COMPLETION FUNCTORS FOR CAUCHY SPACES

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ABSTRACT. Completion functors are constructed on various categories of
Cauchy Spaces by forming the composition of Wyler's completion functor with
suitable modification functors.

*KEY WORDS AND PHRASES. Cauchy Space, Cauchy Filter, Completion Functor,
Modification Functor.*

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1. INTRODUCTION.

Background information on Cauchy spaces and Cauchy space completions is
available in references [3], [4], and [8]. However a review of this material
will be given in this preliminary section.

A Cauchy space $(X, \mathcal{C})$ is a pair consisting of a set X and a collection of
filters $\mathcal{C}$ on X which satisfy the following conditions:

1.   For each $x \in X$, $\dot{x} \in C$, where $\dot{x}$ denotes the fixed ultrafilter

generated by $\{x\}$;

2.   If $\mathcal{F} \in C$ and $\mathcal{G} \geq \mathcal{F}$, then $\mathcal{G} \in C$;

3.   If $\mathcal{F}$, $\mathcal{G} \in C$ and $\mathcal{F} \vee \mathcal{G}$ exists, then $\mathcal{F} \cap \mathcal{G} \in C$.

If (X, C) is a Cauchy space, then the set C is called a Cauchy structure

and its elements Cauchy filters. If (X, C) and $(Y, \mathcal{D})$ are Cauchy spaces, then

(X, C) is finer than $(Y, \mathcal{D})$ (denoted $(Y, \mathcal{D}) \leq (X, C)$) if $X = Y$ and $C \subset \mathcal{D}$.

For each Cauchy space (X, C), there is an associated convergence structure

q_C on X defined as follows: $\mathcal{F} \rightarrow x$ in (X, q_C) if $\dot{x} \cap \mathcal{F} \in C$. A Cauchy space

is said to be Hausdorff if each filter $\mathcal{F}$ converges in (X, q_C) to at most one

point. It will be assumed throughout this paper that all Cauchy spaces are

Hausdorff unless otherwise indicated.

A Cauchy space (X, C) is complete if each Cauchy filter converges. We

shall regard the terms "complete Cauchy space" and "convergence space" as

interchangeable; an axiomatization of "convergence space" is given in [8].

A Cauchy subspace $(Y, \mathcal{D})$ of a Cauchy space (X, C) is a subset Y of X

equipped with a Cauchy structure

$$\mathcal{D} = \{\mathcal{F} : \mathcal{F} \text{ is a filter on } Y, \ \mathcal{F}' \in C\}, \text{ where } \mathcal{F}'$$

denotes the filter generated on X by $\mathcal{F}$ (considered as a filter base on X).

If (X, C) is a complete Cauchy space (i.e. convergence space), then it will

be necessary to distinguish between a convergence subspace (a subspace in the

usual convergence space sense) and a Cauchy subspace (with the meaning defined

above). Note that if (Y, p) is a convergence subspace and $(Y, \mathcal{D})$ a Cauchy

subspace of a complete Cauchy space (X, C), then $q_{\mathcal{D}} = p$.

If (X, C) and $(Y, \mathcal{D})$ are Cauchy spaces, then a function $f : (X, C) \rightarrow (Y, \mathcal{D})$

is said to be Cauchy-continuous if $f(\mathcal{F}) \in \mathcal{D}$ whenever $\mathcal{F} \in C$. Throughout

this paper, the term map will be used exclusively to denote a Cauchy-continuous

function. The terms Cauchy-embedding and Cauchy-homeomorphism are defined

in the obvious way.

For any Cauchy space $(X, \mathcal{C})$, an equivalence relation among Cauchy filters is defined as follows: If $\mathcal{F}, \mathcal{G} \in \mathcal{C}$, then $\mathcal{F} \sim \mathcal{G}$ if $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}$. For $\mathcal{F} \in \mathcal{C}$, let $[\mathcal{F}] = \{\mathcal{G} \in \mathcal{C} : \mathcal{G} \sim \mathcal{F}\}$. Let $X^* = \{[\mathcal{F}] : \mathcal{F} \in \mathcal{C}\}$, and let j be the function defined by : $j(x) = [\dot{x}]$, for all $x \in X$. Note that j is injective under our assumption that $(X, \mathcal{C})$ is Hausdorff.

A <u>completion</u> $((X', \mathcal{C}'),h)$ of a Cauchy space $(X, \mathcal{C})$ consists of a complete Cauchy space $(X', \mathcal{C}')$ and a Cauchy-embedding $h : (X, \mathcal{C}) \rightarrow (X', \mathcal{C}')$ such that $cl_{q_{\mathcal{C}'}} h(X) = X'$. (Notation: cl_q denotes the closure operation for a convergence structure q.) If the last part of the preceding definition is weakened by stating, instead, that some ordinal iteration of the closure of h(X) equals X', then $((X', \mathcal{C}'),h)$ will be called a <u>weak completion</u> of $(X, \mathcal{C})$. A completion $((X', \mathcal{C}'),h)$ of $(X, \mathcal{C})$ is said to be <u>strict</u> if the following additional condition is satisfied: If $\mathcal{F} \in \mathcal{C}'$, then there is $\mathcal{G} \in \mathcal{C}$ such that $\mathcal{F} \geq cl_{q_{\mathcal{C}'}} h(\mathcal{G})$. If $(X', \mathcal{C}')$ is a topological space, then completion, strict completion, and weak completion are equivalent concepts, but in general they are distinct.

Two completions $((X', \mathcal{C}'),h)$ and $((X'', \mathcal{C}''),k)$ of $(X, \mathcal{C})$ are said to be <u>equivalent</u> if there is a Cauchy-homeomorphism φ from $(X', \mathcal{C}')$ onto $(X'', \mathcal{C}'')$ such that the following diagram comutes:

$$(X, \mathcal{C}) \xrightarrow{\ h\ } (X', \mathcal{C}')$$
$$\searrow^{k} \quad \downarrow \varphi$$
$$(X'', \mathcal{C}'')$$

The next result is established in [8].

PROPOSITION 1.1. If $((X', \mathcal{C}'),h)$ is a completion of a Cauchy space $(X, \mathcal{C})$, then there is a complete Cauchy structure $\mathcal{C}''$ on the set X^* of Cauchy equivalence classes relative to $(X, \mathcal{C})$ such that $((X', \mathcal{C}'),h)$ and $((X^*, \mathcal{C}''),j)$
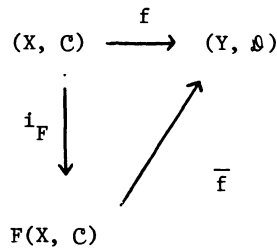
are equivalent completions.

Let $\underset{\smile}{C\,H\,Y}$ be the category with Cauchy spaces as objects and maps

(i.e., Cauchy-continuous functions) as morphisms. Let $\underset{\smile}{L\,C\,H}$ be any full

subcategory of $\underset{\smile}{C\,H\,Y}$, and let $\underset{\smile}{L\,C\,H}*$ be the full subcategory of complete

objects in $\underset{\smile}{L\,C\,H}$.

A <u>completion</u> <u>functor</u> F on $\underset{\smile}{L\,C\,H}$ is a covariant functor $F : \underset{\smile}{L\,C\,H} \rightarrow \underset{\smile}{L\,C\,H}*$

which satisfies the following conditions:

1.   For each $(X,\ C) \in \underset{\smile}{L\,C\,H}$, there is a Cauchy-embedding $i_F : (X,\ C) \rightarrow$

$F(X,\ C)$ such that $(F(X,\ C), i_F)$ is a completion of $(X,\ C)$.

2.   If $f : (X,\ C) \rightarrow (Y,\ \emptyset)$ is a map, with $(X,\ C) \in L\,C\,H$ and $(Y,\ \emptyset) \in \underset{\smile}{L\,C\,H}*$,

then there is a unique map $\overline{f} : F(X,\ C) \rightarrow (Y,\ \emptyset)$ such that the following diagram

commutes:

$$\begin{array}{ccc} (X,\ C) & \xrightarrow{\ \ f\ \ } & (Y,\ \emptyset) \\ \Big\downarrow{\scriptstyle i_F} & \nearrow & \\ F(X,\ C) & \overline{f} & \end{array}$$

If F is a completion functor on $\underset{\smile}{L\,C\,H}$, then it follows that any map

$f : (X_1,\ C_1) \rightarrow (X_2,\ C_2)$ between objects in $\underset{\smile}{L\,C\,H}$ has a unique Cauchy-continuous

extension $\overline{f} : (X_1,\ C_1) \rightarrow F(X_2,\ C_2)$, and $\overline{f} = F(f)$. Thus, two completion

functors F_1 and F_2 on the same category $L\,C\,H$ are equivalent in the sense that,

for each $(X,\ C) \in \underset{\smile}{L\,C\,H}$, the completions $(F_1(X,\ C),\ i_{F_1})$ and $(F_2(X,\ C),\ i_{F_2})$

are equivalent.

A full subcategory $\underset{\smile}{L\,C\,H}$ of $\underset{\smile}{C\,H\,Y}$ which admits a completion functor will

be called a <u>completion</u> <u>subcategory</u> of $\underset{\smile}{C\,H\,Y}$. Examples of completion sub-

categories are the categories of $C^{\wedge}$-embedded spaces and sequentially regular

spaces described in [3]. These and other examples emerge as special cases in

the general theory developed in this paper.

2.   WYLER'S COMPLETION FUNCTOR.

Ellen Reed, [8], constructed a family of completions for any Cauchy

space (X, C). One member of this family, called Wyler's completion, is the

Cauchy space formulation of a completion developed by O. Wyler for uniform

convergence spaces in [10]. Wyler's completion defines a completion functor

whose domain is the whole category C H Y (i.e., C H Y is a completion sub-

category of itself); this completion functor forms the foundation for the

completion theory developed in this paper. Wyler's completion (but not so

named) also appears in a recent abstract by Redfield [7].

Given (X, C) $\in$ C H Y, we define a convergence structure q* on the set X*

of Cauchy equivalence classes as follows: A filter $\mathcal{H}$ on X* q* - converges to

α in X* if there is a filter $\mathcal{F} \in \alpha$ such that $\mathcal{H} \geq (j(\mathcal{F})) \cap \dot{\alpha}$. Let C* be the

complete Cauchy structure on X* consisting of all q* - convergent filters.

Then it is easy to verify that ((X*, C*), j) is a strict completion of (X, C),

and that the only member of C* containing X* - j(X) are fixed ultrafilters.

PROPOSITION 2.1. If f : (X, C) $\rightarrow$ (X', C') is a map and (X', C') is

complete, then there is a unique map $\bar{f}$: (X*, C*) $\rightarrow$ (X', C') such that the

following diagram commutes:

$$
\begin{array}{ccc}
(X, C) & \xrightarrow{\ \ j\ \ } & (X*, C*) \\
 & f \searrow & \downarrow \bar{f} \\
 & & (X', C')
\end{array}
$$

PROOF. If x $\in$ X, define $\bar{f}$([x]) = f(x); if $\alpha \in$ X* - j(X), define f(α) = y

if there is $\mathcal{F} \in \alpha$ such that f($\mathcal{F}$) $\rightarrow$ y in (X', C'). It is a routine matter to

verify that $\bar{f}$ is a unique map, and that the above diagram commutes. ∎

Define the functor W : $\underset{\sim}{C\,H\,Y}$ → $\underset{\sim}{C\,H\,Y}$* as follows: If (X, C) is an object

in $\underset{\sim}{C\,H\,Y}$, then W(X, C) = (X*, C*); if f : (X, C) → (X', C') is a morphism in

$\underset{\sim}{C\,H\,Y}$, then W(f) = $\overline{f}$, where $\overline{f}$: W(X, C) → W(X', C') is the unique extension map

whose existence is guaranteed by Proposition 2.1. It is clear from Proposition

2.1 that W is a completion functor on $\underset{\sim}{C\,H\,Y}$; W will be called Wyler's com-

pletion functor.

PROPOSITION 2.2. Let (Y, $\mathcal{D}$) be a subspace of a Cauchy space (X, C), and

let id : Y → X be the identity embedding. Then the extension $\overline{id}$: W(Y, $\mathcal{D}$) →

W(X, C) is injective.

PROOF. The theorem is an immediate consequence of the following obser-

vation. If $\mathcal{F}$ and $\mathcal{G}$ are filters on Y belonging to $\mathcal{D}$, and if $\mathcal{F}'$ and $\mathcal{G}'$ are

the filters on X generated by $\mathcal{F}$ and $\mathcal{G}$ respectively, then $\mathcal{F}' \cap \mathcal{G}' \in$ C if

and only if $\mathcal{F} \cap \mathcal{G} \in$ C. ∎

Wyler's completion does not, in general, preserve such important pro-

perties as uniformizability, regularity, or total boundedness. By constructing

completion functors on certain subcategories of $\underset{\sim}{C\,H\,Y}$, one obtains completions

which preserve all of the defining properties of the subcategories, and some-

times other properties as well. A general approach to obtaining completion

subcategories of $\underset{\sim}{C\,H\,Y}$ and their completion functors by means of modification

functors is described in the next section.

As a matter of convenience and notational simplicity and since it entails

no loss of generality, we shall adopt the following convention for the

remainder of this paper: For each (X, C) $\in$ $\underset{\sim}{C\,H\,Y}$ and x $\in$ X, we shall identify

x with the element [$\dot{x}$] in X*, and consider (X, C) to be a subspace of W(X, C).

3.   MODIFICATION FUNCTORS.

Our goal is to describe completion subcategories of $\underset{\sim}{C\,H\,Y}$ which are maximal

relative to some Cauchy space property. This is accomplished for properties

which can be characterized by means of modification functors subject to certain

restrictions. Each such modification functor M gives rise to a completion

subcategory $\underset{\sim}{MCHY}$ of $\underset{\sim}{CHY}$, and the composite functor MW is the unique

completion MW is the unique completion functor on $\underset{\sim}{MCHY}$.

Let $\underset{\sim}{MC}$ be a full subcategory of $\underset{\sim}{CHY}$. A modification functor M on $\underset{\sim}{MC}$

is a convariant functor M : $\underset{\sim}{MC} \to \underset{\sim}{CHY}$ with the following properties:

1.   For each object (X, C) $\in \underset{\sim}{MC}$, M(X, C) and (X, C) have the same

underlying set;

2.   For each object (X, C) $\in \underset{\sim}{MC}$, M(M(X, C)) = M(X, C);

3.   For each morphism f $\in \underset{\sim}{MC}$, M(f) = f.

If M is a modification functor on $\underset{\sim}{MC}$, then (X, C) $\in \underset{\sim}{MC}$ is called an

M-space if M(X, C) = (X, C). In what follows, we shall be interested in

modification functors which are subject to the following additional conditions.

(L)   For each object (X, C) $\in \underset{\sim}{MC}$, M(X, C) $\leq$ (X, C).

(H)   If (X, C) $\in \underset{\sim}{MC}$ and (Y,$\mathfrak{D}$) is a subspace of (X, C), then

(Y, $\mathfrak{D}$)   MC. If, in addition, (X, C) is an M-space, then (Y, $\mathfrak{D}$) is also an

M-space.

(C)   If (X, C) $\in \underset{\sim}{MC}$, then W(X, C) $\in \underset{\sim}{MC}$ and MW (X, C) is complete.

For the remainder of this section, we assume that M : $\underset{\sim}{MC} \to \underset{\sim}{CHY}$ is a

modification functor which satisfies conditions (L), (H), and (C).

PROPOSITION 3.1. a. If (X, C) and (X, $\mathfrak{D}$) are in $\underset{\sim}{MC}$ and (X, C) $\leq$ (X, $\mathfrak{D}$)

then M(X, C) $\leq$ M(X, $\mathfrak{D}$).

b.   If (X, C) $\in \underset{\sim}{MC}$, then M(X, C) is the finest M-space coarser than

(X, C).

PROOF. a. Follows immediately by applying M to the identity map from

(X, C) to (X, $\mathfrak{D}$).

b.   Follows easily from (a). ∎

For any object $(X, C) \in \underline{MC}$, we define $M^{\wedge}(X, C)$ to be the Cauchy subspace

of $MW(X, C)$ whose underlying set is X. It follows from (L), (H), and (C)

that $M(X, C)$ is an M-space, and therefore $M^{\wedge}(X, C) \leq M(X, C)$ follows by

Proposition 3.1 (a). Let $\underline{MCHY}$ denote the full subcategory of $\underline{MC}$ whose

objects are those Cauchy spaces (X, C) such that $(X, C) = M^{\wedge}(X, C)$.

THEOREM 3.2. The following statements about a Cauchy space (X, C) are

equivalent.

1. $(X, C) \in \underline{MCHY}$.

2. (X, C) is a subspace of a complete M-space.

3. (X, C) has a weak M-space completion.

PROOF. The only non-obvious implication is $(2) \Rightarrow (1)$. Assume that

(X, C) is a subspace of a complete M-space $(Y, \emptyset)$. Then the identity map

id : $(X, C) \to (Y, \emptyset)$ has an injective extension map $\overline{\text{id}}$: $W(X, C) \to W(Y, \emptyset) =$

$(Y, \emptyset)$ by Proposition 2.2. By Proposition 3.1 (a), $\overline{\text{id}} : MW(X, C) \to (Y, \emptyset)$

is also an injective map. Restricting $MW(X, C)$ and $(Y, \emptyset)$ to X, we obtain

$M^{\wedge}(X, C) \geq (X, C)$. But $M^{\wedge}(X, C) \leq (X, C)$ is always true, and therefore

$(X, C) \in MCHY$. ∎

THEOREM 3.3. The composite functor NW is a completion functor on

$\underline{MCHY}$.

PROOF. Let $(X, C) \in \underline{MCHY}$. In the definition of completion functor,

identify i_{MW} with the identity embedding j : $(X, C) \to W(X, C)$. Since (X, C)

is a dense subspace of $W(X, C)$, $(X, C) = M^{\wedge}(X, C)$ is a subspace of $MW(X, C)$,

and $MW(X, C) \leq W(X, C)$, it follows that (X, C) is a dense subspace of

$MW(X, C)$. Furthermore, $MW(X, C)$ is complete by condition (C).

Let f : $(X, C) \to (Y, \emptyset)$ be a map, where $(X, C) \in \underline{MCHY}$ and $(Y, \emptyset) \in \underline{MCHY}^{\wedge}$.

In the diagram that follows, each unlabeled arrow is the identity map.

$$(X, \ C) \rightarrow W(X, \ C) \rightarrow M \, W \, (X, \ C)$$

$$f \downarrow \qquad \overline{f} \downarrow \qquad \overline{\overline{f}} \downarrow$$

$$(Y, \ \mathfrak{D}) \rightarrow (Y, \ \mathfrak{D}) \rightarrow (Y, \ \mathfrak{D}) = M(Y, \ \mathfrak{D})$$

It follows from the universal property of W and our assumptions governing M

that each of the above maps is Cauchy-continuous and uniquely determined, and

that the diagram commutes. This completes the proof of the theorem. ∎

COROLLARY 3.4. M C H Y is the largest category whose objects consist

only of M-spaces which forms a completion subcategory of C H Y.

PROOF. M C H Y is a completion subcategory of C H Y by Theorem 3.3. The

remainder of the assertion is an immediate consequence of Theorem 3.3. ∎

A Cauchy space (X, C) is <u>regular</u> if $\mathfrak{F} \in$ C implies $\text{cl}_{q_C} \mathfrak{F} \in$ C .

THEOREM 3.5. Let M be a modification functor such that M(X, C) is

regular for each (X, C) $\in$ M C . If $((X', \ C'), h)$ is a strict completion of

an object (X, C) $\in$ M C H Y such that (X', C') is an M-space, then $((X', \ C'), h)$

is equivalent to the completion M W (X, C).

PROOF. In view of Proposition 1.1, we can assume that X' = X* is the set

of all Cauchy equivalence classes relative to (X, C), and, in accordance with

our convention that X is a subset of X*, we can consider h to be the identity

embedding of X into X'.

From the universal property of the functor M W , it follows immediately

that (X', C') $\leq$ M W (X, C). Let $\mathfrak{F} \rightarrow y$ in (X', C'); then by the assumption of

strictness there is a filter $\mathcal{Y} \rightarrow y$ in (X', C') such that X $\in \mathcal{Y}$ and

$\text{cl}_{q_{C'}} \mathcal{Y} \leq \mathfrak{F}$. But it is a simple matter to verify that $\text{cl}_{q_{C'}} \mathcal{Y} = \text{cl}_p \mathcal{Y}$, where

p is the convergence structure on X* associated with M W (X, C). Since

X $\in \mathcal{Y}$, $\mathcal{Y} \rightarrow y$ in M W (X, C), and the regularity of M W (X, C) implies that

$\text{cl}_p \mathcal{Y} \rightarrow y$ in M W (X, C). Consequently, $\mathfrak{F} \rightarrow y$ in M W (X, C), and the two

completions are equivalent. ∎

We note earlier that topological Cauchy space completions are always strict; this fact yields the following corollary.

COROLLARY 3.6. If M is a modification functor as described in Theorem 3.5, then any topological M-space completion of (X, C) is equivalent to MW (X, C).

We conclude this section by remarking that if M is a modification functor on $\underset{\sim}{M C}$ satisfying (H), (L), and (C), then M^ : $\underset{\sim}{M C}$ → MCHY is also a modification functor which satisfies (H), (L), and (C). If (X, C) ∈ MC, then M^(X, C) can be interpreted, in view of Proposition 3.1 (b), as the finest member of $\underset{\sim}{MCHY}$ coarser than (X, C). In general, the modification functors M and M^ are distinct on their common domain category $\underset{\sim}{MC}$; this fact is illustrated in Section 4 in the case where M = R is the regular modification functor. However if (X, C) ∈ $\underset{\sim}{MCHY}$, then M(X, C) = M^(X, C) and MW (X, C) = M W(X, C); thus these two modification functors define the same completion functor.

4.   THE REGULAR COMPLETION FUNCTOR.

The concepts discussed in the preceding section are illustrated in this section using the regular modification functor R in place of the general modification functor M.

If (X, C) ∈ CHY, let C_R be the finest regular Cauchy structure on X which is coarser than C; C_R is commonly called the "regular modification" of C, although it should be noted that (X, C_R) will not be Hausdorff unless additional restrictions are placed on (X, C). Let $\underset{\sim}{RC}$ be the full subcategory of $\underset{\sim}{CHY}$ whose objects are Cauchy spaces (X, C) such that (X*, C^*_R) ∈ $\underset{\sim}{CHY}$, where W(X, C) = (X*, C^*). Define R : $\underset{\sim}{RC}$ → $\underset{\sim}{CHY}$ by R(X, C) = (X, C_R) and R(f) = f for each object (X, C) and morphism f in $\underset{\sim}{RC}$. The R-spaces are the regular objects in $\underset{\sim}{RC}$.

PROPOSITION 4.1.   R is a modification functor on $\underset{\sim}{R\,C}$ which satisfies

conditions (L), (H), and (C).

PROOF. One can verify straightforwardly that R is a modification functor;

it is obvious that (L) is satisfied.

Let (X, C) $\in$ R C and let (Y, $\emptyset$) be a Cauchy subspace of (X, C). Since

regularity is known to be herditary for Cauchy spaces, a subspace of an

R-space is an R-space. It remains to prove (Y, $\emptyset$) $\in$ R C. Put (Y*, $\emptyset$*) =

W(Y, $\emptyset$) and (X*, C*) = W(X, C). Since by Proposition 2.2, the mapping

$\overline{id}$: W(Y, $\emptyset$) $\rightarrow$ W(X, C) is injective, we can consider Y* as a subset of X*.

Denote by $\emptyset$' (respectively, $\emptyset$") the Cauchy structure for Y inherited from

(X*, C*) (respectively, (X*, C*$_R$)). Clearly, (Y*, $\emptyset$*) $\geq$ (Y*, $\emptyset$') $\geq$ (Y*, $\emptyset$"),

and (Y*, $\emptyset$*$_R$) $\geq$ (Y*, $\emptyset$'$_R$) $\geq$ (Y*, $\emptyset$"$_R$) = (Y*, $\emptyset$"). But the last space is

Hausdorff, and hence all finer spaces are Hausdorff, too. Thus R W (Y, $\emptyset$) =

(Y*, $\emptyset$*$_R$) $\in$ R C, and so (H) is satisfied.

If (X, C) $\in$ R C, then R W (X, C) = (X*, C*$_R$) $\in$ $\underset{\sim}{C\,H\,Y}$, and the convergence

structure p on X* determined by C*$_R$ is a regular convergence structure. The

conplete Cauchy structure C' on X* consisting of the p-convergent filters is

also a regular Cauchy structure, and R W (X, C) $\leq$ (X*, C') $\leq$ W(X, C). Thus

R W (X, C) = (X*, C') is complete. ∎

By virute of Theorem 3.2, we can characterize the regular completion

subcategory $\underset{\sim}{R\,C\,H\,Y}$ of $\underset{\sim}{C\,H\,Y}$ as consisting of those Cauchy spaces (X, C) which

are Cauchy subspaces of regular convergence spaces. The completion functor

R W on $\underset{\sim}{R\,C\,H\,Y}$ will be called the regular completion functor. We shall conclude

this section with examples which show that there are regular Cauchy spaces in

$\underset{\sim}{C\,H\,Y}$ which are not R-spaces, and that there are R-spaces in $\underset{\sim}{R\,C}$ which are not

members of $\underset{\sim}{R\,C\,H\,Y}$.

EXAMPLE 4.2. Let (X, p) be a minimal regular topological space which is

not compact; an example of such a space is given in [1]. It is also shown in

[1] that (X, p) cannot be completely regular.

From the results of Section 1 of [5], it follows that there is a regular Cauchy structure C on X compatible with p such that every ultrafilter on X is a member of C (i.e., C is totally bounded), and the non-convergent Cauchy filters form a single equivalence class. Thus W(X, C) is a convergence space one-point compactification of (X, C).

Suppose RW (X, C) is Hausdorff. Then RW (X, C) would be a compact, regular, Hausdorff convergence space, which is shown in [9] to have the same ultrafilter convergence as a compact, Hausdorff topological space. Let (X, q) be the convergence subspace of RW (X, C) determined by the set X. Since (X, q) ≤ (X, p), either (X, p) is completely regular, or else there is a completely regular, Hausdorff topological space coarser than (X, p). In either case, the original assumptions about (X, p) are contradicted. Consequently, RW (X, C) cannot be Hausdorff.

We have shown that (X, C) is a regular member of C H Y which is not in R C, and consequently is not an R-space. More generally, we can assert that R C, is a proper subcategory of the full subcategory of C H Y consisting of those Cauchy spaces whose R-modifications are Hausdorff. ∎

EXAMPLE 4.3. Let X be an infinite set and $\mathfrak{F}$ a free ultrafilter of X. Let C consist of all fixed ultrafilters, along with all finite intersections of free ultrafilters, excluding $\mathfrak{F}$. Thus all ultrafilters are Cauchy except $\mathfrak{F}$, the associated convergence space (X, q_C) is discrete, (X, C) is a regular member of C H Y, and, as in the preceding example, the non-convergent Cauchy filters form a single equivlence class.

Wyler's completion W(X, C) is obtained by adding a single point, call it α, to X. Considering $\mathfrak{F}$ as a filter on X ∪ {α}, we observe that $\mathfrak{F}$ does not converge to α in W(X, C), but $\mathfrak{F}$ does converge to α in RW (X, C). Indeed, RW (X, C) has the same ultrafilter convergence as the topological one-point

compactification of the discrete topological space (X, q_C), which implies that

(X, C) is an R-space. Since (X, C) is not a Cauchy subspace of $RW(X, C)$,

(X, C) has no regular completion; in other words, $(X, C) \notin RCHY$.

Note that $R(X, C) \neq R^{\wedge}(X, C)$. Thus, under the assumptions of Section 3,

M and $M^{\wedge}$ are in general distinct modification functors.

5.   MORE COMPLETION FUNCTORS.

Let (P) be a convergence space property which is both herditary (preserved

under convergence subspaces) and productive (preserved by Cartesian products).

Let PCH be the full category of CHY consisting of Cauchy subspaces of

convergence spaces (considered as complete Cauchy spaces) which have property (P).

LEMMA 5.1. Let (X, C) be a Cauchy space such that there exists

$(X, C') \in$ PCH with $(X, C') \leq (X, C)$. Then there is a finest object

$(X, C_p) \in$ PCH such that $(X, C_p) \leq (X, C)$.

PROOF. Let $\{(X, C_\alpha) : \alpha \in I\}$ be the set of all objects in PCH coarser

than (X, C). Then each (X, C_α) is a Cauchy subspace of a convergence space

(Y_α, q_α) which has property (P). Then X can be regarded in a natural way as

a subset of the Cartesian product (Y, q) of the family $\{(Y_\alpha, q_\alpha) : \alpha \in I\}$.

The Cauchy subspace (X, C'') of (Y, q) determined by X is, by our assumption, a

member of PCH. One can easily verify that $(X, C'') = (X, C_p)$ is the finest

object in PCH coarser than (X, C). ∎

Let PC be the full subcategory of CHY whose objects are those Cauchy

spaces (X, C) such that $(X^*, C_p^*) \in$ CHY. Let P : PC → PCH be defined by

$P(X, C) = (X, C_p)$, and $P(f) = f$ for all morphisms $f \in$ PC. In order for P to

be a functor, it must have the following property: If $f : (X, C) \rightarrow (Y, \emptyset)$ is a

morphism in PC, then $f : P(X, C) \rightarrow P(Y, \emptyset)$ is a morphism in PCH.

THEOREM 5.2. If P is a functor on PC, then P is a modification functor

which satisfies conditions (L), (H), and (C). In this case, PCH = PCHY is

the completion subcategory corresponding to the modification functor P,

and P W is the completion functor on P C H.

    PROOF. If P is a functor, then it is obviously a modification functor

which satisfies condition (L). Furthermore, the arguments used to verify

conditions (H) and (C) in the proof of Proposition 4.1 can be applied to show

that P also satisfies these conditions. Note that P C H consists precisely of

the P-spaces. Since each P-space is, by definition, a Cauchy subspace of

complete P-space, it follows by Theorem 3.2 and Theorem 3.3 that P C Y = P C H Y

is the completion subcategory of C H Y determined by P, and P W the associated

completion functor. ∎

    If P is a functor, then P = P^ (in the notation of Section 3), since every

P-space is in P C H Y . If (P) is the property of being a regular convergence

space, then P is the modification functor R^ rather than R, but P C H = R C H Y

and P W = R W is the regular completion functor.

    For the remainder of this section, we discuss the results of an earlier

paper [3], in the light of the methods developed in Section 3 and 5 of this

paper. The completion functors N and N_S of [3] are both describable as P W ,

where (P) is in the first case the C-embedded property of Binz (see [2]), and

in the second case the sequential regularity of Novak (see [3] and [6]). Both

of these properties are known to be hereditary and productive, and, in each case,

P is a modification functor. In the case where (P) is the C-embedded property,

the P-spaces are the Ĉ-embedded spaces which were originally introduced and

internally characterized in [4]. In the second case, the P-spaces are the

sequentially regular Cauchy spaces which are defined and characterized in [3].

    The completely regular Cauchy spaces of [3] correspond to the category

P C H in the case where (P) is the property of being a completely regular

topological space; by the results of this section they constitute a completion

subcategory of C H Y. The category U C H of uniformizable Cauchy spaces described

in [3] is also a completion subcategory of C H Y, but is not the form discussed

in this section, since the convergence space property of "being compatible with a complete uniformity" is not hereditary. However if U is the modification functor which assigns to each eligible Cauchy space the finest uniformizable Cauchy space coarser than itself, then U satisfies conditions of Section 3, and U C H = U C H Y is the associated completion subcategory of C H Y.

The completion functors determined by the completely regular and uniformizable Cauchy structures, begin toplogical, are clearly strict. It was shown in [3] that the completion functor N_S associated with the sequentially regular Cauchy spaces is not strict. It would be desirable to find some general criterion for determining which of the completion functors generated by the methods described in this paper are strict. Also, in view of the examples of Section 4, it would be desirable to find an internal characterization for the Cauchy spaces which are members of R C H Y.

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