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SOME ANALOGUES OF KNOPP'S CORE THEOREM

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ABSTRACT. Inequalities between certain functionals on the space of bounded real sequences are considered. Such inequalities being analogues of the classical theorem of Knopp on the core of a sequence. Also, a result is given on infinite matrices of bounded linear operators acting on bounded sequences in a Banach space.

KEY WORDS AND PHRASES. *Core theorem, Functionals on the bounded sequences, Infinite matrices.*

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1. INTRODUCTION.

For a real sequence $x = (x_k)$ we write

$$\ell(x) = \lim\inf x_k, \quad L(x) = \lim\sup x_k,$$

$$y(x) = \liminf \frac{x_1 + x_2 + \dots + x_k}{k},$$

$$Y(x) = \limsup \frac{x_1 + x_2 + \dots + x_k}{k},$$

$$w(x) = \inf \{L(x + z) : z \in bs\},$$

$$S(x) = \sup x_k, \quad ||x|| = \sup |x_k|,$$

$$p(x) = \limsup |x_k|, \; q(x) = \liminf |x_k|.$$

In the definition of w we use bs to denote the space of all 'bounded series', more precisely:

$$bs = \{z : \sup_n |\sum_{k=1}^{n} z_k| < \infty\}.$$

If $A = (a_{nk})$ is an infinite matrix of real, or complex, numbers, we write

$$Ax = (\Sigma a_{nk} x_k),$$

where all sums are from $k = 1$ to $k = \infty$, unless otherwise indicated.

Let X be a Banach space with norm $||x||$ and let B(X) be the Banach space of bounded linear operators on X into X with the usual operator norm. The space of bounded X-valued sequences is denoted by $\ell_\infty(X)$, with $||x|| = \sup_n ||x_n||$, for each $x \in \ell_\infty(X)$. By c(X) we denote the space of convergent X-valued sequences.

If G and H are real functionals on $\ell_\infty(X)$, and $M \geq 0$ is a real number, then $G \leq MH$ means that $G(x) \leq MH(x)$ for all $x \in \ell_\infty(X)$.

In connection with a real matrix A, we shall write, for example, $LA \leq L$ to mean that Ax exists for all $x \in \ell_\infty(R)$ and that $L(Ax) \leq L(x)$ for all $x \in \ell_\infty(R)$.

Devi [1] refers to the result that: "$LA \leq L$ if and only if A is regular and almost positive", as Knopp's core theorem, and refers to Cooke [2] for the proof. Strictly speaking the result as stated does not seem to be given by Cooke, though the ingredients for a proof are there. In Section 2 below we indicate, for completeness, a brief proof of the result.

Using Knopp's core theorem, Devi [1] proves that $LA \le w$ if and only if A is strongly regular and almost positive. To say that A is strongly regular is to say that A is regular and

$$\Sigma|a_{nk}-a_{n,k+1}| \to 0 \ (n \to \infty).$$

In Section 2 we prove that $LA \le y$ is impossible, and that $LA \le \ell$ is impossible. Also, necessary and sufficient conditions are given for $pA \le q$.

In Section 3 we give a theorem involving pA for bounded sequences from X, and infinite matrices (A_{nk}) from B(X).

2. REAL BOUNDED SEQUENCES.

We first give exact conditions for $LA \le L$, as mentioned in Section 1.

THEOREM 1. $LA \le L$ *if and only if* A *is regular and*

$$\Sigma|a_{nk}| \to 1 \ (n \to \infty). \tag{2.1}$$

PROOF. For the necessity, let $x \in c(R)$. Then $\ell(x) = L(x) = \lim x_n$ and $L(A(-x)) \le L(-x)$, whence

$$\lim x_n \le \ell(Ax) \le L(Ax) \le L(x) = \lim x_n,$$

and so $Ax \in c(R)$ with $\lim (Ax)_n = \lim x_n$, which implies A is regular. By the Silverman-Toeplitz theorem, see e.g. Maddox [3], p.165, it follows that

$$H = \lim\sup_n \Sigma|a_{nk}| < \infty, \tag{2.2}$$

$$\Sigma a_{nk} \to 1 \ (n \to \infty), \tag{2.3}$$

$$a_{nk} \to 0 \ (n \to \infty, \text{ each fixed } k). \tag{2.4}$$

From (2.2), (2.4), e.g. Agnew [4], there exists $y \in \ell_\infty(R)$ such that $||y|| = 1$ and $L(Ay) = H$. Hence, by (2.3),

$$1 \leq \liminf_n \Sigma|a_{nk}| \leq \limsup_n \Sigma|a_{nk}| \leq L(y) \leq ||y|| \leq 1,$$

which implies (2.1).

For the sufficiency, let $x \in \ell_\infty(R)$, A be regular and let (2.1) hold. If $m > 1$ then

$$\Sigma a_{nk}x_k \leq ||x|| \sum_{k<m} |a_{nk}| + (\sup_{k\geq m} x_k)\, \Sigma|a_{nk}| + ||x||\Sigma(|a_{nk}|-a_{nk}).$$

Applying the operator $\lim_m \limsup_n$ we obtain $L(Ax) \leq L(x)$, which completes the proof.

THEOREM 2. _We have_, _on_ $\ell_\infty(R)$,

$$\ell \leq y \leq Y \leq w \leq L \leq S \leq ||.||.$$

PROOF. By Theorem 1, letting A be the (C,1) matrix, we have $\ell \leq \ell A$, i.e. $\ell \leq y$. It is trivial that $y \leq Y$.

Now take $x \in \ell_\infty(R)$ and $z \in bs$. Then

$$\frac{1}{k}\sum_{i=1}^{k} x_i = \frac{1}{k}\sum_{i=1}^{k} (x_i + z_i) + \varepsilon_k, \tag{2.5}$$

where $\lim \varepsilon_k = 0$. Taking $\limsup_k$ in (2.5), and applying Theorem 1 with $A = (C,1)$, we get $Y(x) \leq L(x + z)$, whence $Y \leq w$ by the definition of w.

Since $\Theta = (0,0,0,\ldots) \in bs$ it is immediate that $w \leq L$, and the remaining inequalities are trivial.

The facts that $LA \leq y$, and $LA \leq \ell$ are impossible are special cases of the following result.

THEOREM 3. <u>Let B be any regular almost positive matrix. Then there is no matrix A such that $LA \leq \ell B$.</u>

PROOF. Suppose, if possible, there exists such an A. Theorem 1 implies $LB \leq L$, and so $LA \leq \ell B \leq LB \leq L$, whence A is regular.

By the theorem of Steinhaus, see e.g. Cooke [2], p.75, there exists $z \in \ell_\infty(R)$ such that $\ell(Az) < L(Az)$. Since $LA \leq LB$ we have $\ell(Bz) \leq \ell(Az)$, and so

$$\ell(Bz) < L(Az) \leq \ell(Bz),$$

a contradiction. This proves the theorem.

The statement prior to Theorem 3 follows on taking B to be either the (C,1) matrix, or the unit matrix.

THEOREM 4. <u>The following are equivalent:</u>

$$pA \leq q, \tag{2.6}$$

A <u>maps bounded sequences into null sequences,</u> (2.7)

$$\Sigma|a_{nk}| \to 0 \ (n \to \infty). \tag{2.8}$$

PROOF. The equivalence of (2.7) and (2.8) is well-known, see e.g. Maddox [3], p.169. We shall prove that (2.6) is equivalent to (2.8).

If (2.8) holds then, for all $x \in \ell_\infty(R)$,

$$\limsup_n |\Sigma a_{nk}x_k| = 0,$$

which implies (2.6). Conversely, let (2.6) hold. Then $\Sigma a_{nk}x_k$ is bounded on the Banach space $\ell_\infty(R)$ whence $\sup_n \Sigma|a_{nk}| < \infty$ by the Banach-Steinhaus theorem. Also, choosing $x_k = 1$, $x_n = 0$ otherwise, we must have (2.4).

Suppose, if possible, that $\lim\sup_n \Sigma|a_{nk}| = d > 0$. Choose $m(1) > 1$ such that $|a_{m(1)1}| < d/10$ and

$$\left|\Sigma|a_{m(1)k}| - d\right| < d/10.$$

Define $k(1) = 1$ and choose $k(2) > 2 + k(1)$ such that

$$\sum_{k(2)}^{\infty} |a_{m(1)k}| < d/10.$$

Next choose $m(2) > m(1)$ such that

$$\sum_{1}^{k(2)} |a_{m(2)k}| < d/10, \quad \left|\Sigma|a_{m(2)k}| - d\right| < d/10,$$

and choose $k(3) > 2 + k(2)$ such that

$$\sum_{k(3)}^{\infty} |a_{m(2)k}| < d/10.$$

Proceeding inductively we now define a sequence x by

$$x_k = \text{sgn } a_{m(r)k} \text{ for } k(r) < k < k(r+1),\ r \geq 1,$$

$$x_k = 0 \qquad \text{for } k = k(r+1), \qquad r \geq 0.$$

Then $||x|| \leq 1$ and $\lim\inf |x_k| = 0$, so (2.6) implies

$$p(Ax) = 0. \tag{2.9}$$

But for $m = m(r)$, with $r > 1$, we have

$$|\Sigma a_{mk} x_k| > \Sigma_1 |a_{mk}| - d/5,$$

where Σ_1 denotes a sum over $k(r) < k < k(r+1)$. Also, we have

$$|\Sigma_1 |a_{mk}| - d| < 3d/10,$$

and so

$$|\Sigma a_{mk} x_k| > d - 3d/10 - d/5 = d/2. \tag{2.10}$$

Since (2.10) holds for infinitely many m it follows that

$$p(Ax) \geq d/2. \tag{2.11}$$

But (2.11) contradicts (2.9), so $d = 0$, and the proof is complete.

3. BOUNDED SEQUENCES IN A BANACH SPACE.

Define, for each $x = (x_k) \in \ell_\infty(X)$,

$$G(X) = \lim\sup ||x_k||,$$

$$H(x) = \inf \{G(x+z) : z \in bs(X)\},$$

where

$$bs(X) = \{z : \sup_n || \sum_{k=1}^{n} z_k || < \infty\}.$$

Thus G and H many be regarded as the Banach space analogues of p and w which appeared earlier.

By $GA \leq MH$ we mean that $G(Ax) \leq MH(x)$ for all $x \in \ell_\infty(X)$, where

$$Ax = (\Sigma A_{nk} x_k),$$

with $A_{nk} \in B(X)$.

It is clear that $bs(X) \subset \ell_\infty(X)$, and that $0 \le H(x) \le G(x) < \infty$ for all $x \in \ell_\infty(X)$.

Also, since $-x \in bs(X)$ whenever $x \in bs(X)$ we have that

$$H(x) = 0 \text{ on } bs(X).$$

In the following theorem we need the ideas of the group norm of a sequence (B_k) from $B(X)$, see e.g. Lorentz and Macphail [5]:

$$||(B_k)|| = \sup || \sum_{k=1}^{n} B_k x_k ||$$

where the supremum is over $n \ge 1$ and x_k in the closed unit sphere of X.

We write

$$R_{nm} = (A_{nm}, A_{n,m+1}, \ldots)$$

for the mth tail of the nth row of $A = (A_{nk})$. Also, we define $\Delta A_{nk} = A_{nk} - A_{n,k+1}$, and

$$\Delta R_{nm} = (\Delta A_{n,m}, \Delta A_{n,m+1}, \ldots).$$

We now prove

THEOREM 5. Let $M \ge 0$. Then $GA \le MH$ if and only if

$$A_{nk} \to 0 \ (n \to \infty, \text{ each } k), \tag{2.12}$$

$$||R_{n1}|| < \infty \text{ and } ||R_{nm}|| \to 0 \ (m \to \infty, \text{ each } n), \tag{2.13}$$

$$\lim_m \limsup_n ||R_{nm}|| \le M, \tag{2.14}$$

$$\lim_m \limsup_n ||\Delta R_{nm}|| = 0 \tag{2.15}$$

PROOF. We remark that, in (2.12), the convergence refers to the topology of pointwise convergence.

For the sufficiency, let $x \in \ell_\infty(X)$, and $z \in bs(X)$. By Maddox [6, THEOREM 1] the conditions (2.12), (2.13), (2.14) imply $GA \le MG$, whence $GA(x+z) \le MG(x+z)$, and so

$$G(Ax) \le MG(x+z) + G(Az). \tag{2.16}$$

Now

$$\sum_{k=1}^{r} A_{nk}z_k = A_{nr}s_r + \sum_{k=1}^{r-1} \Delta A_{nk}s_k, \tag{2.17}$$

where $s_k = z_1 + z_2 + \dots + z_k$. Since $||A_{nr}s_r|| \le ||A_{nr}||\ ||s_r||$, and since $s \in \ell_\infty(X)$, it follows from (2.13) and (2.17) that, for each n,

$$\Sigma A_{nk}z_k = \Sigma\Delta A_{nk}s_k. \tag{2.18}$$

By Maddox [6, COROLLARY to THEOREM 1], the conditions (2.12) - (2.15) imply that $\Delta A : \ell_\infty(X) \to c_o(X)$, where $c_o(X)$ denotes the null X-valued sequences. Hence from (2.18) we have $G(Az) = 0$, whence (2.16) yields $G(Ax) \le MG(x+z)$. It follows that $G(Ax) \le MH(x)$, which proves the sufficiency.

For the necessity, if $GA \le MH$ then $GA \le MG$ so that (2.12) - (2.14) hold by Maddox [6, THEOREM 1].

Now take any $y \in \ell_\infty(X)$ and define $x_1 = y_1$, $x_2 = y_2 - y_1$, ..., so that

$$x_1 + x_2 + \dots + x_n = y_n.$$

Thus $x \in bs(X)$ and

$$\Sigma A_{nk}x_k = \Sigma\Delta A_{nk}y_k.$$

Hence $G(\Delta Ay) = G(Ax) \leq MH(x) = 0$, since $H(x) = 0$ on $\ell_\infty(X)$. Consequently, $G(\Delta Ay) = 0$ on $\ell_\infty(X)$, which implies $\Delta A : \ell_\infty(X) \to c_o(X)$, whence (2.15) holds by [6, COROLLARY TO THEOREM 1]. This proves the theorem.

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