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A GRAPH AND ITS COMPLEMENT WITH SPECIFIED PROPERTIES III: GIRTH AND CIRCUMFERENCE

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ABSTRACT. In this series, we investigate the conditions under which both a graph G and its complement $\overline{G}$ possess certain specified properties. We now characterize all the graphs G such that both G and $\overline{G}$ have the same girth. We also determine all G such that both G and $\overline{G}$ have circumference 3 or 4.

KEY WORDS AND PHRASES. *Graph, Complement, Girth, Circumference.*

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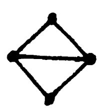
1. NOTATIONS AND BACKGROUND.

In the first paper [2] in this series, we found all graphs G such that both G and its complement $\overline{G}$ have (a) connectivity 1, (b) line-connectivity 1, (c) no cycles, (d) only even cycles, and other properties. Continuing this study, we determined in [3] the graphs G for which G and $\overline{G}$ are (a) block-graphs, (b) middle graphs, (c) bivariegated, and (d) n'th subdivision graphs. Now we concentrate on the two invariants concerning cycle lengths: girth and circumference. We will see that whenever G and $\overline{G}$ have the same girth g, then $g = 3$ or 5 only. For the circumference c, we study only the cases where both G and $\overline{G}$ have $c = 3$ or 4.

Following the notation and terminology of [4], the <u>join</u> $G_1 + G_2$ of two graphs is the union of G_1 and G_2 with the complete bigraph having point sets V_1 and V_2. We will require a related ternary operation denoted $G_1 + G_2 + G_3$ on three disjoint graphs, defined as the union of the two joins $G_1 + G_2$ and $G_2 + G_3$. Thus, this resembles the composition of the path P_3 not with just one other graph but with three graphs, one for each point; Figure 1 illustrates the "random" graph $K_4 - e = K_1 + K_2 + K_1$. Of course the quaternary operation $G_1 + G_2 + G_3 + G_4$ is defined similarly.

Recall that the corona $G \circ H$ of two graphs G with p points v_i, and H is obtained from G and p copies of H by joining each point v_i of G with all the points of the i'th copy of H. Again, for our result on girth we need a ternary operation written $G_1 + G_2 \circ G_3$ which is defined as the union of the join $G_1 + G_2$ with the corona $G_2 \circ G_3$. For example, Figure 2 illustrates the graph $A = K_1 + K_2 \circ K_1$.

$K_4 - e$:



<u>Figure 1</u>. $K_4 - e = K_1 + K_2 + K_1$

A:

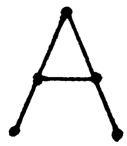


Figure 2. $A = K_1 + K_2 \circ K_1$

2. GIRTH

The girth of a graph G, denoted by $g = g(G)$, is the length of a shortest cycle (if any) in G. Note that this invariant is undefined if G has no cycles. For instance, the tetrahedron K_4, the 3-cube Q_3 and the Petersen graph P illustrated in Figure 3 have girth 3, 4 and 5, respectively.

K_4: Q_3: P:

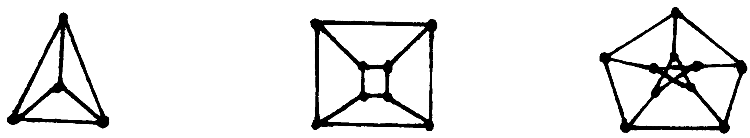


Figure 3. Graphs with small girth

Let $\bar{g}$ denote $g(\bar{G})$. In order to find all graphs G with $g = \bar{g}$, we first develop two lemmas dealing with $g \geq 4$ and with $g = 3$.

LEMMA 1. There are no graphs G other than C_5 such that both G and $\bar{G}$ have girth at least 4.

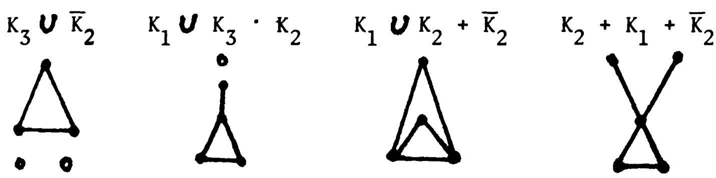
PROOF. If the number of points of G is at least 6, then G or $\bar{G}$ contains C_3 since the ramsey number $r(C_3) = 6$; see [4, p. 16]. On the other hand, the only graphs G with at most 5 points and of girth at least 4 are C_4, $C_4 \cup K_1$, $C_4 \cdot K_2$ and C_5. However, none of their complements except C_5 has girth at least 4.

LEMMA 2. If both G and $\overline{G}$ contain a triangle, then there are two triangles, one in G and the other in $\overline{G}$, which have exactly one common point.

PROOF. Take any pair of triangles T_1 from G, T_2 from $\overline{G}$. Obviously, T_1 and T_2 can have at most one common point. Since the lemma is trivial if T_1 and T_2 have a common point, we may assume that T_1 and T_2 have no common points. Color the lines of T_1 and T_2 with green and red, respectively. Consider the complete bigraph $K_{3,3}$ whose point sets are $V(T_1)$ and $V(T_2)$, and color the lines of $K_{3,3}$ with either green or red arbitrarily. Since there are in $K_{3,3}$ at least 5 lines of the same color, say green, there is a point of $V(T_2)$ with which two green lines of $K_{3,3}$ are incident. Thus, these two lines and a line of T_1 determine a green triangle in G which has a common point v with the red triangle T_2 in $\overline{G}$. □

We can restate Lemma 2 in terms of acquaintances at a party. At any party with at least five people where there are three mutual acquaintances and three mutual strangers, there must be a person who is acquainted with a pair of mutual acquaintances and who is acquainted with neither of two mutual strangers.

A subject related to Lemma 2 is discussed in [5], which specifies all the cases such that there are exactly two monochromatic triangles in the 2-colorings of K_6.



$K_1 + K_2 \circ K_1$ $K_1 + K_1 + P_3$ $K_2 + \overline{K}_3$

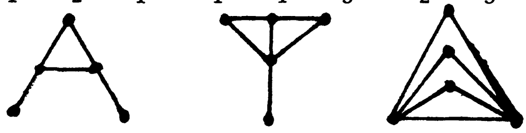


Figure 4. The seven graphs of the iff-induced family for the set of all graphs G with $g = \overline{g} = 3$.

Consider two family of graphs $\underline{\underline{N}}$ and $\underline{\underline{H}}$. In [1], the letter $\underline{\underline{N}}$ was chosen to stand for "necessary subgraphs". However, for the purpose of specifying all graphs G with $g = \overline{g} = 3$, we require the family $\underline{\underline{N}}$ to be both necessary and sufficient in the following sense. We say that $\underline{\underline{N}}$ is an iff-induced family of graphs for $\underline{\underline{H}}$ if:

a) every graph in $\underline{\underline{H}}$ contains some graph in $\underline{\underline{N}}$ as an induced subgraph; and

b) every graph G containing some graph in $\underline{\underline{N}}$ as an induced subgraph must be in $\underline{\underline{H}}$.

We illustrate with Beineke's characterization of line graph in terms of the set $\underline{\underline{N}}$ of nine forbidden induced subgraphs shown in [4, p. 75]. Let $\underline{\underline{H}}$ be the family of all graphs which are not line graphs. Then this set $\underline{\underline{N}}$ is an iff-induced family for $\underline{\underline{H}}$.

THEOREM 1. Let $\underline{\underline{H}}$ be a family of graphs with $g = \overline{g} = 3$. Then the set of seven graphs $K_3 \cup \overline{K}_2$, $K_1 \cup K_3 \cdot K_1$, $K_1 \cup K_2 + \overline{K}_2$, $K_2 + K_1 + \overline{K}_2$, $K_1 + K_2 \cdot K_1$, $K_1 + K_1 + P_3$ and $K_2 + \overline{K}_3$ is an iff-induced family for $\underline{\underline{H}}$.

PROOF. When $g = \overline{g} = 3$, by definition both G and $\overline{G}$ contain a triangle. By Lemma 2, there is a set U of five points of G such that both the induced subgraphs <U> in G and in $\overline{G}$ contain triangles. A graph F of order 5 such that both F and $\overline{F}$ contain a triangle is one of the 7 graphs in Figure 4. Thus, the sufficiency is proved. Since each of the seven graphs and their complements contain a triangle, the necessity also holds. []

3. CIRCUMFERENCE

It is now natural to consider the circumference $c = c(G)$, the length of a longest cycle in G , in place of the girth. However, as it is known that almost all graphs are hamiltonian, see Wright [7], this question is hopeless in general since there will be too many graphs G such that both G and $\overline{G}$ have circumference p , the number of points of G . Hence, we now ask this question only for $c = 3$ and 4 .

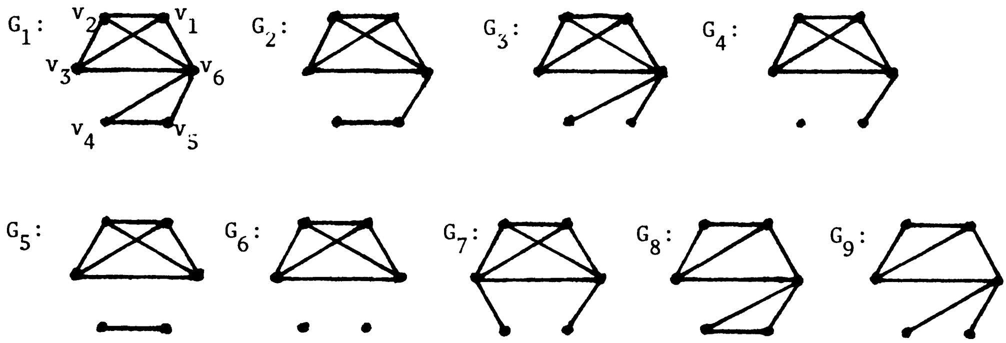


Figure 5. Graphs with $c = \overline{c} = 4$

THEOREM 2. The graph $A = K_1 + K_2 \circ K_1$ is the only graph with $c = \overline{c} = 3$. All the eighteen graphs with $c = \overline{c} = 4$ are $G_1 = K_4 \cdot K_3$, $G_2 = K_1 + K_1 + K_1 + K_3$, $G_3 = \overline{K}_2 + K_1 + K_3$, $G_4 = K_1 \cup K_1 + K_1 + K_3$, $G_5 = K_2 \cup K_4$, $G_6 = K_4 \cup \overline{K}_2$, $G_7 = K_2 + K_2 \circ K_1$, $G_8 = K_2 + K_1 + K_2 + K_1$, and $G_9 = \overline{K}_2 + K_1 + K_2 + K_1$ and their complements.

PROOF. We first settle the condition $c = \overline{c} = 3$. This precludes graphs G of order $p \geq 6$ since the ramsey number $r(C_4) = 6$ as mentioned in [6]. Hence, when $p \geq 6$, G or $\overline{G}$ contains C_4 and so has circumference at least 4 . Thus, if $c = \overline{c} = 3$, then $p \leq 5$. But as K_4 does not have two line-disjoint triangles, we also have $p \geq 5$. Thus, it is sufficient to consider graphs with exactly five points. It is easily verified that the only graph with $c = \overline{c} = 3$ is $A = K_1 + K_2 \circ K_1$ among all graphs of order 5.

We now find all the graphs G with $c = \overline{c} = 4$. Since K_5 does not contain two line-disjoint 4-cycles, the number of points $p(G) \geq 6$. We see by exhaustion that there are exactly 18 graphs G of order 6 such that neither G nor $\overline{G}$ contains C_5 or C_6, namely the nine graphs G_i in Figure 5 and their complements $\overline{G}_i$. It is easily verified that all of them satisfy $c = \overline{c} = 4$ Assume that there exists a graph H of order 7 such that $c = \overline{c} = 4$. Then the graph G obtained by removing a point v of H must be one of the 18 graphs G_i or $\overline{G}_i$, $i = 1,2,\ldots, 9$. However, we now show that there are no graphs H of order 7 such that neither H nor $\overline{H}$ contains a cycle of length at least 5 and $H-v$ is one of the G_i or $\overline{G}_i$. We label the points v_1, v_2, v_3, v_4, v_5 and v_6 of each G_i or $\overline{G}_i$ just as in $\overline{G}_1$ in Figure 5, and denote by v the point of H not belonging to G_i, $i = 1,2,\ldots, 9$. It is convenient to divide the proof into two cases.

CASE 1. Either $H-v$ or $\overline{H-v}$ is one of the G_i, $i = 1,2,\ldots, 7$. Without loss of generality, we may assume that $H-v$ is one of the G_i, $i = 1,2,\ldots, 7$. We see that there are paths of length 3 or 4 joining v_j and v_k in both G_i and $\overline{G}_i$ for any distinct points v_j and v_k, $1 \leq j, K \leq 3$. The point v must be adjacent to at least two points v_j, $j = 1, 2, 3,$ in either H or $\overline{H}$. Thus either H or $\overline{H}$ contains C_5 or C_6, which is a contradiction.

CASE 2. Either $H-v$ or $\overline{H-v}$ is G_8 or G_9.
There are two possibilities. If v is adjacent to v_2 in H, then v is forced to be nonadjacent to v_3 in H, since in G_i there is a path of length 3 joining v_2 and v_3. There is also a path of length 3 in G_i joining v_2 and v_6, and one in $\overline{G}_i$ joining v_3 and v_6. Hence either H or $\overline{H}$ contains C_5, which is a contradiction.

On the other hand, if v is not adjacent to v_2 in H, then v is forced to be adjacent to v_3 in G_i since in $\overline{G}_i$ there is a path of length 4 joining v_2 and v_3. As there is a path in G_i of length 3 joining v_3 and v_6, v is forced to be nonadjacent to v in H. Independent of the adjacency of v and v_4 in H, either H or $\overline{H}$ contains C_5, a contradiction. Since there are no graphs of order 7 with $c = \overline{c} = 4$, no graph of greater order can satisfy this condition. []

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