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CONTINUOUS AND L^p ESTIMATES FOR THE COMPLEX MONGE-AMPÈRE EQUATION ON BOUNDED DOMAINS IN $\mathbb{C}^n$

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Continuous solutions with continuous data and L^p solutions with L^p data are obtained for the complex Monge-Ampère equation on bounded domains, without requiring any smoothness of the domains.

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1. Introduction. Until recently, to solve the Monge-Ampère equation, it was necessary to consider that equation as a boundary value problem—the Dirichlet problem. In [2], we were able to solve the Monge-Ampère equation, without considering it as a boundary value problem, in L^p-Sobolev and Lipschitz spaces on domains with minimally smooth boundaries. There, we had to extend the data from the bounded domain to the whole space, construct a fundamental solution and convolve with the nth root of the data, before solving and estimating. Because of the function spaces that we were dealing with, we could not extend the data by zero outside the bounded domain. Here, we give continuous and L^p solutions of the complex Monge-Ampère equation on arbitrary bounded domains in $\mathbb{C}^n$. The solutions which we obtain here, show that the complex Monge-Ampère equation has viscosity solutions and generalized solutions in the sense of Aleksandrov [3, page 6], on all bounded domains in $\mathbb{C}^n$.

We consider the complex Monge-Ampère equation in the form

$$M_c(u) := \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) = f, \tag{1.1}$$

where at least $f \geq 0$ in Ω—a bounded domain in $\mathbb{C}^n$.

Our results are as follows.

Theorem 1.1. *Let f be a nonnegative continuous function in a bounded domain Ω in $\mathbb{C}^n$, and let $f^{1/n} \in L^1(\Omega)$. Then, there is a continuous u on Ω such that*

$$M_c(u) = f. \tag{1.2}$$

Theorem 1.2. *Let f be a nonnegative function on Ω such that $f^{1/n} \in L^p(\Omega)$, $1 \leq p \leq \infty$, where Ω is a bounded domain in $\mathbb{C}^n$, then there is u in $W^{2,p}(\Omega)$ such that*

$$M_c(u) = f, \qquad \|u\|_{W^{2,p}(\Omega)} \leq \delta \left\|f^{1/n}\right\|_{L^p(\Omega)}, \tag{1.3}$$

where δ is independent of f.

Recall the definition of the space $W^{2,p}(\Omega)$: if Ω is open in $\mathbb{C}^n$, $W^{2,p}(\Omega)$ is the space of functions u which together with their distributional derivatives of order through 2 are in $L^p(\Omega)$, $1 \le p \le \infty$. The norm in $W^{2,p}(\Omega)$ is defined by

$$\|u\|_{W^{2,p}(\Omega)} = \left\{ \int_\Omega \left(\sum_{|\alpha| \le 2} |D^\alpha u|^p \right) d\lambda \right\}^{1/p}, \tag{1.4}$$

λ being Lebesgue measure, and

$$\|u\|_{W^{2,\infty}(\Omega)} = \max_{|\alpha| \le 2} ||D^\alpha u||_{L^\infty(\Omega)}. \tag{1.5}$$

2. Continuous solutions of $M_c(u) = f$. Let e be a fundamental solution of the Laplacian $\triangle$ in $\mathbb{C}$, that is, $\triangle e = \delta$, where δ is the Dirac delta in $\mathbb{C}$. According to [4, page 64], e can be chosen in such a way that it is at least locally integrable in $\mathbb{C}$. So assume that $e \in L^1_{\text{loc}}(\mathbb{C}^n)$.

Define the distribution E_j in $\mathbb{C}^n$ by

$$E_j(\varphi) = e(\varphi(0,0,\dots j \dots 0,0)), \tag{2.1}$$

the action of e being in the jth coordinate; $\varphi \in \mathcal{D}(\mathbb{C}^n)$—a test function.

Let $\Omega_1 \Subset \Omega_2 \Subset \Omega_3 \Subset \cdots$, with $\bigcup_{\nu=1}^\infty \Omega_\nu = \Omega$, be an exhaustion of Ω. Let $\{\varphi_\nu\}_{\nu=1}^\infty$ be a sequence of functions with $\varphi_\nu \in C_0^\infty(\Omega_{\nu+1})$, $\varphi_\nu \equiv 1$ on Ω_ν, $0 \le \varphi_\nu \le 1$.

Define $v_\nu \in C^0(\mathbb{C}^n)$ by

$$v_\nu = \frac{1}{4}(E_1 + E_2 + \cdots + E_n) * (\varphi_\nu \cdot f^{1/n}), \tag{2.2}$$

where $*$ is convolution.

Now, it is clear that $M_c(v_\nu) = f$ in Ω_ν, and $\{v_\nu\}$ tends locally uniformly to a continuous function u on Ω such that

$$M_c(u) = f \quad \text{on } \Omega. \tag{2.3}$$

This proves Theorem 1.1.

3. L^p estimates. To prove Theorem 1.2, let f be defined as zero outside Ω and let E_j, $1 \le j \le n$, be as in (2.2). Define v by

$$v = \frac{1}{4}(E_1 + E_2 + \cdots + E_n) * f^{1/n}, \tag{3.1}$$

where, again, $*$ is convolution.

Then

$$M_c(v) = f, \quad \triangle^n(v) = \left(\frac{n}{4}\right) f^{1/n} \quad \text{on } \mathbb{C}^n, \tag{3.2}$$

where $\triangle^n$ is the Laplacian in C^n.

Now, let u be the restriction of v to Ω, then (3.1), (3.2), and [1, Theorem 4.2, page 47] prove Theorem 1.2.

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