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A GRAPH AND ITS COMPLEMENT WITH SPECIFIED PROPERTIES I: CONNECTIVITY

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Dedicated to Karl Menger

ABSTRACT. We investigate the conditions under which both a graph G and its complement $\overline{G}$ possess a specified property. In particular, we characterize all graphs G for which G and $\overline{G}$ both (a) have connectivity one, (b) have line-connectivity one, (c) are 2-connected, (d) are forests, (e) are bipartite, (f) are outerplanar and (g) are eulerian. The proofs are elementary but amusing.

KEY WORDS AND PHRASES. Graphs, Complement.

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1. CONNECTIVITY.

The connectivity (or line-connectivity) $\kappa = \kappa(G)$ (or $\lambda = \lambda(G)$) of a graph G is the minimum number of points (or lines) whose removal results in a disconnected or a trivial graph. We write $\bar{\kappa}$ (or $\bar{\lambda}$) for $\kappa(\bar{G})$ (or $\lambda(\bar{G})$) where $\bar{G}$ is the complement of G. We follow the graph theoretic terminology and notation of the book [1]. Recall that Δ denotes the maximum degree among all points of G.

LEMMA 1. The complement $\bar{G}$ of a connected graph G is connected if and only if G has no spanning complete bipartite subgraph.

PROOF. If G has a spanning complete bipartite subgraph, then $\bar{G}$ clearly contains no line joining the two parts, hence must be disconnected. Conversely, if $\bar{G}$ is disconnected, then any bipartition of V(G) in which one part consists of the points of precisely one component of $\bar{G}$ gives a spanning complete bipartite subgraph of G.

The next statement is an easy consequence of the lemma.

THEOREM 1. A graph G with p points satisfies the condition $\kappa = \bar{\kappa} = 1$ if and only if G is a graph with either

(1) $\kappa = 1$ and $\Delta = p - 2$, or

(2) $\kappa = 1$, $\Delta \leq p - 3$ and G has a cutpoint v with endline e and endpoint u such that G - u contains a spanning complete bipartite subgraph.

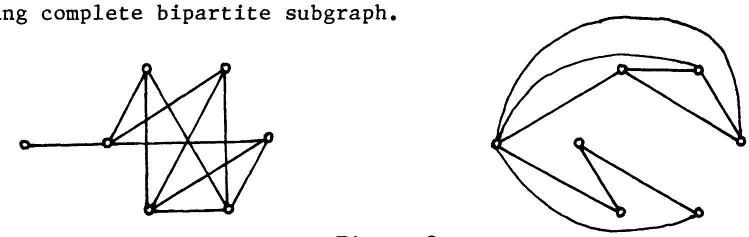
PROOF. We note that if $\kappa = \bar{\kappa} = 1$, then the degree of each point of G is at most p - 2, since otherwise $\bar{G}$ would contain an isolated point which would make $\bar{\kappa} = 0$.

(1) Let G be a graph with $\Delta = p - 2$ and $\kappa = 1$, as in Figure 1a.

(a) Figure 1. (b)

The removal of any cutpoint v from G results in a disconnected graph, so that $\overline{G - v}$ is connected. Since $\Delta = p - 2$ by hypothesis, v is adjacent in $\bar{G}$ to a point of $\overline{G - v}$. Thus $\bar{G}$ is connected. Furthermore $\bar{G}$ has an endline since $\Delta = p - 2$, and hence $\bar{G}$ has a cutpoint (as illustrated in Figure 1b), so that $\bar{\kappa} = 1$.

(2) Let G be a graph with $\kappa = \bar{\kappa} = 1$ and $\Delta \leq p - 3$. By the definition of κ, G is connected and has a cutpoint v. We see that $H = G - v$ has just two components, since otherwise every two points of $\bar{G}$ would lie on a common cycle of $\bar{G}$ and thus $\bar{G}$ would have no cutpoint, contradicting $\bar{\kappa} = 1$. Denote by H_1 and H_2 the two components of H, with p_1 and p_2 points respectively. Assume that both p_1, $p_2 \geq 2$. Then $\bar{G}$ would have no cutpoint since every two points of $\bar{G}$ would lie on a common cycle of $\bar{G}$. Thus it is sufficient to consider only a connected graph G which has a cutpoint with endline e and endpoint u. We now show that G - u contains a spanning complete bipartite subgraph. If G - u does not contain such a subgraph, then $\overline{G - u}$ is connected by Lemma 1. Moreover, the endpoint u of e is adjacent in $\bar{G}$ to every point of $\bar{G}$ lie on a common cycle and so $\bar{G}$ has no cutpoint, which again contradicts $\bar{\kappa} = 1$. Thus G - u contains a spanning complete bipartite subgraph.



(a) Figure 2. (b)

Conversely, let G satisfy the condition (2) as shown in Figure 2a. Then $\bar{G}$ is connected and the removal of the endpoint u from $\bar{G}$ results in at least two components by Lemma 1. Hence we see that $\kappa = \bar{\kappa} = 1$.

A graph G is a block if G is connected and has no cutpoint. From Theorem 1 and Lemma 1, we obtain two consequences whose proofs are ommitted or outlined.

COROLLARY 1a. If G is a block, then $\bar{G}$ is also a block if and only if

(1) $2 \leq \deg v \leq p - 3$ for every point v of G, and

(2) G has no spanning complete bipartite subgraph.

CORLLARY 1b. A graph G with p points satisfies the condition $\lambda = \bar{\lambda} = 1$ if and only if G is a connected graph with a bridge and $\Delta = p - 2$.

PROOF. Let G be a graph with $\lambda = \bar{\lambda} = 1$. Then G satisfies the condition $\kappa = \bar{\kappa} = 1$ by the relation $\kappa \leq \lambda$. Hence the graph G satisfies either (1) or (2) of Theorem 1. It is clear that (2) cannot hold, since $\bar{G}$ can possess an endline only if the spanning bipartite subgraph of G - u is a star, in which case $\Delta = p - 2$, and so (1) must obtain.

Conversely, if G is a graph with $\lambda = 1$ and $\Delta = p - 2$, then $\bar{G}$ is connected and has an endline, that is, $\bar{\lambda} = 1$.

2. BIPARTITE GRAPHS AND OUTERPLANAR GRAPHS.

A graph G is a forest if G has no cycles. An outerplanar graph is planar and can be embedded in the plane so that all its points lie on the same face.

THOEREM 2. All the graphs G such that both G and $\bar{G}$ are bipartite are: are shown in Figure 3.

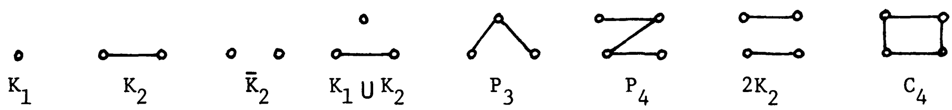


Figure 3.

PROOF. The number k of components of G is at most two, since otherwise $\bar{G}$ would contain a triangle.

CASE 1: $k = 2$. Let G have components G_1 and G_2. Both G_1 and G_2 are complete, since otherwise $\bar{G}$ would contain a triangle. Furthermore, the order of each of the complete graphs G_1 and G_2 is at most two, since otherwise G would contain a triangle. Hence we obtain $G = \bar{K}_2$, $K_1 \cup K_2$ and $2K_2$.

CASE 2: $k = 1$. Since G is bipartite, the point set of G can be partitioned into two subsets V_1 and V_2 such that every line of G joins V_1 with V_2. The cardinalities of V_1 and V_2 are at most two, since otherwise $\bar{G}$ would contain a triangle. Furthermore, each subgraph induced by any three points of G contains one or two lines. Hence we get $G = K_1, K_2, P_3, P_4$, and C_4.

COROLLARY 2a. All the graphs G such that both G and $\bar{G}$ are forests are:

$$G = K_1, K_2, \bar{K}_2, K_1 \cup K_2, P_3 \text{ and } P_4$$

We have determined in Theorem 2 all eight graphs such that both G and $\bar{G}$ are bipartite, and note that for none of these graphs G is both G and $\bar{G}$ have even cycles. We now show that for just two graphs G, both G and $\bar{G}$ have an odd cycle.

THEOREM 3. The two self-complementary graphs of order 5, A and C_5, are the only G such that both G and $\bar{G}$ have only odd cycles (Figure 4).

PROOF. If the number of points of G is at least 6, either G or $\bar{G}$ contains C_4 since the ramsey number $r(C_4) = 6$. It is easily verified that the two self-complementary graphs of order 5, A and C_5 shown in Figure 4, are the only G such that both G and $\bar{G}$ have odd cycles.

THEOREM 4. All the graphs G such that neither G nor $\bar{G}$ are forests but both are outerplanar are the following 32 graphs:

(1) the two self-complementary graphs A and C_5 of order 5 (Figure 4), and

(2) the 15 graphs shown in Figure 5 and their complements.

THEOREM 5. Both G and $\bar{G}$ are eulerian if and only if both are connected, p is odd, and G is eulerian.

Of course p must be odd so that the degree of each point in both G and $\bar{G}$ is even. Lemma 1 already gives a simple condition for both G and $\bar{G}$ to be connected. The result follows at once.

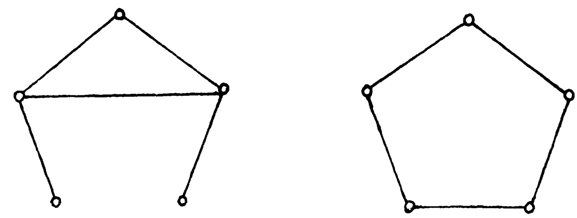


Figure 4.

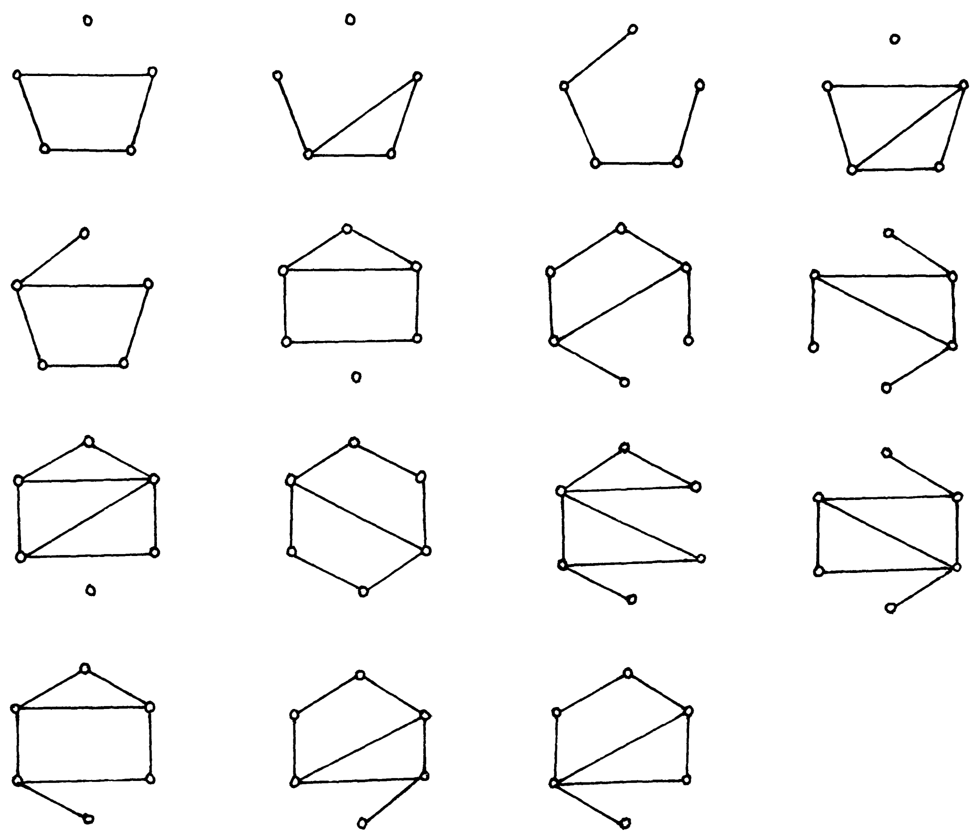


Figure 5.

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