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PURITY OF THE IDEAL OF CONTINUOUS FUNCTIONS WITH PSEUDOCOMPACT SUPPORT

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Let $C_\Psi(X)$ be the ideal of functions with pseudocompact support and let kX be the set of all points in υX having compact neighborhoods. We show that $C_\Psi(X)$ is pure if and only if $\beta X - kX$ is a round subset of βX, $C_\Psi(X)$ is a projective $C(X)$-module if and only if $C_\Psi(X)$ is pure and kX is paracompact. We also show that if $C_\Psi(X)$ is pure, then for each $f \in C_\Psi(X)$ the ideal (f) is a projective (flat) $C(X)$-module if and only if kX is basically disconnected (F'-space).

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1. Introduction. Let X be a completely regular T_1-space, βX the Stone-Čech compactification of X and υX the Hewitt realcompactification of X. Let $C(X)$ be the ring of all continuous real-valued functions defined on X. For each $f \in C(X)$, let $Z(f) = \{x \in X : f(x) = 0\}$, $\operatorname{coz} f = X - Z(f)$, the support of $f = S(f) = \operatorname{cl}_X \operatorname{coz}(f)$, and $S(f^\upsilon) = \operatorname{cl}_{\upsilon X} S(f)$, where f^υ is the extension of f to υX, $S(f^\beta) = \operatorname{cl}_{\beta X} S(f^*)$, where

$$f^*(x) = \begin{cases} 1, & f(x) \geq 1, \\ f(x), & -1 \leq f(x) \leq 1, \\ -1, & f(x) \leq -1, \end{cases} \tag{1.1}$$

and f^β is its extension to βX. If I is an ideal in $C(X)$, then $\operatorname{coz} I = \bigcup_{f \in I} \operatorname{coz} f$.

Let $C_K(X)$, $C_\Psi(X)$, and $I(X)$ be the ideal of functions with compact support, pseudocompact support, and the intersection of all free maximal ideals of $C(X)$, respectively.

The space X is called μ-compact if $C_K(X) = I(X)$, it is called Ψ-compact if $C_K(X) = C_\Psi(X)$, and it is called η-compact if $C_\Psi(X) = I(X)$.

Let μX be the smallest μ-compact subspace of βX containing X, ΨX the smallest Ψ-compact subspace of βX containing X, and ηX the smallest η-compact subspace of βX containing X.

The following diagram illustrates the relationships between these spaces:

$$\begin{tikzcd} & \beta X \arrow[d] & \\ & \upsilon X \arrow[d] & \\ & \Psi X \arrow[dl] \arrow[dr] & \\ \mu X \arrow[dr] & & \eta X \arrow[dl] \\ & X & \end{tikzcd} \tag{1.2}$$

For more information about these spaces the reader may consult [7].

For each subset $A \subseteq \beta X$, let $M^A = \{f \in C(X) : A \subseteq \mathrm{cl}_{\beta X} Z(f)\}$ and $O^A = \{f \in C(X) : A \subseteq \mathrm{Int}_{\beta X}\mathrm{cl}_{\beta X} Z(f)\} = \{f \in C(X) : A \subseteq \mathrm{Int}_{\beta X} Z(f^\beta)\}$. It is well known that $C_K(X) = O^{\beta X - X}$ and $C_\Psi(X) = O^{\beta X - \upsilon X} = M^{\beta X - \upsilon X}$. A subset A of βX is called a round subset of βX if $O^A = M^A$, see [10].

A space X is called locally pseudocompact if every point of X has a pseudocompact neighborhood (nbhd), it is called basically disconnected if for each $f \in C(X)$, $S(f)$ is clopen in X and it is called an F'-space if for each $f, g \in C(X)$ such that $fg = 0$, then $S(f) \cap S(g) = \varnothing$.

An ideal I of $C(X)$ is called pure if for each $f \in I$, there exists $g \in I$ such that $f = fg$. It is clear that in this case $g = 1$ on $S(f)$.

For any undefined terms here the reader may consult [5].

Purity attracted the attention of a lot of people working in ring and module theories. A large class of commutative rings can be classified through the pure ideals of the ring. Purity of some ideals in $C(X)$ was studied by many authors. Kohls [8, Theorem 4.6] called it an ideal with every element having a relative identity. Brookshear [3, page 325] proved that if X is locally compact, then $C_K(X)$ is pure, Brookshear [3] and De Marco [4] studied purity and projectivity, Natsheh and Al-Ezeh [11, Theorem 2.4] characterized pure ideals in $C(X)$ to be the ideals of the form O^A, where A is a unique closed subset of βX, and Abu Osba and Al-Ezeh [1, Theorem 3.2] proved that $C_K(X)$ is pure if and only if $\mathrm{coz}\, C_K(X) = \bigcup_{f \in C_K} S(f)$.

In this paper, we characterize purity of $C_\Psi(X)$ using the subspace kX, the set of all points in υX having compact nbhds, then we use this characterization to study some algebraic properties of this ideal, such as projectivity, when the principal ideal (f) is projective or flat for each $f \in C_\Psi(X)$. We found that if $C_\Psi(X)$ is pure, then it is projective if and only if kX is paracompact, the principal ideal (f) is projective (flat) if and only if kX is basically disconnected (F'-space). An example is given to show that these results are false if $C_\Psi(X)$ is not pure.

The following result is well known and is used very often in this article.

Proposition 1.1. *For each space X, $C(X)$ is isomorphic to $C(\upsilon X)$, and $C_\Psi(X)$ is isomorphic to $C_K(\upsilon X)$.*

Proof. Let $\varphi : C(X) \to C(\upsilon X)$ be defined such that $\varphi(f) = f^\upsilon$. Then φ is the required isomorphism, see [5, Section 8.1] and [6, Theorem 2.1]. □

In this paper, we use the above proposition together with the results we obtained in [1] to characterize purity of the ideal $C_\Psi(X)$ using the subspace kX.

2. The subspace kX. For each ideal I in $C(X)$, define $\theta(I) = \{x \in \beta X : I \subseteq M^x\}$. Then $\theta(I) = \bigcap_{f \in I} \mathrm{cl}_{\beta X} Z(f)$, see [5, Exercise 70.1].

Let $kX = \beta X - \theta(C_\Psi(X)) = \{x \in \beta X : C_\Psi(X)$ is not contained in $M^x\}$. The space kX is important in classifying some properties of X and some of its extensions and it is related to the ideal $C_K(\upsilon X)$. The following propositions and corollaries illustrate this fact.

Proposition 2.1 (see [6, Corollary 3.3 and Theorems 3.1 and 5.3]). *The following statements are equivalent for any space X:*

(i) *X is locally pseudocompact;*
(ii) *$X \subseteq kX$;*
(iii) *ηX is locally compact;*
(iv) *$C_\Psi(X)$ is not contained in any fixed maximal ideal.*

Proposition 2.2 (see [6, Theorems 3.2, 5.1, and 5.2]). *For each space X,*

(i) *$kX = \mathrm{Int}_{\beta X}\upsilon X = \mathrm{Int}_{\beta X}\eta X = \mathrm{Int}_{\beta X}\Psi X$;*
(ii) *$\eta X = X \cup kX$;*
(iii) *$\Psi X - X = \bigcup_{f\in C_\Psi(X)}(S(f^\upsilon) - S(f))$.*

Proposition 2.3 (see [1, Theorem 2.2]). *For each space X, $\operatorname{coz} C_K(X) = \mathrm{Int}_{\beta X} X$.*

The following result is an easy consequence of Propositions 2.2 and 2.3.

Corollary 2.4. *For each space X, $kX = \operatorname{coz} C_K(\upsilon X) = \bigcup_{f\in C_\Psi(X)}\upsilon X - Z(f^\upsilon)$.*

Corollary 2.5. *For each space X, $kX = \bigcup_{f\in C_\Psi(X)}\beta X - Z(f^\beta)$.*

Proof. Let $f \in C_\Psi(X) \subseteq C^*(X)$. For each $p \in \beta X - \upsilon X$, $f \in M^p \cap C^*$.

So $f^\beta(p) = 0$ for each $p \in \beta X - \upsilon X$. Thus $Z(f^\beta) = (\beta X - \upsilon X) \cup Z(f^\upsilon)$, and $\beta X - Z(f^\beta) = \upsilon X \cap (\beta X - Z(f^\upsilon)) = \upsilon X - Z(f^\upsilon)$.

Now, $\bigcup_{f\in C_\Psi(X)}\beta X - Z(f^\beta) = \bigcup_{f\in C_\Psi(X)}\upsilon X - Z(f^\upsilon) = \bigcup_{f^\upsilon\in C_K(\upsilon X)}\upsilon X - Z(f^\upsilon) = kX$, by Corollary 2.4. □

Theorem 2.6. *The space ΨX is locally compact if and only if X is locally pseudocompact and $\theta(C_\Psi(X))$ is a round subset of βX.*

Proof. See [6, Theorem 5.4] and [10, Theorem 3.3]. □

3. Purity of $C_\Psi(X)$.

Here we characterize purity of $C_\Psi(X)$ using the subspace kX. But first we need some preliminaries.

Proposition 3.1 (see [1, Theorem 3.2]). *For each space X, the ideal $C_K(X)$ is pure if and only if $\operatorname{coz} C_K(X) = \bigcup_{f\in C_K(X)} S(f)$.*

It was proved in [11, Theorem 2.4] that an ideal I in $C(X)$ is pure if and only if $I = O^A$ where A is a unique closed subset of βX. In fact, it was proved that A must be the set $\bigcap_{f\in I}\mathrm{cl}_{\beta X} Z(f) = \theta(I)$. Here we show that if the ideal O^A is pure, then A need not be closed, but $O^A = O^{\mathrm{cl}_{\beta X} A}$.

Theorem 3.2. *The ideal O^A is pure if and only if $O^A = O^{\mathrm{cl}_{\beta X} A}$.*

Proof. Suppose that O^A is pure and $f \in O^A$. Then there exists $g \in O^A$ such that $f = fg$. So $f^\beta = f^\beta g^\beta$ which implies that $S(f^\beta) \subseteq \operatorname{coz} g^\beta$. Hence $A \subseteq \mathrm{Int}_{\beta X} Z(g^\beta) \subseteq Z(g^\beta) \subseteq \beta X - S(f^\beta) \subseteq Z(f^\beta)$. Thus $\mathrm{cl}_{\beta X} A \subseteq Z(g^\beta) \subseteq \beta X - S(f^\beta) \subseteq \mathrm{Int}_{\beta X} Z(f^\beta)$ which implies that $f \in O^{\mathrm{cl}_{\beta X} A}$. □

In the following theorem we characterize purity of the ideal $C_\Psi(X)$ using properties of the subspace kX.

Theorem 3.3. *The following statements are equivalent:*
(1) $C_\Psi(X)$ *is pure;*
(2) $C_\Psi(X) = O^{\beta X - kX}$*;*
(3) $\beta X - kX$ *is a round subset of* βX.

Proof. (1)⇔(2). $C_\Psi(X)$ is pure if and only if $C_\Psi(X) = O^{\beta X - \upsilon X} = O^{\mathrm{cl}_{\beta X}(\beta X - \upsilon X)} = O^{\beta X - \mathrm{Int}_{\beta X}\upsilon X} = O^{\beta X - kX}$, see Proposition 2.2 and Theorem 3.2 above.

(2)⇒(3). $M^{\beta X - kX} \supseteq O^{\beta X - kX} = C_\Psi(X) = M^{\beta X - \upsilon X} \supseteq M^{\beta X - kX}$. So $\beta X - kX$ is a round subset of βX.

(3)⇒(1). $O^{\mathrm{cl}_{\beta X}(\beta X - \upsilon X)} = O^{\beta X - \mathrm{Int}_{\beta X}\upsilon X} = O^{\beta X - kX} = M^{\beta X - kX} = M^{\mathrm{cl}_{\beta X}(\beta X - \upsilon X)} = M^{\beta X - \upsilon X} = O^{\beta X - \upsilon X} = C_\Psi(X)$. So it follows by Theorem 3.2 that $C_\Psi(X)$ is pure. □

The following result will be extremely useful throughout the rest of the paper.

Corollary 3.4. *The ideal* $C_\Psi(X)$ *is pure if and only if for each* $f \in C_\Psi(X)$, $S(f^\upsilon) \subseteq kX$.

Proof. The ideal $C_\Psi(X)$ is pure if and only if $C_K(\upsilon X)$ is pure if and only if for each $f \in C_\Psi(X)$, $S(f^\upsilon) \subseteq kX$, see Propositions 1.1 and 3.1. □

Corollary 3.5. *The space* ΨX *is locally compact if and only if* $X \subseteq kX$ *and* $C_\Psi(X)$ *is pure.*

Proof. The result follows from Theorems 2.6 and 3.3. □

It was shown in [1, Theorem 3.2] that $C_K(X)$ is pure if and only if $\mathrm{coz}\, C_K(X) = \bigcup_{f \in C_K(X)} S(f)$. Now, if $C_\Psi(X)$ is pure, then it is easy to see that

$$\mathrm{coz}\, C_\Psi(X) = \bigcup_{f \in C_\Psi(X)} S(f). \tag{3.1}$$

This raises the following question: suppose that $\mathrm{coz}\, C_\Psi(X) = \bigcup_{f \in C_\Psi(X)} S(f)$, does this imply that $C_\Psi(X)$ is pure? The following example shows that this need not be true.

Example 3.6. Let $\mathbf{W}^* = [0, \omega_1]$ be the set of all ordinals less than or equal to the first uncountable ordinal number ω_1. Let $\mathbf{W} = [0, \omega_1)$ and $\mathbf{T}^* = \mathbf{W}^* \times \mathbb{N}^*$, where $\mathbb{N}^*$ denote the one point compactification $\mathbb{N} \cup \{\omega_0\}$ of the natural numbers. Let $t = (\omega_1, \omega_0)$, $\mathbf{T} = \mathbf{T}^* - \{t\}$, let $A = \mathbf{W} \times \{\omega_0\}$ and let $B = \{\omega_1\} \times \mathbb{N}$. Let $\mathbf{S}$ be obtained from $\mathbf{T} \times \mathbb{N}$ by identifying $A \times \{2n-1\}$ with $A \times \{2n\}$ and identifying $B \times \{2n\}$ with $B \times \{2n+1\}$. Then $\mathbf{S}$ is locally compact, since $\mathbf{T}$ is, and $A \cap B = \emptyset$, see [7, Example 7.3] and [9, page 240]. Let $\mathbf{H}$ be obtained from $\mathbf{T}^* \times \mathbb{N}$ by identifying $(A \cup \{t\}) \times \{2n-1\}$ with $(A \cup \{t\}) \times \{2n\}$ and identifying $(B \cup \{t\}) \times \{2n\}$ with $(B \cup \{t\}) \times \{2n+1\}$. Now, $\mathbf{H}$ is σ-compact and so it is realcompact. $\mathbf{H}$ is not locally compact since (ω_1, ω_0, n) has no compact neighborhood for each $n \in \mathbb{N}$. So $\mathbf{S} \subseteq k\mathbf{S} \subseteq \upsilon\mathbf{S} \subseteq \mathbf{H}$. Define $f : \mathbf{S} \to \mathbb{R}$ by $f(\alpha, n, 1) = 1/n$ for all $\alpha \in \mathbf{W}^*$, $n \in \mathbb{N}$ and by zero otherwise. Then $\mathrm{coz}\, f = \mathbf{W}^* \times \mathbb{N} \times \{1\}$ and $S(f) = \mathbf{T} \times \{1\}$ is pseudocompact, noncompact. So $f \in C_\Psi(\mathbf{S}) - C_K(\mathbf{S})$, which implies that $\mathbf{S}$ is not Ψ-compact. Hence $k\mathbf{S} = \mathbf{S} \neq \Psi\mathbf{S}$. Therefore, $\Psi\mathbf{S}$ is not locally compact. So it follows by Corollary 3.5 that $C_\Psi(\mathbf{S})$ is not pure although $\mathbf{S}$ is locally pseudocompact and $\mathrm{coz}\, C_\Psi(\mathbf{S}) = \mathbf{S} = \bigcup_{f \in C_\Psi} S(f)$.

4. Some applications. In this section, we use the characterization obtained in Theorem 3.3 and Corollary 3.4 above for purity of the ideal $C_\Psi(X)$ to characterize when $C_\Psi(X)$ is a projective $C(X)$-module, when every principal ideal of $C_\Psi(X)$ is projective or flat $C(X)$-module, and for which spaces X and Y, the two ideals $C_\Psi(X)$ and $C_\Psi(Y)$ are isomorphic.

Theorem 4.1. *Let $C_\Psi(X)$ and $C_\Psi(Y)$ be pure ideals. Then $C_\Psi(X)$ is isomorphic to $C_\Psi(Y)$ if and only if kX is homeomorphic to kY.*

Proof. If $C_\Psi(X)$ is isomorphic to $C_\Psi(Y)$, then kX is homeomorphic to kY, see [12, Corollary 4.11].

For the converse, we prove that $C_K(\upsilon X)$ is isomorphic to $C_K(\upsilon Y)$, then the result follows from Proposition 1.1.

Suppose $\varphi : kX \to kY$ is a homeomorphism. Let $f \in C_K(\upsilon Y)$, then $f_1 \circ \varphi \in C(kX)$, where $f_1 = f_{|kY}$. But $\operatorname{coz} f = \varphi(\operatorname{coz}(f_1 \circ \varphi))$, which implies that $\varphi^{-1}(\operatorname{coz} f) = \operatorname{coz}(f_1 \circ \varphi)$.

Therefore $\operatorname{cl}_{kX} \operatorname{coz}(f_1 \circ \varphi) = \operatorname{cl}_{kX} \varphi^{-1}(\operatorname{coz} f) = \varphi^{-1}(S(f))$, since $S(f)$ is contained in kY by the purity of $C_\Psi(Y)$. Now, for each $f \in C_K(\upsilon X)$ define

$$g_f : \upsilon X \longrightarrow \mathbb{R} \quad \text{by } g_f(x) = \begin{cases} f_1 \circ \varphi(x), & x \in kX, \\ 0, & x \in \upsilon X - \varphi^{-1}(S(f)). \end{cases} \tag{4.1}$$

Then, $g_f \in C_K(\upsilon X)$, since $S(g_f) = \operatorname{cl}_{kX} \operatorname{coz}(f_1 \circ \varphi)$ is compact.

Define $\bar{\varphi} : C_K(\upsilon Y) \to C_K(\upsilon X)$ by $\bar{\varphi}(f) = g_f$. Then $\bar{\varphi}$ is a ring homomorphism. It remains to show that $\bar{\varphi}$ is bijective.

To see that $\bar{\varphi}$ is one-to-one, suppose $\bar{\varphi}(f) = 0$. Then $f_1 \circ \varphi(x) = 0$ for every $x \in kX$. But $\operatorname{coz}(f_1 \circ \varphi) = \varphi^{-1}(\operatorname{coz} f)$, and so $\varphi^{-1}(\operatorname{coz} f) = \emptyset$. Therefore $f = 0$.

To see that $\bar{\varphi}$ is onto, let $f \in C_K(\upsilon X)$. Define

$$g : \upsilon Y \longrightarrow \mathbb{R} \quad \text{by } g(y) = \begin{cases} f \circ \varphi^{-1}(y), & y \in kY, \\ 0, & y \in \upsilon Y - \varphi(S(f)). \end{cases} \tag{4.2}$$

Then $g \in C(\upsilon Y)$, since $\varphi(S(f))$ is compact. Here again we use the purity of $C_\Psi(X)$, since we assumed that $S(f) \subseteq kX$. Moreover, if $g(y) \neq 0$, then $\varphi^{-1}(y) \in \operatorname{coz} f$. So $\operatorname{coz} g \subseteq \varphi(\operatorname{coz} f)$.

Thus, $\operatorname{cl}_{kY} \operatorname{coz} g \subseteq \operatorname{cl}_{kY} \varphi(\operatorname{coz} f) = \varphi(\operatorname{cl}_{\upsilon X} \operatorname{coz} f) = \varphi(S(f))$. Hence $S(g) = \operatorname{cl}_{kY} \operatorname{coz} g$ is compact. It follows that $g \in C_K(\upsilon Y)$.

Finally, note that

$$\begin{aligned} \bar{\varphi}(g)(x) &= \begin{cases} g_1 \circ \varphi(x), & x \in kX, \\ 0, & x \in \upsilon X - \varphi^{-1}(S(g)); \end{cases} \\ &= \begin{cases} f \circ \varphi^{-1} \circ \varphi(x), & x \in kX, \\ 0, & x \in \upsilon X - \varphi^{-1}(S(g)); \end{cases} \\ &= \begin{cases} f(x), & x \in kX, \\ 0, & \text{otherwise}; \end{cases} \\ &= f(x). \end{aligned} \tag{4.3}$$

Thus $\bar{\varphi}(g) = f$ and so $\bar{\varphi}$ is onto. Hence $C_K(\upsilon X)$ is isomorphic to $C_K(\upsilon Y)$. □

Here we characterize when $C_\Psi(X)$ is a projective $C(X)$-module.

Theorem 4.2. *The ideal $C_\Psi(X)$ is a projective $C(X)$-module if and only if kX is paracompact and $C_\Psi(X)$ is pure.*

Proof. It was proved by Brookshear [3, Theorem 3.10] that $C_K(X)$ is a projective $C(X)$-module if and only if $\operatorname{coz} C_K(X)$ is paracompact and $S(f) \subseteq \operatorname{coz} C_K(X)$ for each $f \in C_K(X)$. Our result now follows from Proposition 3.1 and Corollaries 2.4 and 3.4. $\square$

It was proved in [2, Lemma 2] and [3, Corollary 2.5] that the principal ideal (f) is a projective (flat) $C(X)$-module if and only if $S(f)$ is clopen in X ($\operatorname{Ann}(f)$ is pure). We can use this result to determine when the principal ideal (f) is a projective or a flat $C(X)$-module for each $f \in C_\Psi(X)$.

Theorem 4.3. *For each $f \in C_\Psi(X)$, the ideal (f) is a projective $C(X)$-module if and only if $C_\Psi(X)$ is pure and kX is basically disconnected.*

Proof. Suppose that kX is basically disconnected and $C_\Psi(X)$ is pure. Let $f \in C_\Psi(X)$. Then $S(f^\upsilon) \subseteq kX$ since $C_\Psi(X)$ is pure. Now let $f_1 = f^\upsilon{}_{|kX}$ and note that $\mathrm{cl}_{kX}(kX - Z(f_1)) = S(f^\upsilon)$. Since kX is basically disconnected, $S(f^\upsilon)$ is open in kX and therefore it is open in υX (cf. Proposition 2.2). Thus $S(f) = S(f^\upsilon) \cap X$ is open in X. Hence the ideal (f) is a projective $C(X)$-module.

Conversely, suppose that every principal ideal of $C_\Psi(X)$ is a projective $C(X)$-module. For each $f \in C_\Psi(X)$, $S(f)$ is open in X, so define

$$g(x) = \begin{cases} 1, & x \in S(f), \\ 0, & \text{otherwise}. \end{cases} \tag{4.4}$$

Then $g \in C_\Psi(X)$ and $f = fg$. Thus $C_\Psi(X)$ is a pure ideal.

To demonstrate basic disconnectedness, we first show that for each $f \in C_K(kX)$, $S(f)$ is clopen. Then we will use this result to show that for each $k \in C(kX)$, $S(k)$ is clopen.

Let $f \in C_K(kX)$. Then f can be extended to a function $F \in C_K(\upsilon X)$ with $\mathrm{cl}_{kX}(kX - Z(f)) = S(F)$ which is open, since $C_\Psi(X)$ is isomorphic to $C_K(\upsilon X)$.

Now, let $k \in C(kX)$, and $a \in \mathrm{cl}_{kX}(kX - Z(k)) \subseteq kX$. So there exists an open set U such that $\bar{U}$ is compact, and $a \in U \subseteq \bar{U} \subseteq kX$. There exists $f \in C(kX)$ such that $f(a) = 1$ and $f(kX - U) = 0$. Then $f \in C_K(kX)$.

Thus $a \in (kX - Z(f)) \cap \mathrm{cl}_{kX}(kX - Z(k)) \subseteq \mathrm{cl}_{kX}((kX - Z(f)) \cap \mathrm{cl}_{kX}(kX - Z(k))) = \mathrm{cl}_{kX}((kX - Z(f)) \cap (kX - Z(k))) = \mathrm{cl}_{kX}(kX - Z(fk)) \subseteq \mathrm{cl}_{kX}(kX - Z(k))$. But $\mathrm{cl}_{kX}(kX - Z(fk))$ is compact, and so is clopen since $fk \in C_K(kX)$. So, $\mathrm{cl}_{kX}(kX - Z(k))$ is clopen in kX. Thus kX is basically disconnected. $\square$

Theorem 4.4. *Let X be a space such that $C_\Psi(X)$ is pure. Then for each $f \in C_\Psi(X)$ the principal ideal (f) is a flat $C(X)$-module if and only if kX is an F'-space.*

Proof. Suppose that kX is an F'-space, $f \in C_\Psi(X)$ and $g \in \operatorname{Ann}(f)$. Let $f_1 = f^\upsilon{}_{|kX}$ and $g_1 = g^\upsilon{}_{|kX}$. Then $(kX - Z(f_1)) \cap (kX - Z(g_1)) = \varnothing$. So, $\mathrm{cl}_{kX}(kX - Z(f_1)) \cap \mathrm{cl}_{kX}(kX - Z(g_1)) = \varnothing$, since kX is an F'-space. But $S(f^\upsilon) = \mathrm{cl}_{kX}(kX - Z(f_1))$, since $C_\Psi(X)$ is pure and $S(f^\upsilon) \subseteq kX$. Now if $x \in S(f^\upsilon)$, then $x \in \mathrm{cl}_{kX}(kX - Z(f_1))$, which implies that $x \notin \mathrm{cl}_{kX}(kX - Z(g_1))$. So there exists an open set $U \subseteq kX$ such that $x \in U$ and $U \cap (kX - Z(g_1)) = \emptyset$. Therefore $x \in U \subseteq Z(g_1) \subseteq Z(g^\upsilon)$. But U is open in υX, since kX is. So $U \cap (\upsilon X - Z(g^\upsilon)) = \emptyset$, which implies that $x \notin S(g^\upsilon)$. Thus $S(f^\upsilon) \cap S(g^\upsilon) = \emptyset$.

The compactness of $S(f^\upsilon)$ implies that there exists $k^\upsilon \in C(\upsilon X)$ such that $k^\upsilon(S(f^\upsilon)) = 0$ and $k^\upsilon(S(g^\upsilon)) = 1$. So, $k \in \mathrm{Ann}(f)$, and $g = gk$. Thus the ideal (f) is a flat $C(X)$-module since $\mathrm{Ann}(f)$ is pure.

Conversely, suppose that the principal ideal (f) is a flat $C(X)$-module for each $f \in C_\Psi(X)$. Let $g, k \in C(kX)$ such that $gk = 0$. Suppose $y \in \mathrm{cl}_{kX}(kX - Z(g)) \cap \mathrm{cl}_{kX}(kX - Z(k))$. There exists $f_1 \in C_K(\upsilon X)$ such that $f_1(y) \neq 0$. Let $f = f_1|_{kX}$, then $y \in \mathrm{cl}_{kX}(kX - Z(fg)) \cap \mathrm{cl}_{kX}(kX - Z(fk))$. Define

$$
\begin{aligned}
h_1(x) &= \begin{cases} fg(x), & x \in \mathrm{cl}_{kX}(kX - Z(fg)), \\ 0, & x \in \upsilon X - (kX - Z(fg)); \end{cases} \\
h_2(x) &= \begin{cases} fk(x), & x \in \mathrm{cl}_{kX}(kX - Z(fk)), \\ 0, & x \in \upsilon X - (kX - Z(fk)). \end{cases}
\end{aligned}
\tag{4.5}
$$

Then $h_1, h_2 \in C_K(\upsilon X)$, since $S(h_1)$ and $S(h_2)$ are compact sets. Moreover, $h_1 h_2 = 0$. So, there exists $h_1' \in \mathrm{Ann}(h_2)$ such that $h_1 = h_1 h_1'$. Hence $y \in \mathrm{cl}_{kX}(kX - Z(fg)) = S(h_1) \subseteq \mathrm{coz}\, h_1'$. But $h_1'(S(h_2)) = 0$, so $y \notin S(h_2) = \mathrm{cl}_{kX}(kX - Z(fk))$, a contradiction. Hence $\mathrm{cl}_{kX}(kX - Z(g)) \cap \mathrm{cl}_{kX}(kX - Z(k)) = \emptyset$ and kX is an F'-space. □

Example 4.5. Let $X = [-1, 1]$ with all its points isolated, except for $x = 0$ it has its usual nbhds. Then X is regular, paracompact, and consequently realcompact. So $X = \upsilon X$ and $kX = X - \{0\} \subseteq X$.

Let

$$
f(x) = \begin{cases} x, & x = \dfrac{1}{n},\ n \in \mathbb{Z}^*, \\ 0, & \text{otherwise}. \end{cases}
\tag{4.6}
$$

Then $S(f) = \{1/n : n \in \mathbb{Z}^*\} \cup \{0\}$. So $f \in C_\Psi(X)$ and $S(f)$ is not contained in kX. So $C_\Psi(X)$ is not a pure ideal.

The set $S(f)$ is not open, so the ideal (f) is not projective. $\mathrm{Ann}(f)$ is not pure, since the function

$$
g(x) = \begin{cases} 0, & x = \dfrac{1}{n},\ n \in \mathbb{Z}^*, \\ x, & \text{otherwise}, \end{cases}
\tag{4.7}
$$

belongs to $\mathrm{Ann}(f)$, but $S(g) = X - \{1/n : n \in \mathbb{Z}^*\}$ is not a subset of $\mathrm{coz}\,\mathrm{Ann}(f)$, since for each $h \in \mathrm{Ann}(f)$, $h(0) = 0$. So the ideal (f) is not a flat $C(X)$-module. Let $Y = X - \{0\}$, then $kX = Y = kY$, but $C_\Psi(X)$ is not isomorphic to $C_\Psi(Y)$, since the latter is pure.

This example shows that Theorems 4.1, 4.2, 4.3, and 4.4 need not be true if $C_\Psi(X)$ is not pure.

References

[1] E. A. Abu Osba and H. Al-Ezeh, *Purity of the ideal of continuous functions with compact support*, Math. J. Okayama Univ. **41** (1999), 111–120.

[2] H. Al-Ezeh, M. A. Natsheh, and D. Hussein, *Some properties of the ring of continuous functions*, Arch. Math. (Basel) **51** (1988), no. 1, 60–64.

[3] J. G. Brookshear, *Projective ideals in rings of continuous functions*, Pacific J. Math. **71** (1977), no. 2, 313–333.

[4] G. De Marco, *Projectivity of pure ideals*, Rend. Sem. Mat. Univ. Padova **69** (1983), 289–304.

[5] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Graduate Texts in Mathematics, no. 43, Springer-Verlag, New York, 1976.

[6] D. Harris, *The local compactness of υX*, Pacific J. Math. **50** (1974), 469–476.

[7] D. G. Johnson and M. Mandelker, *Functions with pseudocompact support*, General Topology and Appl. **3** (1973), 331–338.

[8] C. W. Kohls, *Ideals in rings of continuous functions*, Fund. Math. **45** (1957), 28–50.

[9] J. E. Mack and D. G. Johnson, *The Dedekind completion of $C(\mathscr{X})$*, Pacific J. Math. **20** (1967), 231–243.

[10] M. Mandelker, *Round z-filters and round subsets of βX*, Israel J. Math. **7** (1969), 1–8.

[11] M. A. Natsheh and H. Al-Ezeh, *Characterization of pure ideals in $C(X)$*, J. Fac. Sci. UAE Univ. **3** (1991), no. 1, 17–23.

[12] D. Rudd, *On isomorphisms between ideals in rings of continuous functions*, Trans. Amer. Math. Soc. **159** (1971), 335–353.

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