^n \|u_i\| \, \|v_i\|$. Then we can obtain

$$||(I + i\alpha T)a||_p \le ||I_1 + i\alpha T_1|| \, \|a\|_p \quad \forall a \in U \otimes_p V. \tag{2.8}$$

This gives $\|(I + i\alpha T)\| \le \|I_1 + i\alpha T_1\|$.

On the other hand, let $u \in U$ with $\|u\| = 1$. Choose $v \in V$ such that $\|v\| = 1$. Then $\|u \otimes v\|_p = 1$.

Now,

$$\begin{aligned}
||(I + i\alpha T)|| &= \sup\{||(I + i\alpha T)a\|_p : \|a\|_p = 1\} \\
&\ge ||(I + i\alpha T)(u \otimes v)||_p \\
&= ||u \otimes v + (i\alpha T_1 u) \otimes v||_p \\
&= ||[(I_1 + i\alpha T_1)u] \otimes v||_p \\
&= ||(I_1 + i\alpha T_1)u|| \, \|v\| \\
&= ||(I_1 + i\alpha T_1)u||.
\end{aligned} \tag{2.9}$$

Thus, $\|I + i\alpha T\| \ge \|(I_1 + i\alpha T_1)u\|$, for all $u \in U$ with $\|u\| = 1$. This yields $\|I + i\alpha T\| \ge \|I_1 + i\alpha T_1\|$ and so $\|I + i\alpha T\| = \|I_1 + i\alpha T_1\|$.

Therefore, $\lim_{\alpha \to 0+}((\|I + i\alpha T\| - 1)/\alpha) = \lim_{\alpha \to 0+}((\|I_1 + i\alpha T_1\| - 1)/\alpha) = 0$ and hence T is Hermitian.

Step 3. From Step 1 we can obtain that $\|Ta\|_p \le \|T_1\| \, \|a\|_p$, for each $a \in U \otimes_p V$. This implies that $\|T\| \le \|T_1\|$. The converse inequality $\|T_1\| \le \|T\|$ can also be established without much difficulty. This establishes that $\|T\| = \|T_1\|$.

Step 4. We now concentrate on the result $\mathrm{sp}(T_1) = \mathrm{sp}(T)$.

Let

$$\begin{aligned}\lambda_1 \in \mathrm{sp}\,(T_1) &\Longrightarrow T_1 - \lambda_1 I_1 \text{ is singular} \\ &\Longrightarrow \exists \text{ a nonzero vector } u \in U \text{ such that } (T_1 - \lambda_1 I_1)u = 0.\end{aligned} \tag{2.10}$$

Let $v \in V$ be a vector with $v \ne 0$. Then $u \otimes v \in U \otimes_p V$ with $u \otimes v \ne 0$.

Now,

$$\begin{aligned}(T - \lambda_1 I)(u \otimes v) &= T(u \otimes v) - \lambda_1 I(u \otimes v) \\ &= (T_1 u) \otimes v - (\lambda_1 u) \otimes v \\ &= (T_1 - \lambda_1 I_1)u \otimes v = 0.\end{aligned} \tag{2.11}$$

Consequently, $\lambda_1 \in \mathrm{sp}(T)$ and thus $\mathrm{sp}(T_1) \subseteq \mathrm{sp}(T)$. On the other hand, let $\lambda \in \mathrm{sp}(T)$. Then there exists a nonzero vector $a = \sum_{j=1}^{k} u_j \otimes e_j \in U \otimes_p V$, such that

$$\begin{aligned}(T - \lambda I)a = 0 \Longrightarrow T\left(\sum_{j=1}^{k} u_j \otimes e_j\right) - \lambda \sum_{j=1}^{k} u_j \otimes e_j = 0 \\ \Longrightarrow \sum_{j=1}^{k} (T_1 u_j) \otimes e_j - \sum_{j=1}^{k} (\lambda u_j) \otimes e_j = 0 \\ \Longrightarrow \sum_{j=1}^{k} (T_1 - \lambda I_1) u_j \otimes e_j = 0 \\ \Longrightarrow (T_1 - \lambda I_1) u_j = 0 \quad \forall j = 1, 2, \dots, k.\end{aligned} \tag{2.12}$$

Since $u \ne 0$, there exists at least one j such that $u_j \ne 0$. Hence, λ is an eigenvalue of T_1. So, $\mathrm{sp}(T) \subseteq \mathrm{sp}(T_1)$. Ultimately we have $\mathrm{sp}(T) = \mathrm{sp}(T_1)$. □

Remark 2.3. (i) If both U and V are infinite-dimensional Banach algebras, then the compactness of T_1 may not imply that of T. For example, choose a sequence $\{v_n\}$ in V with $\|v_n\| = 1$ so that $\{v_n\}$ has no convergent subsequence. (To wit, let $v = 1_2$, $v_n = e_n$, where e_n is a sequence with nth place equal to 1 and 0 elsewhere. Then $\{e_n\}$ cannot have a convergent subsequence.)

Let $u \in U$ with $\|u\| = 1$. Then $\{u \otimes v_n\}$ is a bounded sequence in $U \otimes_p V$. Now, $T(u \otimes v_n) = T_1 u \otimes v_n$ and

$$||T(u \otimes v_n) - T(u \otimes v_m)||_p = ||T_1 u \otimes v_n - T_1 u \otimes v_m||_p = ||T_1 u|| \, ||v_n - v_m||. \tag{2.13}$$

Equation (2.13) shows that $\{T(u \otimes v_n)\}$ has a convergent subsequence only if the sequence $\{v_n\}$ has a convergent subsequence. This ascertains that T cannot be compact. So, we are forced to consider V to be finite dimensional.

(ii) Although $\mathrm{sp}(T)$ is, in general, a larger set than $\mathrm{sp}(T_1)$, the result $\mathrm{sp}(T) = \mathrm{sp}(T_1)$, indicates that many eigenvalues of T repeat the same eigenvalue. For the sake of completeness, we illustrate the situation *with an example.*

Let $\mathscr{A}$ be the Banach algebra of 2×2 real square matrices with $\|A\| = \max\{|u_i| : i = 1,2,3,4\}$, where

$$A = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix} \in \mathscr{A}. \tag{2.14}$$

Let $U = V = \mathscr{A}$. Put

$$e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad e_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{2.15}$$

Then $\beta = \{e_1, e_2, e_3, e_4\}$ is a basis of $\mathscr{A}$.

Define a map $T_1 : \mathscr{A} \to \mathscr{A}$ by

$$T_1 A = \begin{bmatrix} \sum_{i=1}^{4} \alpha_i u_i & \sum_{i=1}^{4} \beta_i u_i \\ \sum_{i=1}^{4} \gamma_i u_i & \sum_{i=1}^{4} \delta_i u_i \end{bmatrix}. \tag{2.16}$$

Then it is evident that T_1 is a linear operator on $\mathscr{A}$ and the matrix representation of T_1 with respect to the basis β is

$$[T_1]_\beta = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \delta_1 & \delta_2 & \delta_3 & \delta_4 \end{bmatrix}. \tag{2.17}$$

Next, $\{e_i \otimes e_j : i, j = 1,2,3,4\}$ is a basis for $\mathscr{A} \otimes_p \mathscr{A}$. If we define an operator $T : \mathscr{A} \otimes_p \mathscr{A} \to \mathscr{A} \otimes_p \mathscr{A}$ by $T(\sum_i u_i \otimes v_i) = \sum_i (T_1 u_i) \otimes v_i$, then the matrix representation of T is a 16×16 square matrix, that is, exhibited below:

$$\begin{bmatrix}
a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 \\
0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & a_4 & 0 & 0 \\
0 & 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & a_4 & 0 \\
0 & 0 & 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & a_4 \\
b_1 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & b_4 & 0 & 0 & 0 \\
0 & b_1 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & b_4 & 0 & 0 \\
0 & 0 & b_1 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & b_4 & 0 \\
0 & 0 & 0 & b_1 & 0 & 0 & 0 & b_2 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & b_4 \\
c_1 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & c_4 & 0 & 0 & 0 \\
0 & c_1 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & c_4 & 0 & 0 \\
0 & 0 & c_1 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & c_4 & 0 \\
0 & 0 & 0 & c_1 & 0 & 0 & 0 & c_2 & 0 & 0 & 0 & c_3 & 0 & 0 & 0 & c_4 \\
d_1 & 0 & 0 & 0 & d_2 & 0 & 0 & 0 & d_3 & 0 & 0 & 0 & d_4 & 0 & 0 & 0 \\
0 & d_1 & 0 & 0 & 0 & d_2 & 0 & 0 & 0 & d_3 & 0 & 0 & 0 & d_4 & 0 & 0 \\
0 & 0 & d_1 & 0 & 0 & 0 & d_2 & 0 & 0 & 0 & d_3 & 0 & 0 & 0 & d_4 & 0 \\
0 & 0 & 0 & d_1 & 0 & 0 & 0 & d_2 & 0 & 0 & 0 & d_3 & 0 & 0 & 0 & d_4
\end{bmatrix}. \tag{2.18}$$

For the matrix representation of T_1 and T, we found that if T_1 is a Hermitian operator, that is, T_1 has a diagonal matrix representation, then so is T. Moreover, putting different values of α_i's, β_i's, γ_i's, and δ_1's, we can see that T_1 and T have the same eigenvalues. For instance, if we substitute

$$\begin{array}{llll} \alpha_1 = 1, & \alpha_2 = -1, & \alpha_3 = 2, & \alpha_4 = -2; \\ \beta_1 = -1, & \beta_2 = 2, & \beta_3 = 3, & \beta_4 = -3; \\ \gamma_1 = 2, & \gamma_2 = 3, & \gamma_3 = 3, & \gamma_4 = 1; \\ \delta_1 = -2, & \delta_2 = -3, & \delta_3 = 1, & \delta_4 = 4; \end{array} \tag{2.19}$$

then the eigenvalues of T_1 are 4.456658, 6.930083, 2.234990, and -3.621735 (correct up to six decimal places). Then the eigenvalues of T are also the same each repeating four times (verified by computer).

To show some applications of Theorem 2.2, we want to concentrate on the study of the Hermitian approximation property and compact analytic semigroups on $U \otimes_p V$. We first recall some definitions.

(I) A Banach space X is said to have the *Hermitian approximation property* (HAP), if for each compact subset C of X and every $\varepsilon > 0$, there is a compact Hermitian operator H on X such that $\|Hx - x\| < \varepsilon$ for every $x \in C$, and $\|H\| \le 1$.

To wit, the spaces 1_2, c_0, and so forth have the HAP.

If X is separable, HAP is equivalent to the existence of a sequence $\{h_m\}$ with $\|h_m\| \le 1$, for every $m \in \mathbb{N}$ of compact Hermitian operators on X such that

$$||h_m x - x|| \longrightarrow 0 \quad \text{as } m \longrightarrow \infty \ \forall x \in X. \tag{2.20}$$

(II) Let $S_\alpha = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ and } |\mathrm{Arg}(z)| < \alpha\}$ be a sector in $\mathbb{C}$, where α lies in $(0, \pi/2]$. *An analytic semigroup* T^z on X is a family of bounded linear operators $T^z : X \to X$ defined for $z \in S_\alpha$, where α is fixed, satisfying the following conditions:

(i) $T^{z_1} T^{z_2} = T^{z_1 + z_2}$ for all z_1, z_2 in S_α;

(ii) T^z is an analytic-valued function of $z \in S_\alpha$;

(iii) if $x \in X$ and $\varepsilon > 0$, then $\lim_{z \to 0} T^z x = x$ provided z remains within $S_{\alpha - \varepsilon}$.

We define the generator Z of T^z by $Zx = \lim_{t \uparrow 0} t^{-1}(T^t x - x)$ where $t > 0$ and $\mathrm{Dom}(Z)$ is the set of x for which the limit exists. If all T^z are compact operators, then we call it a *compact analytic semigroup* on X. In the following, $\mathbb{C}_+$ denotes the right-hand half plane of the complex plane, $\mathrm{KL}(X)$ the algebra of compact operators on X and "$-$" the closure of a set. As a first shot, a little but interesting lemma is set forth.

Lemma 2.4. *If U and V are arbitrary separable Banach algebras, then $U \otimes_p V$ is also separable.*

Proof. The proof is straightforward. □

The principal theorem in this section is the following.

Theorem 2.5. *If U and V are separable Banach algebras, then the following results are true:*

(i) *if U has the* HAP, *then $U \otimes_p V$ has the* HAP,

(ii) *if there is a compact analytic semigroup*

$$z \longrightarrow T_1^z : \mathbb{C}_+ \longrightarrow \mathrm{KL}(U) \tag{2.21}$$

such that

$$(T_1^z U)^- = U, \quad \|T_1^z\| \le 1 \ \forall z \in \mathbb{C}_+, \tag{2.22}$$

then there exists a compact analytic semigroup

$$z \longrightarrow T^z : \mathbb{C}_+ \longrightarrow \mathrm{KL}\left(U \otimes_p V\right) \tag{2.23}$$

having the same kind of properties, viz,

$$\left(T^z\left(U \otimes_p V\right)\right)^- = U \otimes_p V, \quad \|T^z\| \le 1 \ \forall z \in \mathbb{C}_+. \tag{2.24}$$

Proof. (i) Let U have the Hermitian approximation property. Then there exists a sequence $\{T_m^1\}$ with $\|T_m^1\| \le 1$ for all $m \in \mathbb{N}$ of a compact Hermitian operators on U such that $\|T_m^1 u - u\| \to 0$ for all $u \in U$. Then as usual for every $m \in \mathbb{N}$, define

$$T_m : U \otimes_p V \longrightarrow U \otimes_p V \quad \text{by } T_m\left(\sum_i u_i \otimes v_i\right) = \sum_i \left(T_m^1 u_i\right) \otimes v_i. \tag{2.25}$$

Then by Theorem 2.2, T_m is a compact Hermitian operator on $U \otimes_p V$ such that

$$\|T_m\| = \|T_m^1\| \le 1 \quad \forall m \in \mathbb{N}. \tag{2.26}$$

Further,

$$\begin{aligned}
\left\| T_m\left(\sum_i u_i \otimes v_i\right) - \sum_i \left(u_i \otimes v_i\right) \right\|_p &= \left\| \sum_i \left(T_m^1 u_i\right) \otimes v_i - \sum_i u_i \otimes v_i \right\|_p \\
&= \left\|\left(T_m^1 u_i - u_i\right) \otimes v_i\right\|_p \\
&\le \sum_i \left\|T_m^1 u_i - u_i\right\| \left\|v_i\right\| \\
&\longrightarrow 0 \quad \text{as } m \longrightarrow \infty,
\end{aligned} \tag{2.27}$$

(we can choose $\|v_i\| = 1$, for all i as in Lemma 2.1). Since by Lemma 2.4, $U \otimes_p V$ is separable, $U \otimes_p V$ has the HAP.

(ii) For each $z \in \mathbb{C}_+$, define

$$T^z\left(\sum_i u_i \otimes v_i\right) = \sum_i \left(T_1^z u_i\right) \otimes v_i. \tag{2.28}$$

We first show that T^z is an analytic semigroup.

(1) Let $z_1, z_2 \in \mathbb{C}_+$. Then

$$\begin{aligned}
\left(T^{z_1} T^{z_2}\right) a = T^{z_1}\left(\sum_i \left(T_1^{z_2} u_i\right) \otimes v_i\right) &= \sum_i T_1^{z_1}\left(T_1^{z_2} u_i\right) \otimes v_i \\
&= \sum_i T_1^{z_1+z_2} u_i \otimes v_i \quad \text{(because } T_1 \text{ is an analytic semigroup)} \\
&= T^{z_1+z_2}(a).
\end{aligned} \tag{2.29}$$

So, $T^{z_1} T^{z_2} = T^{z_1+z_2}$.

(2) Let $a = \sum_i u_i \otimes v_i \in U \otimes_p V$.

Since T_1^z is analytic,

$$\begin{aligned}
&\lim_{h\to 0} \frac{T_1^{z+h}u_i - T_1^z u_i}{h} \quad \text{exists for every } u_i \text{ (w.r.t. the norm on } U)\\
&\Longrightarrow \lim_{h\to 0}\left(\frac{T_1^{z+h}-T_1^z}{h}\right)u_i\otimes v_i \quad \text{exists for every pair } u_i\in U,\ v_i\in V\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(w.r.t. the norm on } U\otimes_p V)\\
&\Longrightarrow \sum_i\left[\lim_{h\to 0}\left(\frac{T_1^{z+h}-T_1^z}{h}\right)\right]u_i\otimes v_i \quad \text{exists}\\
&\Longrightarrow \lim_{h\to 0}\left(\frac{T^{z+h}-T^z}{h}\right)a \quad \text{exists.}
\end{aligned} \tag{2.30}$$

So, T^z is an analytic semigroup of $z \in \mathbb{C}_+$.

Also,

$$\begin{aligned}
\lim_{z\to 0} T^z a &= \lim_{z\to 0}\sum_i (T_1^z u_i)\otimes v_i = \sum_i\left(\lim_{z\to 0} T_1^z u_i\right)\otimes v_i\\
&= \sum_i (u_i\otimes v_i) \quad \text{because } T_1^z \text{ is analytic}\\
&= a.
\end{aligned} \tag{2.31}$$

Consequently, T^z is an analytic semigroup. By Theorem 2.2, the compactness of T_1^z implies that of T^z. To show that $(T^z(U\otimes_p V))^- = U\otimes_p V$, let $a = \sum_{j=1}^k u_j\otimes e_j$ by any element of $U\otimes_p V$ and $\varepsilon > 0$. $(T_1^z U)^- = U$ provides an element $u_j' \in U$ such that $\|T_1^z u_j' - u_j\| < \varepsilon/k$.

Let $a' = \sum_{j=1}^k u_j'\otimes e_j$. Then

$$\begin{aligned}
||T^z a' - a||_p &= \left\|\sum_{j=1}^k T_1 u_j'\otimes e_j - \sum_{j=1}^k T_1 u_j\otimes e_j\right\|_p\\
&\le \sum_{j=1}^k ||T_1 u_j' - u_j||\,||e_j||\\
&< \varepsilon.
\end{aligned} \tag{2.32}$$

This guarantees that $(T^z(U\otimes_p V))^- = U\otimes_p V$, for all $z\in\mathbb{C}_+$. Further, $\|T_1^z\| = \|T^z\|$ ensures that $\|T^z\| \le 1$, for all $z \in \mathbb{C}_+$. □

Our next goal is to study the orthogonality of the null space of T with its range.

We again recall some definitions. Let X be a normed linear space and $x, y \in X$. If $\|x - \lambda y\| \ge \|\lambda y\|$ for all $\lambda \in \mathbb{C}$, then x is said to be *orthogonal* to y. Let M and N be two subspaces of X. If $\|m+n\| \ge \|n\|$ for all $m \in M$ and for all $n \in N$, then M is said to be *orthogonal* to N, and then we write $M \perp N$ (for details, see [2]).

Theorem 2.6. (i) *If N_1 and R_1 are the null space and the range of T_1, then $N = N_1 \otimes V$ and $R = R_1 \otimes V$ are the respective null space N and the range R of T.*

(ii) *If $N_1 \perp R_1$, then $N \perp R$, provided the basis $\{e_j\}$ is chosen in such a way that*

$$\|a\| = \left\| \sum_{j=1}^{k} u_j \otimes e_j \right\| = \sum_{j=1}^{k} ||u_j|| \quad \forall a \in U \otimes_p V. \tag{2.33}$$

Proof. It is not so hard to establish the results. □

The converse problems. The main objective here is to investigate the possibility of studying the converse of the above results. To be precise, for a given compact Hermitian operator T on $U \otimes_p V$, can we obtain a compact Hermitian operator T_1 on U such that $\|T\| = \|T_1\|$ and $\operatorname{sp}(T) = \operatorname{sp}(T_1)$? Some possibilities are highlighted below. We first state a lemma whose proof is straightforward, and hence omitted.

Lemma 2.7. *For each j $(j = 1, 2, \ldots, k)$, let T_j be a map from $U \otimes_p V$ into U defined by $T_j(a) = u_j$ for every element $a = \sum_{j=1}^{k} u_j \otimes e_j$ in $U \otimes_p V$. Then T_j's are continuous linear transformation.*

Theorem 2.8. *Every compact Hermitian bounded operator T on $U \otimes_p V$ gives rise to k^2 number of linear operators $T^{i,j}$ $(i, j = 1, 2, \ldots, k)$ on U such that*

(i) *$T^{i,j}$ is compact for all i, j.*

(ii) *If $T^{i,j} = 0$ for $i \neq j$, then $T^{i,j}$ is Hermitian, $\|T^{i,j}\| \le \|T\|$ for all $j = 1, 2, \ldots, k$ and $\bigcup_{j=1}^{k} \operatorname{sp}(T^{j,j}) = \operatorname{sp}(T)$.*

Proof. Let T be a compact Hermitian operator on $U \otimes_p V$. For fixed i, j $(i, j = 1, 2, \ldots, k)$, we define an operator $T^{i,j}$ as follows.

Let u be an arbitrary element of U and let $T(u \otimes e_i) = \sum_{\ell=1}^{k} u_\ell \otimes e_\ell$. Then $T^{i,j} : U \to U$ is a map defined by $T^{i,j}(u) = u_j$. Since the expression $\sum_{\ell=1}^{k} u_\ell \otimes e_\ell$ for every element $a \in U \otimes_p V$ is unique, $T^{i,j}$ is well defined.

(1) The linearity of $T^{i,j}$ is obvious.

(2) We wish to show that $T^{i,j}$ is bounded.

Now, for each $u \in U$, $T(u \otimes e_i) = \sum_{\ell=1}^{k} T^{i,\ell}(u) \otimes e_\ell$. For a fixed j $(j = 1, 2, 3, \ldots, k)$, we define a map f_j by $f_j : U \otimes_p V \to U$ and $f_j(a) = u_j$, where $a = \sum_{\ell=1}^{k} u_\ell \otimes e_\ell$. Then f_j is a bounded linear operator by Lemma 2.7.

So,

$$\begin{aligned} ||T^{i,j} u|| &= ||f_j(T(u \otimes e_i))|| \\ &\le ||f_j|| \, ||T(u \otimes e_i)|| \\ &\le ||f_j|| \, \|T\| \, ||u \otimes e_i||_p \\ &= K\|u\|, \quad \text{where } K = ||f_j|| \, \|T\|. \end{aligned} \tag{2.34}$$

Hence, $T^{i,j}$ is bounded.

(i) Next, our purpose is to prove the compactness of $T^{i,j}$. Let $\{u_n\}$ be a sequence in U with $\|u_n\| \le 1$, for all n. Then for a fixed i $(i = 1, 2, \ldots, k)$,

$$||u_n \otimes e_i||_p = ||u_n|| \, ||e_i|| = ||u_n|| \le 1 \quad \forall n. \tag{2.35}$$

Hence for a fixed i, $\{u_n \otimes e_i\}_{n=1}^{\infty}$ is a bounded sequence in $U \otimes_p V$. The compactness of T yields that $\{T(u_n \otimes e_i)\}$ has a convergent subsequence, say $\{T(u_{n_m} \otimes e_i)\}$

converging to

$$x = \sum_{\ell=1}^{k} x_\ell \otimes e_\ell \quad \text{in } U \otimes_p V. \tag{2.36}$$

Now,

$$\begin{aligned}
&\lim_{m\to\infty} \left\| T(u_{n_m} \otimes e_i) - \sum_{\ell=1}^{k} x_\ell \otimes e_\ell \right\|_p = 0 \\
&\Longrightarrow \lim_{m\to\infty} \left\| \sum_{\ell=1}^{k} \left(T^{i,\ell} u_{n_m}\right) - \sum_{\ell=1}^{k} x_\ell \otimes e_\ell \right\|_p = 0 \\
&\Longrightarrow \left\| \sum_{\ell=1}^{k} \left[\left(\lim_{m\to\infty} T^{i,\ell} u_{n_m} - x_\ell \right) \otimes e_\ell \right] \right\|_p = 0 \quad \text{because } T^{i,\ell} \text{ are continuous} \\
&\Longrightarrow \sum_{\ell=1}^{k} \lim_{m\to\infty} \left(T^{i,\ell} u_{n_m} - x_\ell\right) \otimes e_\ell = 0.
\end{aligned} \tag{2.37}$$

Hence $\lim_{m\to\infty} T^{i,\ell} u_{n_m} = x_\ell$, for all $\ell = 1,2,\dots,k$. So, for $\ell = j$, we have that $T^{i,j}$ is compact.

(ii) Now suppose that $T^{i,j} = 0$ for $i \neq j$.

(I) We want to show that $T^{j,j}$ is Hermitian for each $j = 1,2,\dots,k$.

T is Hermitian and so $\|e^{i\alpha T}\| = 1$, for all $\alpha \in \mathbb{R}$. Let $u \in U$ with $\|u\| = 1$. Then for fixed j, $\|u \otimes e_j\|_p = 1$.

Now,

$$\begin{aligned}
1 = \|e^{i\alpha T}\| &= \sup\left\{\|e^{i\alpha T} a\|_p : \|a\|_p = 1\right\} \geq \|e^{i\alpha T}(u \otimes e_j)\|_p \\
&= \left\| \left[1 + i\alpha T + \frac{(i\alpha T)^2}{2!} + \frac{(i\alpha T)^3}{3!} + \cdots \right](u \otimes e_j) \right\|_p \\
&= \left\| \left[u + i\alpha T^{i,j} u + \frac{(i\alpha T^{i,j})^2}{2!} u + \frac{(i\alpha T^{j,j})^3}{3!} u + \cdots \right] \otimes e_j \right\|_p \\
&= \left\| e^{i\alpha T^{j,j}} u \right\| \quad \text{because } \|e_j\| = 1.
\end{aligned} \tag{2.38}$$

Thus, $\|e^{i\alpha T^{j,j}} u\| \leq 1$, for all $u \in U$ with $\|u\| = 1$. This gives $\|e^{i\alpha T^{j,j}}\| \leq 1$. Again, $e^{i\alpha T^{j,j}} e^{-i\alpha T^{j,j}} = I$, so, $1 = \|I\| = \|e^{i\alpha T^{j,j}} e^{-i\alpha T^{j,j}}\|_p \leq \|e^{i\alpha T^{j,j}}\|_p \|e^{-i\alpha T^{j,j}}\|_p$.

This yields $\|e^{-i\alpha T^{j,j}}\| \geq 1/\|e^{i\alpha T^{j,j}}\| \geq 1$. Since $\alpha \in \mathbb{R}$ then $-\alpha \in \mathbb{R}$, we can obtain $\|e^{i\alpha T^{j,j}}\| \geq 1$ and hence $\|e^{i\alpha T^{j,j}}\| = 1$. This implies that $T^{j,j}$ is Hermitian.

(II) Again for fixed j,

$$\|u \otimes e_j\|_p = 1, \quad \text{whenever } u \in U,\ \|u\| = 1. \tag{2.39}$$

Now, $T(u \otimes e_j) = (T^{j,j} u) \otimes e_j$. So, $\|T\| \geq \|T(u \otimes e_j)\|_p = \|T^{j,j}(u \otimes e_j)\|_p = \|T^{j,j} u\|$. This gives $\|T^{j,j}\| \leq \|T\|$, $j = 1,2,\dots,k$.

(III) We wish to show that $\bigcup_{j=1}^{k} \operatorname{sp}(T^{j,j}) = \operatorname{sp}(T)$.

Let

$$\begin{aligned}\lambda \in \bigcup_{j=1}^{k} \operatorname{sp}(T^{j,j}) &\Longrightarrow \lambda \in \operatorname{sp}(T^{j,j}) \quad \text{for some } j\\&\Longrightarrow (T^{j,j} - \lambda I) \quad \text{is singular}\\&\Longrightarrow \text{there exists a nonzero vector } u \in U \text{ such that } (T^{j,j} - \lambda I)u = 0\\&\Longrightarrow (T^{j,j} - \lambda I)u \otimes e_j = 0\\&\Longrightarrow (T - \lambda I)u \otimes e_j = 0 \quad \text{because } u \neq 0,\ e_j \neq 0 \Longrightarrow u \otimes e_j \neq 0\\&\Longrightarrow \lambda \in \operatorname{sp}(T).\end{aligned} \tag{2.40}$$

This gives $\bigcup_{j=1}^{k} \operatorname{sp}(T^{j,j}) \subseteq \operatorname{sp}(T)$.

On the other hand,

$$\begin{aligned}\lambda \in \operatorname{sp}(T) &\Longrightarrow T - \lambda I \text{ is singular}\\&\Longrightarrow \exists a \quad \text{nonzero vector } a = \sum_{\ell=1}^{k} u_\ell \otimes e_\ell \text{ such that } (T - \lambda I)a = 0\\&\Longrightarrow \sum_{\ell=1}^{k} T(u_\ell \otimes e_\ell) - \lambda \sum_{\ell=1}^{k} u_\ell \otimes e_\ell = 0\\&\Longrightarrow \sum_{\ell=1}^{k} T^{\ell,\ell} u_\ell \otimes e_\ell - \lambda \sum_{\ell=1}^{k} u_\ell \otimes e_\ell = 0\\&\Longrightarrow \sum_{\ell=1}^{k} (T^{\ell,\ell} - \lambda I) u_\ell \otimes e_\ell = 0\\&\Longrightarrow (T^{\ell,\ell} - \lambda I) u_\ell = 0; \quad \forall \ell = 1, 2, \ldots, k.\end{aligned} \tag{2.41}$$

Since $a \neq 0$, there exists at least one j such that $u_j \neq 0$. For this u_j, we have

$$\begin{aligned}(T^{j,j} - \lambda I)u_j = 0 &\Longrightarrow \lambda \in \operatorname{sp}(T^{j,j})\\&\Longrightarrow \lambda \in \bigcup_{j=1}^{k} \operatorname{sp}(T^{j,j}).\end{aligned} \tag{2.42}$$

Thus,

$$\bigcup_{j=1}^{k} \operatorname{sp}(T^{j,j}) = \operatorname{sp}(T). \tag{2.43}$$

This completes the proof of the theorem. □

We are now in a position to pose some open problems.

(i) Does the HAP in $U \otimes_p V$ give the existence of the HAP in U?

(ii) If the null space of T is orthogonal to its range, is the null space $T^{i,j}$ orthogonal to its range? What are the orthogonal complements of the null space and the range of T?

(iii) Suppose that the operator T is the derivation (for definition of a derivation, see [5]). Is $T^{i,j}$ a derivation?

(iv) Can we make the analogous study in case of other tensor products, viz, the injective tensor products, the Haagerup tensor product, and so forth, [1]?

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T. K. Dutta, H. K. Nath, and H. K. Sarmah: Department of Mathematics, Gauhati University, Guwahati-781014, Assam, India