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A COMBINATORIAL COMMUTATIVITY PROPERTY FOR RINGS

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We study commutativity in rings R with the property that for a fixed positive integer n, $xS = Sx$ for all $x \in R$ and all n-subsets S of R.

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1. Introduction. In [2], we discussed P_∞-rings R, which were defined by the property that

$$XY = YX \tag{1.1}$$

for all infinite subsets X, Y of R; and in an earlier paper [1], the first author discussed P_n-rings, defined by the property that (1.1) holds for all n-subsets X, Y of R. For a fixed positive integer n, we now define a Q_n-ring to be a ring R with the property that

$$xS = Sx \quad \forall x \in R, \ \forall n\text{-subsets } S \text{ of } R. \tag{1.2}$$

Clearly, every commutative ring is a Q_n-ring for arbitrary n; moreover, there exist badly noncommutative Q_n-rings, since every ring with fewer than n elements is a Q_n-ring. Our purpose is to identify conditions which force Q_n-rings to be commutative or nearly commutative.

It is obvious that every Q_n-ring is a P_n-ring and every P_n-ring is a P_∞-ring. We make no use of the results on P_n-rings in [1], and most of our results are of a different sort than those in [1]. However, a special case of the theorem on P_∞-rings in [2] plays a crucial role in our study.

2. Preliminaries. We begin with some notation. Let R be an arbitrary ring, not necessarily with 1. The symbols D, N, Z, and $C(R)$ denote the set of zero divisors, the set of nilpotent elements, the center, and the commutator ideal, respectively; and $|R|$ denotes the cardinal number of R. For Y being an element or subset of R, the symbols $C_R(Y)$, $A_\ell(Y)$, $A_r(Y)$, and $A(Y)$ denote the centralizer of Y and the left, right, and two-sided annihilators of Y. For $x, y \in R$, the set $L_{x,y}$ is defined to be $\{w \in R \mid xy = wx\}$.

We give three lemmas, the first of which is rather trivial and the other two of which are not.

Lemma 2.1. *Let R be a Q_n-ring with $|R| \geq n$. Then*

(i) *for all $x \in R$, $xR = Rx$ and $|A_\ell(x)| = |A_r(x)|$;*

(ii) *all idempotents of R are central;*

(iii) *N is an ideal;*

(iv) *if R is not commutative and $x \notin Z$, then $R\backslash(A_\ell(x) \cup C_R(x))$ and $R\backslash(A_r(x) \cup C_R(x))$ are nonempty.*

Proof. (i) is obvious; and if e is idempotent, the fact that $eR = Re$ yields $ex = exe = xe$ for all $x \in R$, so $e \in Z$. Moreover, (i) enables us to prove (iii) by adapting the standard proof that N is an ideal in commutative rings. Finally, if $x \notin Z$ then $C_R(x)$ is a proper subgroup of $(R,+)$; and (i) implies that $A_\ell(x)$ and $A_r(x)$ are also proper subgroups of $(R,+)$. Since a group cannot be the union of two proper subgroups, (iv) is immediate. □

Lemma 2.2. *If R is an infinite Q_n-ring, then R is commutative.*

Proof. Since every Q_n-ring is a P_∞-ring, we could simply invoke the theorem of [2], which states that every P_∞-ring is either finite or commutative. However, the proof in [2] is long and involved, so we prefer to give a more elementary proof.

Let R be a noncommutative Q_n-ring. We may assume that R is not a Q_m-ring for any $m < n$. Since all Q_1-rings are commutative, $n > 1$, and there exist $x \in R$ and an $(n-1)$-subset H of R such that $xH \neq Hx$; and we may assume that xH is not a subset of Hx. We may also assume that $R\backslash H \neq \varnothing$, since otherwise R is finite.

For any $a \in R\backslash H$, $x(H \cup \{a\}) = (H \cup \{a\})x$, so if we take $h \in H$ for which $xh \notin Hx$, we have

$$xh = ax. \tag{2.1}$$

Since (2.1) holds for all $a \in R\backslash H$, it follows that for fixed $b \in R\backslash H$, $R\backslash H \subseteq b + A_\ell(x)$. Moreover, if $c \in A_\ell(x)$, then $xh = bx = (b+c)x$, so $b + c \notin H$. Therefore $R\backslash H = b + A_\ell(x)$, hence $|R\backslash H| = |A_\ell(x)|$ and $|R\backslash A_\ell(x)| = |H|$. But since $A_\ell(x)$ is a proper subgroup of R, $|R\backslash A_\ell(x)| \geq |A_\ell(x)|$, that is, $|H| \geq |R\backslash H|$; and the finiteness of H yields the finiteness of R. □

Lemma 2.3 (see [4]). *If R is a finite ring with $N \subseteq Z$, then R is commutative.*

In view of Lemma 2.2, we assume henceforth that R is finite.

3. Commutativity of Q_n-rings with 1

Theorem 3.1. *If R is any Q_n-ring with 1 such that $|R| > n$, then R is commutative.*

Proof. By Lemma 2.3, we need only to show that $N \subseteq Z$; and since $u \in N$ implies $1+u$ is invertible, it suffices to prove that invertible elements are central.

Suppose, then, that x is a noncentral invertible element and $y \notin C_R(x)$. If H is any $(n-1)$-subset of R which excludes y, the condition $x(\{y\} \cup H) = (\{y\} \cup H)x$ yields $z \in H$ such that

$$xy = zx. \tag{3.1}$$

Since x is invertible, there is a unique $z \in R$ satisfying (3.1); and we have shown that every $(n-1)$-subset contains either y or z. But $|R\backslash\{y,z\}| \geq n-1$; therefore noncentral invertible elements cannot exist. □

The bound on $|R|$ in Theorem 3.1 is best possible, as the following example shows. The rings of this example were introduced by Corbas in [3].

Example 3.2. Let $n = p^{2k}$, where p is prime and $k > 1$. Let ϕ be a nonidentity automorphism of $GF(p^k)$. Let $R = GF(p^k)\times GF(p^k)$, with addition being componentwise and multiplication given by $(a,b)(c,d) = (ac, ad + b\phi(c))$. It is easily shown that R is a ring with $|R| = n$ and $D = \{(0,b) \mid b \in GF(p^k)\}$; hence if $a \neq 0$, (a,b) is invertible. Thus, if $a \neq 0$, $(a,b)R = R(a,b) = R$; and if $b \neq 0$, $(0,b)R = \{(0,b\phi(c)) \mid c \in GF(p^k)\}$ and $R(0,b) = \{(0,bc) \mid c \in GF(p^k)\}$, so that $(0,b)R = R(0,b) = D$. Thus, R is a Q_n-ring. Obviously, R is noncommutative and $(1,0)$ is a multiplicative identity element.

4. Commutativity of Q_n-rings: the general case.

We begin this section with a near-commutativity theorem, which is reminiscent of [1, Theorem 6].

Theorem 4.1. *If $n \leq 16$ and R is any Q_n-ring, then $C(R)$ is nil.*

Proof. Since every Q_k-ring is a Q_{k+1}-ring, we may assume that $n = 16$. If $|R| \geq 16$, then N is an ideal by Lemma 2.1(iii); and R/N is a finite ring with no nonzero nilpotent elements, hence is commutative. If $|R| < 16$, it follows easily from the Wedderburn-Artin structure theory that $C(R)$ is nil. □

We proceed to our major commutativity theorems.

Theorem 4.2. *Let $n \geq 4$, and let R be a Q_n-ring. If $|R| > 2n-2$, or if n is even and $|R| > 2n-4$, then R is commutative.*

Proof. Let R be a Q_n-ring which is not commutative, and let $x \notin Z$. Our aim is to show that $|R| \leq 2n-2$ or $|R| \leq 2n-4$; and since $n-1 < 2n-4$, we may suppose that $|R| \geq n$. By Lemma 2.1(iv), there exists $y \in R\backslash(A_r(x)\cup C_R(x))$. If H is any $(n-1)$-subset which does not contain y, we have $x(\{y\}\cup H) = (\{y\}\cup H)x$; and since $xy \neq yx$, there exists $z \in H$ such that $xy = zx$—that is, $H \cap L_{x,y} \neq \varnothing$. We have argued that any $(n-1)$-subset of R must either contain y or intersect $L_{x,y}$—a condition that cannot hold if $|R\backslash L_{x,y}| \geq n$; thus,

$$|R| \leq \left|L_{x,y}\right| + n - 1. \tag{4.1}$$

We now investigate $|L_{x,y}|$. If $w \in L_{x,y}$, then $L_{x,y} = w + A_\ell(x)$, hence $|L_{x,y}| = |A_\ell(x)|$. By Lemma 2.1(iv), $A_\ell(x) \neq R$, so $|L_{x,y}| = |R|/k$ for some $k \geq 2$. Substituting into (4.1) gives

$$|R| \leq \frac{k}{k-1}(n-1) \leq 2n-2. \tag{4.2}$$

Suppose now that n is even. If $A_\ell(x)$ has index at least 3 in $(R,+)$, (4.2) yields $|R| \leq \lfloor 3(n-1)/2 \rfloor \leq 2n-4$. Thus, we may assume that $|A_\ell(x)| = |R|/2$ and show that $|R| \neq 2n-2$.

Suppose, then, that $|A_\ell(x)| = n-1$, so that $|A_r(x)| = n-1$ by Lemma 2.1(i). Note that $A_\ell(x)$ is an $(n-1)$-subset not intersecting $L_{x,y}$, so y must be in $A_\ell(x)$; and since

$y \notin A_r(x)$, $A_\ell(x) \neq A_r(x)$, so $A_r(x)x \neq \{0\}$. Now $x(y \cup A_r(x)) = (y \cup A_r(x))x$ and therefore $A_r(x)x \subseteq \{xy, 0\}$; hence $A_r(x)x = \{0, xy\}$ is an additive subgroup of order 2. Therefore the map $\phi : A_r(x) \to A_r(x)x$ given by $w \mapsto wx$ has kernel of index 2 in $A_r(x)$. But $|A_r(x)|$ is odd, so we have a contradiction; hence $|R| \le 2n-4$. □

As we will see later, the bounds on $|R|$ in Theorem 4.2 are best possible; however, under various restrictions, a smaller bound holds.

Theorem 4.3. *Let $n \ge 4$ and let R be a Q_n-ring with $|R| > (3/2)(n-1)$. Then R is commutative if one of the following is satisfied:*

(i) *$|R|$ is odd;*

(ii) *$(R, +)$ is not the union of three proper subgroups;*

(iii) *N is commutative;*

(iv) *$R^3 \neq \{0\}$.*

Proof. Again we suppose that R is not commutative and $x \notin Z$. Since $|R| > (3/2)(n-1) > n$, the arguments in the proof of Theorem 4.2 show that $|A_\ell(x)| = |A_r(x)| = |R|/2$—a fact which proves that (i) implies commutativity of R.

Applying the first isomorphism theorem for groups shows that $|xR| = |Rx| = 2$; hence for any $u \in R \backslash A_r(x)$ and $v \in R \backslash A_\ell(x)$, $xR = \{0, xu\}$ and $Rx = \{0, vx\}$. Since $xR = Rx$ by Lemma 2.1(i), it follows that if $y \in R \backslash (A_\ell(x) \cup A_r(x))$, then $\{0, xy\} = xR = Rx = \{0, yx\}$ and therefore $y \in C_R(x)$. Thus $R = A_\ell(x) \cup A_r(x) \cup C_R(x)$, and we have proved that (ii) implies commutativity of R.

We now show that $x \in N$. Since R is not commutative, it follows from Theorem 3.1 that R does not have 1, hence $R = D$; and if $x \notin N$, some power of x is an idempotent zero divisor $e \neq 0$. Since $A_\ell(x) \subseteq A_\ell(e)$ and $A_\ell(e) \neq R$, we must have $A_\ell(x) = A_\ell(e)$ and similarly $A_r(x) = A_r(e)$. But e is central by Lemma 2.1(ii), hence $A_\ell(x) = A_r(x) = A(x) \subseteq C_R(x)$. Thus, if $y \notin A(x)$, $\{0, xy\} = xR = Rx = \{0, yx\}$ and y is also in $C_R(x)$, contrary to the assumption that $x \notin Z$. But x was an arbitrary noncentral element; hence, if there exist two noncommuting elements, both must be nilpotent. Thus (iii) forces commutativity of R.

To complete our proof, we show that our assumption that R is not commutative forces $R^3 = \{0\}$. For $x \notin Z$, the fact that $x \in N$ yields $A_r(x^2) \supsetneq A_r(x)$, so $A_r(x^2) = R$; hence $x^2R = Rx^2 = \{0\}$. If we choose $y \in R \backslash (A_r(x) \cup C_R(x))$ and $w \in R \backslash (A_\ell(x) \cup C_R(x))$, then $y^2R = Ry^2 = \{0\}$; moreover, $\{0, xy\} = xR = Rx = \{0, wx\}$, so $xy = wx$. Thus, $xR^2 = xyR = wxR = \{wxy, 0\} = \{xy^2, 0\} = \{0\}$. If $z \in Z$, then $x + z \notin Z$ so $(x+z)R^2 = \{0\}$; therefore $R^3 = \{0\}$ as required. □

We now give examples showing that the bounds on $|R|$ in Theorems 4.2 and 4.3 are best possible.

Example 4.4. Let R be the algebra over $GF(2)$ with basis x, y, x^2 and multiplication defined by $xy = x^2 = y^2$, $0 = yx = x^2y = yx^2 = xx^2 = x^2x = x^2x^2$. Then $\{0, x^2\} = A(R)$. It is easily verified that for any $u \notin A(R)$, the sets $A_\ell(u)$, $A_r(u)$, $\{w \in R \mid uw = x^2\}$ and $\{w \in R \mid wu = x^2\}$ all have 4 elements; hence for any 5-subset S of R, $uS = Su = \{0, x^2\}$. Therefore R is a Q_5-ring, and hence a Q_6-ring, with $|R| = 8$. Thus, in general, neither bound in Theorem 4.2 can be improved.

Example 4.5. Let R be the algebra over $GF(3)$ with basis x, y, x^2 and multiplication defined as in the previous example. An argument similar to the one above shows that R is a Q_{19}-ring with $|R| = 27$, so the bound $(3/2)(n-1)$ in Theorem 4.3 cannot be reduced.

5. Further results for small n. By definition all Q_1-rings are commutative, and it is easy to see that all Q_2-rings are commutative; and since there exist rings of order 4 which are not commutative, not all Q_5-rings are commutative. It is natural to ask: what is the largest n such that all Q_n-rings are commutative? Our next theorem gives the answer.

Theorem 5.1. *If $n \le 4$, all Q_n-rings are commutative.*

Proof. Since every Q_k-ring is a Q_{k+1}-ring, we may assume $n = 4$. By Theorem 4.2 any counterexample R would have $|R| \le 4$; and since all rings of order less than 4 are commutative, we need only to consider rings of order 4.

Suppose, then, that R is a counterexample and x and y are noncommuting elements with $xy \ne 0$. Then $R = \{0, x, y, x+y\}$. Since idempotents are central, any of $x^2 = x$, $x^2 = y$, $x^2 = x+y$ would force x and y to commute; hence $x^2 = 0$. It is now easily checked that the condition $xR = Rx$ cannot hold. □

Not surprisingly, a better result holds for rings with 1.

Theorem 5.2. *If $n \le 8$, then every Q_n-ring with 1 is commutative.*

Proof. We may assume that $n = 8$. Suppose that R is a counterexample. By Theorem 3.1, $|R| \le 8$; and since all rings with 1 having fewer than 8 elements are commutative, $|R| = 8$ and R is indecomposable. Since idempotents are central, we therefore have no idempotents except 0 and 1; hence every element is either nilpotent or invertible. Since $u \in N$ implies $1+u$ is invertible, it follows from Lemma 2.3 that there exists a pair x, y of noncommuting invertible elements. The group of units is not commutative and has at most 7 elements, hence is isomorphic to S_3. Thus, there exists a unique nonzero nilpotent element u, which by Lemma 2.3 is not central; and there is therefore an invertible element w such that $uw \ne wu$. But in view of Lemma 2.1(iii), wu and uw are nonzero nilpotents, so we have a contradiction. □

Theorem 5.2 is best possible; the ring of upper-triangular 2×2 matrices over $GF(2)$ is a Q_9-ring with 1 which is not commutative.

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