IJMMS 30:1 (2002) 1–8
PII. S0161171202012607
http://ijmms.hindawi.com

A MEASURE OF GRAPH VULNERABILITY: SCATTERING NUMBER

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Received 2 April 2001 and in revised form 20 August 2001

The *scattering number* of a graph G, denoted $\mathrm{sc}(G)$, is defined by $\mathrm{sc}(G) = \max\{c(G-S) - |S| : S \subseteq V(G)$ and $c(G-S) \neq 1\}$ where $c(G-S)$ denotes the number of components in $G-S$. It is one measure of graph vulnerability. In this paper, general results on the *scattering number* of a graph are considered. Firstly, some bounds on the *scattering number* are given. Further, *scattering number* of a binomial tree is calculated. Also several results are given about *binomial trees* and *graph operations*.

2000 Mathematics Subject Classification: 05C40, 05C85.

1. Introduction. In a communication network, vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations or communication links. To measure vulnerability we have some parameters that are *toughness, binding number, vertex integrity, and scattering number* [5]. In this paper, we discuss the *scattering number* of a graph.

The *scattering number* of a graph G, denoted $\mathrm{sc}(G)$, was introduced in [4]. Formally the *scattering number* is defined by

$$\mathrm{sc}(G) = \max\{c(G-S) - |S| : S \subseteq V(G),\ c(G-S) \neq 1\}, \tag{1.1}$$

where $c(G-S)$ denotes the number of components in $G-S$. A cutset S of a graph G fulfilling $\mathrm{sc}(G) = c(G-S) - |S|$ is said to be a *scattering set*. The problem "given a graph G, decide whether the scattering number is larger than zero" is NP-complete.

The scattering number of a graph is closely related to the toughness of a graph and to the existence of Hamilton cycles and paths. The *toughness* of a graph G, denoted $t(G)$, was defined by Chvátal [1]: for the complete graph K_n we have $t(K_n) = \infty$; if G is not complete, then $t(G) = \min\{|S|/c(G-S) : S \subseteq V(G), c(G-S) > 1\}$. A graph G is said to be *t-though* if $t(G) \geq t$, that is, $|S| \geq tc(G-S)$ for any cutset S. It follows from the definitions that $t(G) \geq 1$ if and only if $\mathrm{sc}(G) \leq 0$ for any graph G [3]. Moreover, Jung [4] calls the scattering number the "additive dual" of the toughness.

A Hamilton cycle in a graph G is a cycle containing every vertex of G. Similarly, a Hamilton path in a graph G is a path that contains every vertex of G. If a graph G has a Hamilton cycle, then $\mathrm{sc}(G) \leq 0$; and if a graph G has a Hamilton path, then $\mathrm{sc}(G) \leq 1$ [3].

The following theorem is given by Deogun et al. [3].

Theorem 1.1. (a) *The scattering number of a graph G $\mathrm{sc}(G) \geq \mathrm{sc}(G-V') - |V'|$ holds for every subset $V' \subseteq V(G)$ in any graph G;*

(b) $\operatorname{sc}(G) \geq \operatorname{sc}(G-v)-1$ *for every vertex* $v \in V(G)$ *in any graph* G;

(c) *let* G *be a connected graph. Then there is a vertex* $v \in V(G)$ *such that* $\operatorname{sc}(G) \leq \operatorname{sc}(G-v)-1$;

(d) *for every connected graph* G $\operatorname{sc}(G) = \max_{v \in V(G)} \operatorname{sc}(G-v)-1$.

The *path cover number* of a graph G is the smallest number of disjoint paths covering the vertex set of G and is denoted by $\pi(G)$. For the next theorem a short proof is given in [3] and this theorem was also proven by Lehel without using order-theoretic tools [6].

Theorem 1.2 (see [3]). *If G is cocomparability graph, then* $\pi(G) = \max(1, \operatorname{sc}(G))$.

Now we give some definitions.

Definition 1.3. The *connectivity* $\kappa = \kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph.

Definition 1.4. A subset X of V is called a covering of G if every edge of G has at least one end in X. A covering X is a *minimum covering* if G has no covering X' with $|X'| < |X|$. The *covering number*, $\alpha(G)$, is the number of vertices in a minimum covering of G.

Definition 1.5. A subset X of V is called an *independent set* of G if no two vertices of X are adjacent in G. An independent set X is maximum if G has no independent set X' with $|X'| > |X|$. The *independence number* of G, $\beta(G)$, is the number of vertices in a maximum independent set of G.

In Section 2, some bounds on the scattering number are given. Section 3 gives several results about the scattering number and graph operations.

2. Bounds.

Firstly, we give two theorems showing the relation between the toughness and the *scattering number*.

Theorem 2.1. *If* $t(G) \leq 0$, *then* $\operatorname{sc}(G) \leq (\alpha(G)/t(G))(1-t(G))$.

Proof. For any cutset S, we have

$$t(G) = \min\left\{\frac{|S|}{c(G-S)}\right\} = \min\left\{1-\frac{c(G-S)-|S|}{c(G-S)}\right\} = 1-\max\left\{\frac{c(G-S)-|S|}{c(G-S)}\right\}, \quad (2.1)$$

and so

$$\max\left\{\frac{c(G-S)-|S|}{c(G-S)}\right\} = 1-t(G). \quad (2.2)$$

Hence,

$$\frac{c(G-S)-|S|}{c(G-S)} \leq 1-t(G) \Longrightarrow c(G-S)-|S| \leq (1-t(G))c(G-S). \quad (2.3)$$

On the other hand,

$$t(G) = \min\left\{\frac{|S|}{c(G-S)}\right\} \Longrightarrow \frac{|S|}{c(G-S)} \geq t(G), \quad c(G-S) \leq \frac{|S|}{t(G)} \tag{2.4}$$

for every cutset S. Since S is a cutset, we have $|S| \leq \alpha(G)$ and $c(G-S) \leq \alpha(G)/t(G)$. If $t(G) \leq 0$, then

$$(1-t(G))c(G-S) \leq \frac{\alpha(G)}{t(G)}(1-t(G)). \tag{2.5}$$

By (2.3) and (2.5),

$$c(G-S) - |S| \leq \frac{\alpha(G)}{t(G)}(1-t(G)), \quad t(G) \leq 0, \tag{2.6}$$

and so

$$\mathrm{sc}(G) = \max_S \{c(G-S) - |S|\} \leq \frac{\alpha(G)}{t(G)}(1-t(G)). \tag{2.7}$$

The proof is completed. □

Theorem 2.2. *If $t(G) > 0$, then $\mathrm{sc}(G) \leq \alpha(G)/t(G) - \kappa(G)$.*

Proof. Consider any cutset S. Then $\kappa(G) \leq |S| \leq \alpha(G)$, obviously. Since $t(G) = \min\{|S|/c(G-S)\}$, we have $c(G-S) \leq |S|/t(G)$ and so

$$c(G-S) - |S| \leq \frac{|S|}{t(G)} - |S| \leq \frac{\alpha(G)}{t(G)} - \kappa(G). \tag{2.8}$$

Hence

$$\mathrm{sc}(G) = \max_S \{c(G-S) - |S|\} \leq \frac{\alpha(G)}{t(G)} - \kappa(G). \tag{2.9}$$

The proof is completed. □

Next, we give two theorems containing the relation between some graph parameters and the scattering number.

Theorem 2.3. *If a graph G does not contain graph $2K_2$ as an induced subgraph, then*

$$\mathrm{sc}(G) = \begin{cases} \beta(G) - \alpha(G), & \text{if } \kappa(G) = \alpha(G), \\ \beta(G) - \alpha(G) + 1, & \text{if } \alpha(G) = \kappa(G) + 1. \end{cases} \tag{2.10}$$

Proof. If a graph G does not contain graph $2K_2$ as an induced subgraph, then we have $\alpha(G) = \kappa(G)$ or $\alpha(G) = \kappa(G) + 1$. That is, $|S|$ must be $\alpha(G)$ or $\alpha(G) - 1$.

$$\begin{aligned} &\text{If } |S| = \alpha(G), \quad \text{then } c(G-S) = \beta(G), \\ &\text{If } |S| = \alpha(G) - 1, \quad \text{then } c(G-S) = \beta(G). \end{aligned} \tag{2.11}$$

By (2.11) the proof is completed. □

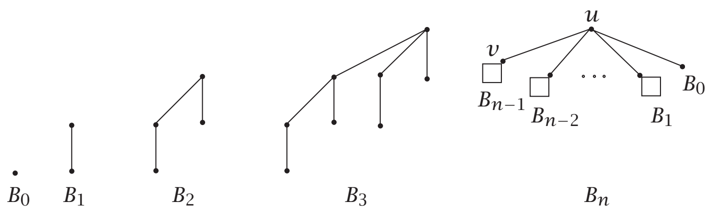

FIGURE 3.1

THEOREM 2.4. *For any graph G,* $\mathrm{sc}(G) \le \beta(G) - \kappa(G)$.

PROOF. For every $S \subset V(G)$, we have $c(G-S) \le \beta(G)$. If S is a cutset, then $|S| \ge \kappa(G)$ and $c(G-S) - |S| \le \beta(G) - \kappa(G)$. So

$$\max_S \{c(G-S) - |S|\} \le \beta(G) - \kappa(G). \tag{2.12}$$

The proof is completed. □

3. Binomial trees and scattering number. In this section, we consider the *binomial tree* B_n (Figure 3.1) (see [2]). The binomial tree B_n is an ordered tree defined recursively. The binomial tree B_0 consists of a single vertex. The binomial tree B_n consists of two binomial trees B_{n-1} that are linked together: the root of one is the leftmost child of the root of the other.

Now we give the scattering number of a binomial tree.

THEOREM 3.1. *Let $n \ge 3$ be a positive integer. Then* $\mathrm{sc}(B_n) = 2^{n-2}$.

PROOF. In Figure 3.1, we call the vertex u top vertex of B_n. Let r be the number of removing vertices of B_n. If we remove top vertex u of B_n, then $B_{n-1}, B_{n-2}, \dots, B_1, B_0$ are components. Hence the number of components is n. Now if we remove top vertex of B_{n-1}, then we obtain the components $B_{n-2}, B_{n-3}, \dots, B_1, B_0$. Then we have $2(n-1)$ components. If we continue to remove the top vertex of each component, then we have two cases.

CASE 1. If $r = 2^i$ where $0 \le i \le n-1$, then the number of remaining components is exactly $(n-i)2^i$ where $0 \le i \le n-1$. Hence

$$\mathrm{sc}(B_n) = \max_{0 \le i \le n-1} \{(n-i)2^i - 2^i\}. \tag{3.1}$$

CASE 2. If $2^{i-1} < r < 2^i$ where $2 \le i \le n-1$, then the number of remaining components is exactly $(n-(i-1))2^{i-1} + (r - 2^{i-1})(n-(i+1))$. Hence

$$\mathrm{sc}(B_n) = \max_{2 \le i \le n-1} \{(n-(i-1))2^{i-1} + (r-2^{i-1})(n-(i+1)) - r\}. \tag{3.2}$$

Now we can show that

$$\max_{2 \le i \le n-1} \{(n-(i-1))2^{i-1} + (r-2^{i-1})(n-(i+1)) - r\} \le \max_{0 \le i \le n-1} \{(n-i)2^i - 2^i\}. \tag{3.3}$$

Then

$$\operatorname{sc}(B_n) = \max_{0\le i<n} \{(n-i)2^i - 2^i\}. \tag{3.4}$$

The function $(n-i)2^i - 2^i$ takes its maximum value at $i = \lceil n-1/\ln 2-1\rceil$. It is obvious that $\lceil n-1/\ln 2-1\rceil = n-2$ for every $n \ge 3$. Hence if we substitute this value in the function $(n-i)2^i - 2^i$, then the proof is completed. □

Definition 3.2. The tensor product of two graphs $G = (V(G),E(G))$ and $H = (V(H),E(H))$, denoted by $G\otimes H$, has the vertex set $V(G)\times V(H)$, the Cartesian product of $V(G)$ and $V(H)$, and an edge between vertices (x,y) and (u,v), if and only if $\{x,u\}\in E(G)$ and $\{y,v\}\in E(H)$.

Theorem 3.3. *Let $m \ge 4$ and $n \ge 4$ be positive integers. Then*

$$\operatorname{sc}(B_m\otimes B_n) = \max\{2^{n-1}, 2^{m-1}\}. \tag{3.5}$$

Proof. The graph $B_m\otimes B_n$ has the graphs B_m and B_n as subgraphs. We consider these graphs, respectively. Let r be the number of removing vertices from $B_m\otimes B_n$. Then we have two cases, depending on B_m or B_n.

Case 1. Let $u_1,u_2,\dots,u_{2^m}$ be the vertices of B_m and let v be the top vertex of B_n. If we remove the vertices u_iv $(i = 1,2,\dots,2^m)$, then the remaining components are $B_m\otimes B_{n-1}$, $B_m\otimes B_{n-2},\dots,B_m\otimes B_1$, and 2^mB_0. Now let the top vertex of B_{n-1} be v'. If we remove the vertices u_iv' $(i = 1,2,\dots,2^m)$, then we obtain the components $B_m\otimes B_{n-2},\dots,B_m\otimes B_1$ and 2^mB_0. If we continue to remove the vertices as mentioned above, then we obtain the following cases for r.

(a) If $r = 2^m2^i$ where $0\le i\le n-2$, then the number of components is $(n-i)2^i+2^m2^i$ and

$$\operatorname{sc}(B_m\otimes B_n) = \max_{0\le i\le n-2}\{2^i(n-i)\}. \tag{3.6}$$

(b) If $r = k2^m$ where $2^{i-1} < k < 2^i$ and $2\le i\le n-2$, then the number of components is $(n-(i-1))2^{i-1}+(k-2^{i-1})(n-(i+1))+2^mk$ and

$$\operatorname{sc}(B_m\otimes B_n) = \max_{2^{i-1}<k<2^i,\,2\le i\le n-2}\{2^i+k(n-i-1)\}. \tag{3.7}$$

(c) If $r = k2^m$ and $2^{n-2}+1\le k\le 2^{n-1}$, then the number of components is $k2^m + 2(2^{n-1}-k)$ and

$$\operatorname{sc}(B_m\otimes B_n) = \max_{2^{n-2}+1\le k\le 2^{n-1}}\{2(2^{n-1}-k)\}. \tag{3.8}$$

But we can show that

$$\begin{gathered}\max_{2^{i-1}<k<2^i,\,2\le i\le n-2}\{2^i+k(n-i-1)\} \le \max_{0\le i\le n-2}\{2^i(n-i)\},\\ \max_{2^{n-2}+1\le k\le 2^{n-1}}\{2(2^{n-1}-k)\} \le \max_{0\le i\le n-2}\{2^i(n-i)\}.\end{gathered} \tag{3.9}$$

Consequently, $\mathrm{sc}(B_m \otimes B_n) = \max_{0\le i\le n-2}\{2^i(n-i)\}$. The function $2^i(n-i)$ takes its maximum value at $i = \lceil n-1/\ln 2\rceil$. It is obvious that $\lceil n-1/\ln 2\rceil = n-1$ for every $n \ge 4$ and so

$$\mathrm{sc}\left(B_m \otimes B_n\right) = 2^{n-1}. \tag{3.10}$$

Case 2. Let $v_1, v_2, \dots, v_{2^n}$ be the vertices of B_n and let u be the top vertex of B_m. If we remove the vertices uv_i $(i = 1,2,\dots,2^n)$, then the remaining components are $B_{m-1}\otimes B_n$, $B_{m-2}\otimes B_n, \dots, B_1 \otimes B_n$, and $2^n B_0$. Now let the top vertex of B_{m-1} be u'. If we remove the vertices $u'v_i$ $(i = 1,2,\dots,2^n)$, then we obtain the components $B_{m-2}\otimes B_n, \dots, B_1\otimes B_n$ and $2^n B_0$. If we continue to remove the vertices as mentioned above, then we obtain the following cases for r.

(a) If $r = 2^n 2^i$ where $0 \le i \le m-2$, then the number of components is $(m-i)2^i + 2^n 2^i$ and

$$\mathrm{sc}\left(B_m \otimes B_n\right) = \max_{0\le i\le m-2}\left\{2^i(m-i)\right\}. \tag{3.11}$$

(b) If $r = k2^n$ where $2^{i-1} < k < 2^i$ and $2 \le i \le m-2$, then the number of components is $(m-(i-1))2^{i-1} + (k-2^{i-1})(m-(i+1)) + 2^n k$ and

$$\mathrm{sc}\left(B_m \otimes B_n\right) = \max_{2^{i-1}<k<2^i,\,2\le i\le m-2}\left\{2^i + k(m-i-1)\right\}. \tag{3.12}$$

(c) If $r = k2^n$ and $2^{m-2}+1 \le k \le 2^{m-1}$, then the number of components is $k2^n + 2(2^{m-1}-k)$ and

$$\mathrm{sc}\left(B_m \otimes B_n\right) = \max_{2^{m-2}+1\le k\le 2^{m-1}}\left\{2\left(2^{m-1}-k\right)\right\}. \tag{3.13}$$

But we can show that

$$\begin{aligned}\max_{2^{i-1}<k<2^i,\,2\le i\le m-2}\left\{2^i + k(m-i-1)\right\} &\le \max_{0\le i\le m-2}\left\{2^i(m-i)\right\},\\ \max_{2^{m-2}+1\le k\le 2^{m-1}}\left\{2\left(2^{m-1}-k\right)\right\} &\le \max_{0\le i\le m-2}\left\{2^i(m-i)\right\}.\end{aligned} \tag{3.14}$$

Consequently, $\mathrm{sc}(B_m \otimes B_n) = \max_{0\le i\le m-2}\{2^i(m-i)\}$. The function $2^i(m-i)$ takes its maximum value at $i = \lceil m-1/\ln 2\rceil$. It is obvious that $\lceil m-1/\ln 2\rceil = m-1$ for every $m \ge 4$ and so

$$\mathrm{sc}\left(B_m \otimes B_n\right) = 2^{m-1}. \tag{3.15}$$

By (3.10) and (3.15) we have $\mathrm{sc}(B_m \otimes B_n) = max\{2^{m-1}, 2^{n-1}\}$.

The proof is completed. □

Definition 3.4. Let G_1 and G_2 be two graphs. The *union* $G = G_1 \cup G_2$ has $V(G) = V(G_1)\cup V(G_2)$ and $E(G) = E(G_1)\cup E(G_2)$. The *join* is denoted $V(G_1)+V(G_2)$ and consists of $V(G_1)\cup V(G_2)$ and all edges joining $V(G_1)$ with $V(G_2)$. For three or more disjoint graphs $G_1, G_2, \dots, G_n$, the *sequential join* $G_1 + G_2 + \cdots + G_n$ is $(G_1+G_2)\cup(G_2+G_3)\cup\cdots\cup(G_{n-1}+G_n)$.

Theorem 3.5. *If n is an even number, then* $\mathrm{sc}(B_0+B_1+\cdots+B_n) = 4/3 - 2^n/3$.

Proof. To prove this theorem we have two cases.

Case 1. If we remove the vertices of graphs $B_1, B_3, \ldots, B_{n-1}$, then the remaining components are $B_0, B_2, \ldots, B_n$ and the number of removing vertices is $\sum_{i=1}^{n/2} |V(B_{2i-1})| = \sum_{i=1}^{n/2} 2^{2i-1}$. Moreover, we must delete $2^{2(i-1)}$ more vertices from each B_{2i} where $0 < i \le n/2$ (except B_0). Hence $2 * 2^{2(i-1)}$ components are obtained from each B_{2i} where $0 < i \le n/2$ (except B_0). Then the number of removing vertices is exactly

$$|S| = \sum_{i=1}^{n/2} 2^{2i-1} + \sum_{i=1}^{n/2} 2^{2(i-1)} \tag{3.16}$$

and the number of components is exactly

$$c\left((B_0+B_1+\cdots+B_n)-S\right) = 2\sum_{i=1}^{n/2} 2^{2(i-1)} + 1. \tag{3.17}$$

So

$$\mathrm{sc}\left(B_0+B_1+\cdots+B_n\right) = 1 - \frac{1}{4}\sum_{i=1}^{n/2} 2^{2i}. \tag{3.18}$$

Case 2. If we remove the vertices of graphs $B_0, B_2, \ldots, B_n$, then the remaining components are $B_1, B_3, \ldots, B_{n-1}$ and the number of removing vertices is $\sum_{i=0}^{n/2} |V(B_{2i})| = \sum_{i=0}^{n/2} 2^{2i}$. Moreover, we must delete 2^{2i-3} more vertices from each B_{2i-1} where $1 < i \le n/2$ (except B_1). Hence $2 * 2^{2i-3}$ components are obtained from each B_{2i-1} where $1 < i \le n/2$ (except B_1). Then the number of removing vertices is exactly

$$|S| = \sum_{i=1}^{n/2} 2^{2(i-1)} + \sum_{i=0}^{n/2} 2^{2i} \tag{3.19}$$

and the number of components is exactly

$$c\left((B_0+B_1+\cdots+B_n)-S\right) = 2\sum_{i=2}^{n/2} 2^{2i-3} + 1. \tag{3.20}$$

So

$$\mathrm{sc}\left(B_0+B_1+\cdots+B_n\right) = -5 - \sum_{i=2}^{n/2} 2^{2i}. \tag{3.21}$$

By (3.18) and (3.21), we have

$$\mathrm{sc}\left(B_0+B_1+\cdots+B_n\right) = \max_i \left\{1 - \frac{1}{4}\sum_{i=1}^{n/2} 2^{2i}, -5 - \sum_{i=2}^{n/2} 2^{2i}\right\}. \tag{3.22}$$

Since $\sum a^t = a^t/(a-1) + c(t)$ where $a \ne 1$, we have

$$\mathrm{sc}\left(B_0+B_1+\cdots+B_n\right) = \max_i \left\{\frac{4}{3} - \frac{2^n}{3}, \frac{1}{3} - \frac{4}{3}2^n\right\} = \frac{4}{3} - \frac{2^n}{3}. \tag{3.23}$$

The proof is completed. □

Theorem 3.6. *If n is an odd number, then* $\mathrm{sc}(B_0 + B_1 + \cdots + B_n) = 2/3 - 2^n/3$.

Proof. The proof follows directly from Theorem 3.5. □

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