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EQUIVALENCE RESULTS FOR DISCRETE ABEL MEANS

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We present theorems showing when the discrete Abel mean and the Abel summability method are equivalent for bounded sequences and when two discrete Abel means are equivalent for bounded sequences.

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1. Introduction and notation. The well-known Abel summability method is a sequence-to-function transformation which is defined as follows: for a sequence $s := \{s_n\}$ of complex numbers, define

$$f(x) := (1-x)\sum_{k=0}^{\infty} s_k x^k, \tag{1.1}$$

for all x for which the series converges. If $f(x)$ exists for each $x \in (0,1)$ and $\lim_{x\to 1^-} f(x) = L$, then the sequence s is Abel summable to L. The discrete Abel mean is a sequence-to-sequence transformation given by the summability matrix A_λ whose nkth entry is

$$A_\lambda[n,k] := \frac{1}{\lambda(n)}\left(1-\frac{1}{\lambda(n)}\right)^k, \quad n,k = 0,1,2,3,\dots, \tag{1.2}$$

where $\lambda := \{\lambda(n)\}$ is a strictly increasing sequence of real numbers such that $\lambda(0) \ge 1$ and $\lambda(n) \to \infty$. Then the sequence s is A_λ-summable to L provided that

$$\lim_{n\to\infty} (A_\lambda s)_n = \lim_{n\to\infty} \frac{1}{\lambda(n)}\sum_{k=0}^{\infty} s_k\left(1-\frac{1}{\lambda(n)}\right)^k = L. \tag{1.3}$$

In [1], Armitage and Maddox proved inclusion and Tauberian theorems for the discrete Abel mean. In this paper, we expand upon the work of these authors by examining equivalence properties of the A_λ method for bounded sequences.

For a given sequence s, define a sequence a by $a_0 := s_0$ and $a_n := s_n - s_{n-1}$ for $n \ge 1$. Then, $s_n = \sum_{k=0}^{n} a_k$ and for every n,

$$(A_\lambda s)_n = \frac{1}{\lambda(n)}\sum_{k=0}^{\infty} s_k\left(1-\frac{1}{\lambda(n)}\right)^k = \sum_{k=0}^{\infty} a_k\left(1-\frac{1}{\lambda(n)}\right)^k. \tag{1.4}$$

Also, define the sequence t by

$$t_n := \sum_{k=1}^{n} k a_k. \tag{1.5}$$

A straightforward induction argument yields

$$t_n = \sum_{k=0}^{n} (s_n - s_k). \tag{1.6}$$

If B and C are two summability methods, then C includes B, denoted $B \subset C$, provided that every sequence which is B-summable is also C-summable to the same limit. If $B \subset C$ and $C \subset B$, then B and C are equivalent, denoted $B \sim C$.

2. Equivalence results. For any sequence λ, A_λ is clearly a regular (i.e., limit preserving) method. In [1], Armitage and Maddox proved the following inclusion results for the A_λ method.

Theorem 2.1 (see [1]). *Let $E(\lambda) := \{\lambda(n) : n = 0,1,2,\dots\}$ and $E(\mu) := \{\mu(n) : n = 0,1,2,\dots\}$. Then*

(1) *$A_\lambda \subset A_\mu$ if and only if $E(\mu)\setminus E(\lambda)$ is a finite set;*
(2) *$A_\mu \sim A_\lambda$ if and only if the symmetric difference $E(\lambda) \triangle E(\mu)$ is a finite set.*

Corollary 2.2 (see [1]). *For every λ, A_λ strictly includes the Abel method.*

The main result of this section is that A_λ is equivalent to the Abel method for bounded sequences provided that $\lambda(n+1)/\lambda(n) \to 1$. To show this we need the following two lemmas.

Lemma 2.3 (see [1]). *If $\sum_{k=0}^{\infty} a_k x^k$ converges for all $x \in (0,1)$, then*

$$\sum_{k=1}^{\infty} a_k x^k = \sum_{k=1}^{\infty} t_k \Delta\left(\frac{x^k}{k}\right), \quad 0 < x < 1, \tag{2.1}$$

where $\Delta(x^k/k) = x^k/k - x^{k+1}/(k+1)$.

Lemma 2.4. *If s is a bounded sequence, then $t_n = O(n)$.*

Proof. Let s be a bounded sequence. By (1.6),

$$\begin{aligned} |t_n| = \left|\sum_{k=0}^{n}(s_n - s_k)\right| &= \left|(n+1)s_n - \sum_{k=0}^{n} s_k\right| \\ &\le (n+1)\|s\|_\infty + \sum_{k=0}^{n} |s_k| \\ &\le (n+1)\|s\|_\infty + (n+1)\|s\|_\infty \\ &= O(n). \end{aligned} \tag{2.2}$$

□

Theorem 2.5. *If $\lim_{n\to\infty}(\lambda(n+1)/\lambda(n)) = 1$, then A_λ is equivalent to the Abel method for bounded sequences.*

Proof. By Corollary 2.2, A_λ includes the Abel method. So assume that

$$\lim_{n\to\infty}\left(\frac{\lambda(n+1)}{\lambda(n)}\right)=1, \tag{2.3}$$

let s be a bounded sequence, that is, A_λ-summable to L, and let a be the sequence such that $s_n=\sum_{k=0}^{n}a_k$. Let $x_n:=1-1/\lambda(n)$. Then, for a given $x\in(x_0,1)$, there exists an n such that $x_n<x\le x_{n+1}$. By (1.1) and (1.4),

$$\begin{aligned}\left|f(x)-(A_\lambda s)_n\right| &= \left|(1-x)\sum_{k=0}^{\infty}s_kx^k-\frac{1}{\lambda(n)}\sum_{k=0}^{\infty}s_k\left(1-\frac{1}{\lambda(n)}\right)^k\right|\\ &=\left|\sum_{k=0}^{\infty}a_kx^k-\sum_{k=0}^{\infty}a_k{x_n}^k\right|.\end{aligned} \tag{2.4}$$

By Lemma 2.3, this becomes

$$\begin{aligned}\left|f(x)-(A_\lambda s)_n\right| &= \left|\sum_{k=1}^{\infty}t_k\Delta\left(\frac{x^k}{k}\right)-\sum_{k=1}^{\infty}t_k\Delta\left(\frac{x_n^k}{k}\right)\right|\\ &=\left|\sum_{k=1}^{\infty}t_k\int_{x_n}^{x}t^{k-1}(1-t)\,dt\right|\\ &\le\sum_{k=1}^{\infty}\left|t_k\right|\int_{x_n}^{x_{n+1}}t^{k-1}(1-t)\,dt.\end{aligned} \tag{2.5}$$

By Lemma 2.4, there exists an $M>0$ such that $|t_k|\le kM$. Hence,

$$\begin{aligned}\left|f(x)-(A_\lambda s)_n\right| &\le M\sum_{k=1}^{\infty}k\int_{x_n}^{x_{n+1}}t^{k-1}(1-t)\,dt\\ &=M\int_{x_n}^{x_{n+1}}(1-t)\sum_{k=1}^{\infty}kt^{k-1}\,dt\\ &=M\int_{x_n}^{x_{n+1}}\frac{1}{1-t}\,dt\\ &=-M\left(\log\left(1-x_{n+1}\right)-\log\left(1-x_n\right)\right)\\ &=-M\left(\log\left(\frac{1}{\lambda(n+1)}\right)-\log\left(\frac{1}{\lambda(n)}\right)\right)\\ &=M\log\left(\frac{\lambda(n+1)}{\lambda(n)}\right)\\ &=o(1).\end{aligned} \tag{2.6}$$

Since s is A_λ-summable to L, we see that $\lim_{x\to1^-}f(x)=L$. That is, s is Abel summable to L, and hence, A_λ is equivalent to the Abel method for bounded sequences. □

The next theorem presents an equivalence relationship between the discrete Abel means when λ and μ are asymptotic.

Theorem 2.6. *Let λ and μ be strictly increasing sequences of real numbers such that $\lambda(0) \geq 1$, $\mu(0) \geq 1$, $\lambda(n) \to \infty$, $\mu(n) \to \infty$, and $\lim_{n\to\infty}(\mu(n)/\lambda(n)) = 1$. Then A_λ is equivalent to A_μ for bounded sequences.*

Proof. We proceed as in the proof of Theorem 2.5. Let s be a bounded sequence and let a be the sequence such that $s_n = \sum_{k=0}^{n} a_k$. Let $M(n) := \max\{\lambda(n), \mu(n)\}$, $m(n) := \min\{\lambda(n), \mu(n)\}$, $x_n := 1 - 1/m(n)$, and $y_n := 1 - 1/M(n)$. Then $0 \leq x_n \leq y_n < 1$ and for a given n,

$$\begin{aligned} |(A_\mu s)_n - (A_\lambda s)_n| &= \left| \frac{1}{\mu(n)} \sum_{k=0}^{\infty} s_k \left(1 - \frac{1}{\mu(n)}\right)^k - \frac{1}{\lambda(n)} \sum_{k=0}^{\infty} s_k \left(1 - \frac{1}{\lambda(n)}\right)^k \right| \\ &= \left| \frac{1}{M(n)} \sum_{k=0}^{\infty} s_k \left(1 - \frac{1}{M(n)}\right)^k - \frac{1}{m(n)} \sum_{k=0}^{\infty} s_k \left(1 - \frac{1}{m(n)}\right)^k \right| \\ &= \left| \sum_{k=0}^{\infty} a_k {y_n}^k - \sum_{k=0}^{\infty} a_k {x_n}^k \right|. \end{aligned} \tag{2.7}$$

By Lemma 2.3,

$$\begin{aligned} |(A_\mu s)_n - (A_\lambda s)_n| &= \left| \sum_{k=1}^{\infty} t_k \Delta\left(\frac{y_n^k}{k}\right) - \sum_{k=1}^{\infty} t_k \Delta\left(\frac{x_n^k}{k}\right) \right| \\ &= \left| \sum_{k=1}^{\infty} t_k \int_{x_n}^{y_n} t^{k-1}(1-t)\,dt \right| \\ &\leq \sum_{k=1}^{\infty} |t_k| \int_{x_n}^{y_n} t^{k-1}(1-t)\,dt. \end{aligned} \tag{2.8}$$

By Lemma 2.4, there exists an $M > 0$ such that $|t_k| \leq kM$. Hence,

$$\begin{aligned} |(A_\mu s)_n - (A_\lambda s)_n| &\leq M \sum_{k=1}^{\infty} k \int_{x_n}^{y_n} t^{k-1}(1-t)\,dt \\ &= M \int_{x_n}^{y_n} (1-t) \sum_{k=1}^{\infty} k t^{k-1}\,dt \\ &= M \int_{x_n}^{y_n} \frac{1}{1-t}\,dt \\ &= -M\left(\log(1-y_n) - \log(1-x_n)\right) \\ &= -M\left(\log\left(\frac{1}{M(n)}\right) - \log\left(\frac{1}{m(n)}\right)\right) \\ &= M \log\left(\frac{M(n)}{m(n)}\right) \\ &= o(1), \end{aligned} \tag{2.9}$$

since $\lim_{n\to\infty}(M(n)/m(n)) = \lim_{n\to\infty}(\mu(n)/\lambda(n)) = 1$. Hence, if s is A_λ-summable to L, then

$$0 \leq |(A_\mu s)_n - L| \leq |(A_\mu s)_n - (A_\lambda s)_n| + |(A_\lambda s)_n - L| = o(1) + o(1) = o(1). \tag{2.10}$$

Similarly, if s is A_μ-summable to L, then

$$0 \le \left|(A_\lambda s)_n - L\right| \le \left|(A_\lambda s)_n - (A_\mu s)_n\right| + \left|(A_\mu s)_n - L\right| = o(1) + o(1) = o(1). \tag{2.11}$$

Thus, A_λ and A_μ are equivalent for bounded sequences. □

To see that $\lim_{n\to\infty}(\mu(n)/\lambda(n)) = 1$ is not a necessary condition in Theorem 2.6, simply consider the sequences $\lambda(n) := n^2$ and $\mu(n) := n^3$. Then

$$\lim_{n\to\infty} \frac{\lambda(n+1)}{\lambda(n)} = \lim_{n\to\infty} \frac{\mu(n+1)}{\mu(n)} = 1, \tag{2.12}$$

and hence, by Theorem 2.5, A_λ, A_μ, and the Abel method are all equivalent for bounded sequences. However, λ and μ are not asymptotic.

References

[1] D. H. Armitage and I. J. Maddox, *Discrete Abel means*, Analysis **10** (1990), no. 2-3, 177–186.

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