Internat. J. Math. & Math. Sci.
Vol. 2 No. 2 (1979) 251-260

THE FOUR KNOWN BIPLANES WITH k = 11

CHESTER J. SALWACH

Department of Mathematics
Lafayette College
Easton, Pennsylvania 18042

JOSEPH A. MEZZAROBA

Department of Mathematics
Villanova University
Villanova, Pennsylvania 19085

(Received May 21, 1978)

ABSTRACT. The four known biplanes of order 9 (k = 11) are described in terms of their ovals, λ-chain structures, and automorphism groups. An exhaustive computer search for all biplanes of order 9 with certain chain structures has produced but two, one of which is new. None of these four biplanes yield the putative plane of order 10.

1. INTRODUCTION

Only finitely many biplanes (projective designs with $\lambda = 2$) are presently known and it is conjectured that for any fixed $\lambda \geq 2$ but finitely many projective designs exist. The authors know of four biplanes of order 9. All are self-dual. The most symmetric of these has repeatedly been uncovered. A group-theoretical treatment of this design was given by Hall, Lane, Wales [5]. The authors discovered a second with the aid of a computer, and R. H. F. Denniston has constructed two more [4]. A description of these will be provided in sections 3 and 5.

There exists an algebraic coding theoretic technique whereby certain projective planes of even order can be constructed via biplanes of odd order and it is successful in obtaining the planes of orders 2, 4, and 8 from biplanes or orders 1, 3, and 7, respectively [1]. Unfortunately, the four known biplanes of order 9 fail to yield the putative plane of order 10 since none admit a sufficient number of ovals. An exhaustive computer research for all biplanes of order 9 with chain structures consisting of (4-4-3)-chains, conjectured to admit the largest number of ovals, failed to produce any additional biplanes.

2. DEFINITIONS AND REMARKS. A (v, k, λ) design or projective design on a v-set S (a set consisting of v elements, called points) is a collection, D, of k-subsets of S (called blocks) such that every 2-subset of S is contained in precisely λ elements of D. The order of a projective design is $m = k-\lambda$. The incidence matrix of a projective design is a $v \times v$ matrix $(a_{B,P})$ where $a_{B,P}=1$ if P is contained in B and 0 otherwise. The automorphism group of the design is $\{\sigma \in \mathrm{Sym}(S) \mid \sigma \cdot B \in D$, for all $B \in D\}$. The dual of a projective design is obtained by switching the roles of blocks and points. A projective design is called a projective plane if $\lambda = 1$, and a biplane if $\lambda = 2$.

Assmus and van Lint [3] have generalized the notion of an oval in a projective plane to arbitrary projective designs. An oval in a biplane of even order, m, is a set of $\frac{m+4}{2}$ points, and in a biplane of odd order of $\frac{m+3}{2}$ points, such that no three points are contained in a block. A block is tangent to an oval if it meets the oval in precisely one point. An oval in a biplane of even order has no tangents, whereas there passes precisely one tangent through each point of an oval in an odd order biplane. In the odd order case, the set of all blocks tangent to an oval forms an oval in the dual biplane [3].

A linear (n,ℓ) code C is an ℓ-dimensional subspace of F^n, F a finite field. The dual of C, $C^{\perp}$, is the subspace of F^n consisting of all vectors orthogonal to each vector of C. The weight of a code vector is the number of non-zero coordinates and the minimum weight of D is $\text{Min}\{\text{weight}(v) \mid v \in C, v \neq 0\}$.

Pick a block of the biplane, index its point from 1 to k, and call it the indexing block. Each remaining block can then be indexed by the 2-subset in which it meets the indexing block, and each point not incident with the indexing block can be denoted by a λ-chain, which is simply a permutation on k letters written in cycle notation, where two numbers are adjacent if the point is incident with the block indexed by the resulting 2-subset. The chain structure of the biplane for a given indexing block is then the number of chains of each type of cycle structure. Further information on the notion of λ-chains can be found in [7] and [9].

3. THE FOUR KNOWN BIPLANES OF ORDER 9.

A biplane of order 9 is a (56, 11, 2) projective design. Let I be the incidence matrix of a projective design. Since $Rk_p(I)$ depends only on the parameters of the projective design unless $p|m$ [6], only the mod 3 rank can aid in distinguishing biplanes of order 9.

If I is the incidence matrix of a biplane, then $\det(I) = k\, m^{\frac{v-1}{2}}$. Let d_i, $1 \le i \le v$, be the elementary divisors of I. For $k \neq 2$, $d_v = \frac{km}{2}$, if m is even, and km otherwise. Moreover, if m is a prime not equal to 2, then $Rk_m(I) = \frac{v+1}{2}$ [8]. For $m = 9$, $d_v = d_{56} = 11 \cdot 3^2$, and $\det(I) = 11 \cdot 3^{55}$. Thus for a biplane of order 9, $Rk_3(I) \le 28$, since in the case of equality the elementary divisors are 1^{28}, 3^1, 9^{26}, $9 \cdot 11$ (where x^n means that x occurs as an elementary divisor n times).

The 3-rank does distinguish the four known biplanes oforder 9. The Hall-Lane-Wales biplane is of 3-rank 20, the one the authors discovered has a 3-rank of 22, and the two constructed by Denniston have 3-ranks of 24 and 26, respectively.

Hence we shall denote the four by B_{20}, B_{22}, B_{24}, and B_{26} respectively.

The collection of chain structures for each of these four biplanes can be found in Appendix A. Of course, the only possible λ-chains are of type (11), (8-3), (7-4), (6-5), (5,3,3), (4,4,3).

Notice that as the 3-rank increases, the cycle sizes become larger. B_{22} admits only two chain structures consisting entirely of (4-4-3)-chains. For B_{24}, all but one type of structure contains (11)-chains, and this remaining type consists of (7-4), (5-3-3), and (4-4-3)-chains. In the case of B_{26}, the single structure type devoid of (11)-chains consists of (8-3) and (4-4-3)-chains.

Biplanes of order 4 afford the only other known example in which biplanes of a given order are distinguished by rank. The three biplanes of order 4 have 2-ranks of 6, 7, and 8, respectively. The chain structures for B_6, B_7, and B_8 contain 0, 4, and 6 (6)-chains, respectively. Again, there exists a relationship between rank and chain cycle sizes. This relationship could conceivably hold for biplanes in general, or at least for all biplanes of order 9.

The codes generated by the rows of the incidence matrices over GF(2) of the three biplanes of order 4 are nested in the sense that span $(B_6) \subset$ span (B_7) $\subset$ span (B_8) [2]. Such a nesting does not occur over GF(3) for the known biplanes of order 9 since a computer calculation demonstrated that each of the codes contains but 56 weight-11 vectors where all non-zero coordinates are one.

4. BIPLANES OF ORDER 9 AND THE PUTATIVE PLANE OF ORDER 10.

Let G be the v×2v matrix obtained by preceding the incidence matrix, I, of a biplane of odd order by a v×v identity matrix, and let G' be obtained by preceding the identity matrix by I^t. The mod 2 spans of G and G' are identical and this subspace is a self-dual (2v,v) code over GF(2) with minimum weight k+1. By selecting the minimal-weight vectors containing a 1 in a fixed coordinate, one sometimes obtains a plane of orders 2, 4, and 8 [1]. However, none of the four known biplanes of order 9 yield the putative plane of order 10.

It is shown [3] that the minimal-weight vectors which are neither rows of G nor rows of G' are in fact the chatacteristic functions on sets of the form $\theta^d \cup \theta$, where θ is an oval of the biplane.

Thus, for a biplane of order 9 to yield the plane of order 10 it is necessary that a point be contained in 99 ovals, since 12 weight-12 vectors that "contain" the point are rows of G or G' and the plane order 10 has 111 lines. The number of ovals in each biplane and the number of ovals through each point of the four biplanes is tabulated in Appendix B. As the 3-rank increases, the number of ovals in the biplane decreases. The same is seen to hold true with the three biplanes of order 4, B_6, B_7, and B_8, which admit 60, 28 and 12 ovals, respectively [3]. All four biplanes of order 7 necessarily have a 7-rank of 19 and all but one admit 63 ovals. This special biplane is the difference set biplane constructed via the bi-quadratic residues mod 37 and contains no ovals.

The biplane of order 9 with the lowest 3-rank, B_{20}, admitted the most ovals and all its chain structures consisted of (4-4-3)-chains. An exhaustive computer search was performed which produced all biplanes with a chain structure consisting entirely of (4-4-3)-chains. Only two such biplanes exist, namely, B_{20} and B_{22}. It was also possible to exhaustively search for all biplanes with a chain structure consisting of (5-3-3) and (4-4-3)-chains. No additional biplanes were found.

5. AUTOMORPHISM GROUPS.

We now provide a description of the biplane we discovered, B_{22}, and the two discovered by Denniston, B_{24} and B_{26}, in terms of their automorphism groups. These groups we obtained by a judicious mixture of computer and hand calculation.

In the following, let $\{P_i\}$ and $\{B_i\}$ denote base points and base blocks, respectively. B_{22} contains four point and block orbits, the other two each contain three. Also, let $G_i = G_{P_i}$ and $H_i = G_{B_i}$, where G is the automorphism group of the biplane. Each of these three biplanes contains an orbit of points (and of blocks) on which no subgroup of G acts regularly.

$\underline{B_{22}}$

$$G = \mathrm{Aut}(B_{22}) = T\ \Psi\ \Phi$$

$$= \{\tau^i\psi^j\phi^k \mid \tau^6 = \psi^6 = \phi^8 = 1,\ \psi\tau = \tau^5\psi^5,\ \psi\tau^2 = \tau^2\psi,$$

$$\phi\tau = \tau^5\psi^4\phi^3,\ \phi\tau^2 = \psi^4\phi,\ \phi\psi = \tau^2\psi^5\phi,\ \phi\psi^2 = \tau^4\psi^4\phi\}$$

Let $A = \{1, \phi\}$.

$$S = \{P_1 < \psi^3 >,\ P_2 < \tau^2 > \Psi,\ P_3 < \tau^2 > \Psi A\}$$

$$D = \{B_1 < \psi^3 >,\ B_2 < \tau^2 > \Psi,\ B_3 < \tau^2 > \Psi A\}$$

$$G_1 = T < \psi^2 > < \psi^3\phi > \qquad H_1 = T < \psi^2 > \Phi$$

$$G_2 = < \tau^3 > \Phi \qquad H_2 = < \tau^3 > < \psi^3\phi >$$

$$G_3 = H_3 = < \tau^3 > < \psi^3\phi^2 >$$

The incidence structure is defined by:

$$B_1 = \{P_1 < \psi^3 >,\ P_2 < \tau^2 > < \psi^2 >\}$$

$$B_2 = \{P_1,\ P_2 < \psi^3 >,\ P_3\tau^2\psi^2,\ P_3\tau^2\psi^3,\ P_3\tau^4\psi^3,\ P_3\tau^4\psi^4,\ P_3\tau^2\phi,$$
$$P_3\tau^4\phi,\ P_3\tau^4\psi\phi,\ P_3\tau^2\psi^5\phi\}$$

$$B_3 = \{P_2\tau^2,\ P_2\tau^4,\ P_2\tau^4\psi,\ P_2\tau^2\psi^5,\ P_3,\ P_3\tau^4\psi,\ P_3\tau^2\psi^3,\ P_3\tau^4\psi^3,$$
$$P_3\tau^2\psi^5,\ P_3\phi,\ P_3\psi^3\phi\}$$

$\underline{B_{24}}$

$G = \mathrm{Aut}(B_{24}) = \Delta\Sigma T\Psi$

$= \{\delta^i\sigma^j\tau^k\psi^\ell \mid \delta^2 = \sigma^2 = \tau^2 = \psi^8 = 1,\ \sigma\delta = \delta\sigma,\ \tau\delta = \delta\tau\psi^4,$

$\psi\delta = \delta\psi^3, \tau\sigma = \sigma\tau,\ \psi\sigma = \sigma\psi^3,\ \phi\tau = \tau\psi^7$

Let $A = \{1, \psi, \psi^2, \psi^3\}$.

$S = \{P_1\Sigma T,\ P_2\ A,\ P_3\ \Delta\Psi,\ P_4\ \Sigma\ T\ \Psi\}$

$D = \{B_1\Sigma T,\ B_2\ A,\ B_3\ \Delta\Psi,\ B_4\ \Sigma\ T\ \Psi\}$

$G_1 = H_1 = \Delta < \sigma\tau\psi >$

$G_2 = H_2 = \Delta T <\sigma\tau\psi^2>$

$G_3 = H_3 = \Sigma T$

$G_4 = H_4 = \Delta$

The incidence structure is defined by:

$B_1 = \{P_1\sigma,\ P_1\tau,\ P_1\sigma\tau,\ P_4\psi,\ P_4\psi^3,\ P_4\psi^5,\ P_4\psi^7,\ P_4\sigma\tau < \psi^2 >\}$

$B_2 = \{P_2\psi,\ P_2\psi^2,\ P_2\psi^3,\ P_4\sigma < \psi^4 >,\ P_4\sigma\tau < \psi^4 >,\ P_4T\psi^2,\ P_4T\psi^6\}$

$B_3 = \{P_3\psi^2,\ P_3\psi^4,\ P_3\psi^6,\ P_4\tau\psi,\ P_4\sigma\psi^2,\ P_4\tau\psi^2,\ P_4\sigma\tau\psi^3,\ P_4\sigma\psi^5,\ P_4\psi^6,$

$P_4\sigma\tau\psi^6,\ P_4\psi^7\}$

$B_4 = \{P_1, P_2, P_3\psi,\ P_3\delta\psi^2,\ P_3\delta\psi^3,\ P_3\psi^6,\ P_4,\ P_4\sigma,\ P_4\psi^5,\ P_4\tau\psi^6,\ P_4\psi^7\}$

$\underline{B_{26}}$

$G = \mathrm{Aut}(B_{26}) = T\Psi\Phi$

$= \{\tau^i\psi^j\phi^k | \tau^3 = \psi^6 = \phi^8 = 1,\ \psi\tau = \tau^2\psi^5,\ \psi\tau^2 = \tau\psi^3,\ \phi\tau = \tau\psi^4\phi, \phi\psi = \tau\psi^3\phi^3,$

$\phi\psi^2 = \tau^2\phi\}$

Let $A = \{1, \phi^2\}$.

$S = P_1 <\psi^3>,\ P_2T\Psi,\ P_3T\ \Psi\ A\}$

$D = \{B_1 <\psi^3>,\ B_2T\Psi,\ B_3T\ \Psi\ A\}$

$G_1 = H_1 = T <\psi^2> \Phi$

$G_2 = H_2 = \Phi$

$G_3 = H_3 = <\psi^3\phi^3>$

The incidence structure is defined by:

$B_1 = \{P_1 <\psi^3>,\ P_2T <\psi^2>\}$

$B_2 = \{P_1,\ P_2 <\psi^3>,\ P_3\tau\psi A,\ P_3\tau\psi^2A,\ P_3\tau^2\psi^4A,\ P_3\tau^2\psi^5A\}$

$B_3 = \{P_2\tau^2\psi^2,\ P_2\tau\psi^3,\ P_2\tau^2\psi^3,\ P_2\tau\ \psi^4,\ P_3\tau\psi^2,\ P_3\tau^2\psi^2,\ P_3\psi^3,\ P_3\tau\psi^4,\ P_3\tau^2\psi^4,$

$P_3\phi^2,\ P_3\psi^3\phi^2\}$

REFERENCES

1. Assmus, E. F., J. A. Mezzaroba, and C. J. Salwach, Planes and Biplanes, Proceedings of the 1976 Berlin Combinatories Conference, Vance-Reidle, 1977, 205-212.

2. Assmus, E. F., and C. J. Salwach, The (16,6,2) designs. (in preparation).

3. Assmus, E. F. and J. H. van Lint, Ovals in Projective Designs, J. Combinatorial Theory, (to appear).

4. Denniston, R. H. F., (private communicaton to E. F. Assmus, Jr.).

5. Hall, M., R. Lane, and D. Wales, Designs Derived from Permutation Groups, J. Combinatorial Theory 8 (1970) 12-22.

6. Hamada, N., On the p-rank of the Incidence Matrix of a Balanced or Partially Balanced Incomplete Bolck Design and its Applications to Error Correcting Codes, Hiroshima Math. J. 3 (1973) 154-226.

7. Hussain, Q. M., On the Totality of the Solutions for the Symmetrical Incomplete Block Designs λ=2, k=5 or 6, Sankhya 7 (1945) 204=208.

8. Salwach, C. J., Biplanes and Projective Planes, Ph.D. Dissertation, Lehigh University, Bethlehem, Pa. May, 1976.

9. Salwach, C. J. and J. A. Mezzaroba, The Four Biplanes with k=9, J. Combinatorial Theory (A), 24 (1978) 141-145.

KEY WORDS AND PHRASES. *Biplane, projective plane, λ-chain structures, ovals, automorphism groups, linear code, incidence matrix, elementary divisors, computer search.*

AMS (MOS) SUBJECT CLASSIFICATION (1970) CODES. *Primary 05-02, 05-04, 05B05, 05B20, 20B25; Secondary 15A15, 94A10.*

Appendix A

B_{20}

56 with 45 (4-4-3)-chains

B_{24}

4 with 24 (7-4)-chains
8 (5-3-3)-chains
13 (4-4-3)-chains

4 with 24 (11)-chains
21 (4-4-3)-chains

16 with 18 (11)-chains
6 (8-3)-chains
8 (7-4)-chains
4 (6-5)-chains
4 (5-3-3)-chains
5 (4-4-3)-chains

32 with 18 (11)-chains
4 (8-3)-chains
10 (7-4)-chains
2 (6-5)-chains
4 (5-3-3)-chains
7 (4-4-3)-chains

B_{22}

2 with 45 (4-4-3)-chains

18 with 24 (7-4)-chains
8 (5-3-3)-chains
13 (4-4-3)-chains

36 with 16 (11)-chains
8 (8-3)-chains
8 (7-4)-chains
13 (4-4-3)-chains

B_{26}

2 with 9 (8-3)-chains
36 (4-4-3)-chains

18 with 16 (11)-chains
17 (8-3)-chains
8 (7-4)-chains
4 (4-4-3)-chains

36 with 12 (11)-chains
8 (8-3)-chains
12 (7-4)-chains
8 (6-5)-chains
5 (4-4-3)-chains

Appendix B

B_{20} has 336 ovals

56 in 36 ovals

B_{24} has 64 ovals

4 in 12 ovals
16 in 8 ovals
32 in 6 ovals
4 in 4 ovals

B_{22} has 120 ovals

18 in 16 ovals
36 in 12 ovals
2 in 0 ovals

B_{26} has 48 ovals

2 in 18 ovals
36 in 6 ovals
18 in 2 ovals