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APPLICATION OF UNIFORM ASYMPTOTICS TO THE FIFTH PAINLEVÉ TRANSCENDANT

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We apply the uniform asymptotics method to the fifth Painlevé transcendants, find its asymptotics of the form $y = -1+t^{-1/2}A(t)$ as $t \to \infty$ along the positive t-axis, and obtain the corresponding monodromy data.

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1. Introduction. We study the general fifth Painlevé equation

$$\begin{aligned}\frac{d^2y}{dt^2} &= \left(\frac{1}{2y}+\frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} \\ &\quad + \frac{(y-1)^2}{t^2}\left(\alpha y+\frac{\beta}{y}\right)+\frac{\gamma y}{t}+\frac{\delta y(y+1)}{y-1},\end{aligned} \tag{1.1}$$

where α, β, γ, and δ are parameters, and its solution of the form

$$\begin{aligned} y(t) &= -1+4t^{-1/2}A(t), \\ y'(t) &= -2t^{-3/2}A(t)+4t^{-1/2}A'(t) = 4t^{-1/2}A'(t)+O\left(t^{-3/2}\right), \end{aligned} \tag{1.2}$$

with $A(t) = O(1)$ as $t \to \infty$.

The fifth Painlevé equation (1.1) can be obtained as the compatibility condition of the following linear systems of equations (see [2, 3]):

$$Y'_z(z) = \begin{pmatrix} \dfrac{t}{2}+\dfrac{2v+\theta_0}{2z}-\dfrac{w}{z-1} & -\dfrac{u(v+\theta_0)}{z}+\dfrac{uy(2w-\theta_1)}{2(z-1)} \\ \dfrac{v}{uz}-\dfrac{2w+\theta_1}{2uy(z-1)} & -\dfrac{t}{2}-\dfrac{2v+\theta_0}{2z}+\dfrac{w}{z-1} \end{pmatrix} Y(z), \tag{1.3}$$

$$Y'_t(z) = \begin{pmatrix} \dfrac{1}{2} & \dfrac{u}{z}\left[v+\theta_0-y\left(w-\dfrac{\theta_1}{2}\right)\right] \\ \dfrac{1}{uz}\left[v-\dfrac{1}{y}\left(w+\dfrac{\theta_1}{2}\right)\right] & -\dfrac{1}{2} \end{pmatrix} Y(z), \tag{1.4}$$

where

$$w = v+\frac{1}{2}\left(\theta_0+\theta_\infty\right), \tag{1.5}$$

$$\begin{aligned} t\frac{dy}{dt} &= ty-2v(y-1)^2-\frac{1}{2}(y-1)\left[\left(\theta_0-\theta_1+\theta_\infty\right)y-\left(3\theta_0+\theta_1+\theta_\infty\right)\right], \\ t\frac{du}{dt} &= u\left\{-2v-\theta_0+y\left[v+\frac{1}{2}\left(\theta_0-\theta_1+\theta_\infty\right)\right]+\frac{1}{y}\left[v+\frac{1}{2}\left(\theta_0+\theta_1+\theta_\infty\right)\right]\right\}, \end{aligned} \tag{1.6}$$

with

$$\alpha = \frac{1}{2}\left(\frac{\theta_0-\theta_1+\theta_\infty}{2}\right)^2, \quad \beta = -\frac{1}{2}\left(\frac{\theta_0-\theta_1-\theta_\infty}{2}\right)^2, \quad \gamma = 1-\theta_0-\theta_1, \quad \delta = -\frac{1}{2}. \tag{1.7}$$

The canonical solutions of system (1.3) are defined in [2] by

$$\begin{gathered} -\frac{3\pi}{2}+k\pi \le \arg\lambda < -\frac{\pi}{2}+k\pi, \\ Y_k(\lambda) \sim \hat{Y}_\infty(\lambda)e^{(t\lambda/2-\log\lambda)\sigma_3}, \end{gathered} \tag{1.8}$$

where $\sigma_3 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$, and the Stokes multiplier G_1 is defined in [2] by

$$Y_2(\lambda) = Y_1(\lambda)G_1, \tag{1.9}$$

where $G_1 = \left(\begin{smallmatrix}1 & 0\\ s & 1\end{smallmatrix}\right)$ and its entry s is independent of t and y.

2. Reduction of the problem. Generally, if $(d/dz)\left(\begin{smallmatrix}Y_1\\ Y_2\end{smallmatrix}\right) = \left(\begin{smallmatrix}A & B\\ C & -A\end{smallmatrix}\right)\left(\begin{smallmatrix}Y_1\\ Y_2\end{smallmatrix}\right)$, $\phi = B^{-1/2}Y_1$, and $\psi = C^{-1/2}Y_2$, then

$$\begin{aligned} \frac{d^2\phi}{dz^2} &= \left(A^2+BC+A'-B'B^{-1}A+\frac{3}{4}B^{-2}B'^2-\frac{1}{2}B^{-1}B''\right)\phi, \\ \frac{d^2\psi}{dz^2} &= \left(A^2+BC-A'+C'C^{-1}A+\frac{3}{4}C^{-2}C'^2-\frac{1}{2}C^{-1}C''\right)\psi. \end{aligned} \tag{2.1}$$

We first apply the transformation

$$\hat{Y} = \begin{pmatrix}1 & i\\ i & 1\end{pmatrix} u^{-(1/2)\sigma_3}Y \tag{2.2}$$

to system (1.3) to get

$$\frac{d\hat{Y}}{dz} = \frac{1}{2}\begin{pmatrix}L & N\\ M & -L\end{pmatrix}\hat{Y}, \tag{2.3}$$

where

$$\begin{aligned} L &= \frac{i(2v+\theta_0)}{z} - \frac{i(1/y+y)w+i(1/y-y)(\theta_1/2)}{z-1}, \\ M &= \frac{i(2v+\theta_0)-\theta_0}{z} + \frac{(y-1/y-2i)w-(y+1/y)(\theta_1/2)}{z-1}+it, \\ N &= -\frac{i(2v+\theta_0)+\theta_0}{z} + \frac{(y-1/y+2i)w-(y+1/y)(\theta_1/2)}{z-1}-it. \end{aligned} \tag{2.4}$$

Applying (2.1) to (2.3), we get

$$\begin{aligned}\frac{d^2\phi}{dz^2} = \Bigg\{&\left(\frac{t}{2}+\frac{2v+\theta_0}{2z}-\frac{w}{z-1}\right)^2+\left[-\frac{u(v+\theta_0)}{z}+\frac{uy(2w-\theta_1)}{2(z-1)}\right]\left[\frac{v}{uz}-\frac{2w+\theta_1}{2uy(z-1)}\right]\\
&-\frac{i(2v+\theta_0)}{2z^2}+\frac{i\left(\frac{1}{y}+y\right)w+i\left(\frac{1}{y}-y\right)\frac{\theta_1}{2}}{2(z-1)^2}\\
&-\frac{\left[\frac{2iv+(i+1)\theta_0}{z^2}-\frac{\left(y-\frac{1}{y}+2i\right)w-\left(y+\frac{1}{y}\right)\frac{\theta_1}{2}}{(z-1)^2}\right]\frac{L}{2}}{N}\\
&+\frac{3}{4}\frac{\left[\frac{2iv+(i+1)\theta_0}{z^2}-\frac{\left(y-\frac{1}{y}+2i\right)w-\left(y+\frac{1}{y}\right)\frac{\theta_1}{2}}{(z-1)^2}\right]^2}{N^2}\\
&-\frac{-\frac{2iv+(i+1)\theta_0}{z^3}+\left(y-\frac{1}{y}+2i\right)w-\left(y+\frac{1}{y}\right)\frac{\theta_1}{2}}{N}\Bigg\}\phi.\end{aligned} \tag{2.5}$$

Now, using (1.6), the following asymptotics can be obtained:

$$\begin{aligned}v(t)&=-\frac{1}{8}t-\frac{1}{2}t^{1/2}A'(t)+\frac{1}{2}A^2(t)-2A(t)A'(t)-\frac{1}{2}\theta_0-\frac{1}{4}\theta_\infty+O(t^{-1/2}),\\
u(t)&=Ce^{t/2}(1+O(t^{-1/2})).\end{aligned} \tag{2.6}$$

Substituting (2.6) into (2.5), we get the following second-order equation:

$$\begin{aligned}\frac{d^2\phi}{dz^2}&=-t^2\Bigg\{-\frac{(2z-1)^2}{16z(z-1)}-t^{-1}\Bigg[-\frac{(2z-1)\theta_\infty}{4z(z-1)}-\frac{A^2+4A'^2}{4z(z-1)}+\frac{i(2z^2-2z+1)}{8z^2(z-1)^2}\\
&\qquad+\frac{i(2z-1)[(it/4)(2z-1)+t^{1/2}Az^2]}{8z^2(z-1)^2[-it^{1/2}A'-t^{1/2}Az-it(z-1/2)^2]}\Bigg]+O(t^{-3/2})\Bigg\}\phi\\
&=-t^2F(z,t)\phi.\end{aligned} \tag{2.7}$$

Equation (2.7) has two turning points

$$z_j=\frac{1}{2}\pm t^{-1/2}\sqrt{A^2+4A'^2+i}\,(1+o(1)),\quad j=1,2 \tag{2.8}$$

which merge to $1/2$ as $t\to\infty$, and Stokes directions

$$\operatorname{Re}\left(\sqrt{z(z-1)}\right)=0. \tag{2.9}$$

Now, we define a constant α by

$$\frac{1}{2}\pi i\alpha^2=\int_{-\alpha}^{\alpha}(\tau^2-\alpha^2)^{1/2}d\tau=\int_{z_1}^{z_2}F^{1/2}(z,t)dz \tag{2.10}$$

and a new variable ζ by

$$\int_{\alpha}^{\zeta}(\tau^2-\alpha^2)^{1/2}d\tau=\int_{z_1}^{z}F^{1/2}(s,t)ds. \tag{2.11}$$

Using [1, Theorem 1], we have the following theorem.

Theorem 2.1. *Given any solution ϕ of (2.7), there exist constants c_1 and c_2 such that, uniformly for z on the Stokes curve, as $t\to\infty$,*

$$\left(\frac{\zeta^2-\alpha^2}{F(z,t)}\right)^{-1/4}\phi(z,t)=\left\{(c_1+o(1))D_\nu\left(e^{\pi i/4}\sqrt{2t}\zeta\right)+(c_2+o(1))D_{-\nu-1}\left(e^{-\pi i/4}\sqrt{2t}\zeta\right)\right\}, \tag{2.12}$$

where $\nu=-1/2+(1/2)it\alpha^2$ and $D_\nu(z)$, $D_{-\nu-1}(z)$ are solutions of the parabolic cylinder equation.

3. Monodromy data and asymptotics

Theorem 3.1. *For large t and z,*

$$\begin{aligned}\frac{1}{2}\zeta^2&-\frac{A^2+4A'^2+i}{2t}\log\zeta+o(t^{-1})\\&=\frac{iz}{2}-\frac{i\theta_\infty}{2t}\log(4z)+\frac{i}{2t}\log\frac{2it^{1/2}}{2iA'+A}+\frac{1-i}{4}-\frac{\pi\theta_\infty}{4t}\\&\quad-\frac{\pi i\alpha^2}{4}+o(t^{-1})+O(z^{-1}).\end{aligned} \tag{3.1}$$

Proof. Carrying out the integration on the left-hand side of (2.11), we have

$$\frac{1}{2}\zeta^2-\frac{\alpha^2}{2}\log(2\zeta)-\frac{\alpha^2}{4}+\frac{\alpha^2}{2}\log\alpha+O(\alpha^4\zeta^{-2})=\int_{z_1}^{z}F^{1/2}(s,t)ds. \tag{3.2}$$

Because we are going to calculate the higher-order part of the right-hand side, we will simply ignore the lower-order part in $F(z,t)$, and split the right-hand side into two integrals

$$\int_{z_1}^{z}F^{1/2}(s,t)ds=\left(\int_{z_1}^{z^*}+\int_{z^*}^{z}\right)F^{1/2}(x,t)dx=I_1+I_2, \tag{3.3}$$

where $z^* = 1/2 + Tt^{-1/2}$ and T is a large parameter to be specified later. Using the substitution

$$x - \frac{1}{2} = st^{-1/2}, \tag{3.4}$$

I_1 can be evaluated as follows:

$$\begin{aligned} I_1 &= \frac{1}{t}\int_{\sqrt{A^2+4A'^2}}^{T} \left(\sqrt{s^2-(A^2+4A'^2+i)}+o(1)\right)ds \\ &= \frac{T^2}{2t} - \frac{A^2+4A'^2+i}{4t} - \frac{A^2+4A'^2+i}{2t}\log(2T) \\ &\quad + \frac{A^2+4A'^2+i}{4t}\log\left(A^2+4A'^2+i\right)+o(t^{-1}). \end{aligned} \tag{3.5}$$

Using the formula

$$\begin{aligned} &\int \frac{2ax+b}{(ax^2+bx+c)\sqrt{x^2-\frac{1}{4}}}dx \\ &= 2a\frac{\left(\operatorname{arctanh}\dfrac{a+2bx-2\sqrt{(b^2-4ac)}x}{\sqrt{\left(2b^2-2b\sqrt{(b^2-4ac)}-4ac-a^2\right)}\sqrt{(4x^2-1)}}\right)}{\sqrt{\left(2b^2-2b\sqrt{(b^2-4ac)}-4ac-a^2\right)}} \\ &\quad + 2a\frac{\operatorname{arctanh}\dfrac{a+2bx+2\sqrt{(b^2-4ac)}x}{\sqrt{\left(2b^2+2b\sqrt{(b^2-4ac)}-4ac-a^2\right)}\sqrt{(4x^2-1)}}}{\sqrt{\left(2b^2+2b\sqrt{(b^2-4ac)}-4ac-a^2\right)}}, \end{aligned} \tag{3.6}$$

we find the asymptotic expression of I_2

$$\begin{aligned} I_2 &= \int_{z^*}^{z}\Bigg\{-\frac{(2s-1)^2}{16s(s-1)} - t^{-1}\Bigg[-\frac{(2s-1)\theta_\infty}{4s(s-1)} - \frac{A^2+4A'^2}{4s(s-1)} + \frac{i(2s^2-2s+1)}{8s^2(s-1)^2} \\ &\qquad - \frac{i(2s-1)\left[(1/2)it^{1/2}(s-1/2)+As^2\right]}{8z^2(s-1)^2\left[iA'+As+it^{1/2}(s-1/2)^2\right]}\Bigg]\Bigg\}^{1/2} ds \\ &= \int_{z^*}^{z}\frac{i(2s-1)}{4\sqrt{s(s-1)}}\Bigg\{1+\frac{16s(s-1)}{t(2s-1)^2}\Bigg[-\frac{(2s-1)\theta_\infty}{4s(s-1)} - \frac{A^2+4A'^2}{4s(s-1)} + \frac{i(2s^2-2s+1)}{8s^2(s-1)^2} \\ &\qquad + \frac{i(2s-1)\left[(it/4)(2s-1)+t^{1/2}As^2\right]}{8s^2(s-1)^2\left[-it/4-t^{1/2}A'-t^{1/2}As-its(s-1)\right]}\Bigg]\Bigg\}^{1/2} ds \end{aligned}$$

$$
\begin{aligned}
&= \int_{z^*}^{z} \left[\frac{i(2s-1)}{4\sqrt{s(s-1)}} - \frac{i\theta_\infty}{2t\sqrt{s(s-1)}} - \frac{i(A^2+4A'^2+i)}{2t(2s-1)\sqrt{s(s-1)}} \right.\\
&\qquad \left. - \frac{2it^{1/2}(s-1/2)+A}{4t\sqrt{(s-1/2)^2-1/4}[iA'+(1/2)A+A(s-1/2)+it^{1/2}(s-1/2)^2]} \right] dz\\
&\quad + O\left(t^{-2}\int_{z^*}^{z} \frac{|ds|}{|2s-1|^3}\right) + O\left(\frac{1}{z}\right)\\
&= \frac{iz}{2} - \frac{i}{4} - \frac{i\theta_\infty}{2t}\log(4z) - \frac{i\pi(A^2+4A'^2+i)}{4t} + \frac{1}{4} - \frac{T^2}{2t} - \frac{\pi\theta_\infty}{4t}\\
&\quad - \frac{A^2+4A'^2+i}{4t}\log\frac{t}{T^2} + \frac{i}{2t}\log\frac{2it^{1/2}}{2iA'+A+O(t^{-1/4})} + O(z^{-1})\\
&\quad + O(T^{-2}t^{-1}) + O(T^4t^{-2}).
\end{aligned}
\tag{3.7}
$$

Using definition (2.10) and setting $T = -\sqrt{A^2+4A'^2}$ in I_1, we have the following expression for α:

$$
\alpha^2 = -\frac{A^2+4A'^2+i}{t} + o(t^{-1}). \tag{3.8}
$$

Substituting (3.8) into (3.2), setting $T < t^{1/4}$, and combining it with (3.5) and (3.7), the theorem is proved. □

Knowing [4] that

$$
D_\nu(z) \sim \begin{cases} z^\nu e^{-(1/4)z^2}, & \text{if } |\arg z| < \frac{3}{4}\pi,\\ z^\nu e^{-(1/4)z^2} - \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi\nu} z^{-\nu-1} e^{(1/4)z^2}, & \text{if } \arg z = \frac{3}{4}\pi, \end{cases} \tag{3.9}
$$

and $\arg(e^{\pi i/4}\sqrt{2t}\zeta) \sim \pi/4$ when $z \to -\infty$, we can choose, as $z \to -\infty$,

$$
\begin{aligned}
\hat{Y}_2^{(11)}(z) &\sim z^{-\theta_\infty/2} e^{tz/2}\\
&\sim 2^{\theta_\infty} t^{1/4}\sqrt{2iA'+A}\, e^{-(\pi i/4)(1/2+(3it/2)\alpha^2)} e^{(1/4)t+i[t/4-(1/4)(A^2+4A'^2)\log(2t)+\theta_\infty/4]}\\
&\quad \times \zeta^{1/2} D_\nu\left(e^{\pi i/4}\sqrt{2t}\zeta\right).
\end{aligned}
\tag{3.10}
$$

Because $\arg(e^{\pi i/4}\sqrt{2t}\zeta) \sim 3\pi/4$ when $z \to \infty$, we have the following asymptotics for $\hat{Y}_2^{(11)}(z)$ as $z \to \infty$:

$$
\begin{aligned}
\hat{Y}_2^{(11)}(z) &\sim z^{-\theta_\infty/2} e^{tz/2}\\
&\quad - z^{\theta_\infty/2} e^{-tz/2} \frac{4^{\theta_\infty}\sqrt{\pi}(A^2+2iA'^2)}{\Gamma(1/2-(ti/2)\alpha^2)} e^{(\pi i/4)(2ti\alpha^2-3)+(1/2)t+i[t/2-(1/2)(A+4A'^2)\log(2t)]}.
\end{aligned}
\tag{3.11}
$$

By (1.9) and (2.2), the Stokes multiplier can be defined by $\hat{Y}_2 = \hat{Y}_1 G_1$. Therefore, the monodromy data is

$$s = \frac{4^{\theta_\infty}\sqrt{\pi}(A+2iA')}{\Gamma(1/2-(ti/2)\alpha^2)} e^{(\pi i/4)(2ti\alpha^2-5)+i[t/2-(1/2)(A+4A'^2)\log(2t)]}. \tag{3.12}$$

Taking the square of the absolute value of both sides of this equation, we find that $A^2+4A'^2 \sim d^2$ where d is a constant. Solving (3.12) for $A+2iA'$, we have

$$A+2iA' = \left(\pm\sqrt{(A^2+4A'^2)}+O(t^{-1/2})\right)e^{i[t/2-(1/2)(A+4A'^2)\log(2t)+\theta]}. \tag{3.13}$$

Taking the real part and the imaginary part of (3.13), we obtain the following theorem.

Theorem 3.2. *Equation (1.1) has a solution with the following asymptotics:*

$$\begin{aligned} y(t) &\sim -1+4t^{-1/2}(d+O(t^{-1/2}))\cos\left(\frac{t}{2}-\frac{1}{2}d^2\log(2t)+\theta\right), \\ y'(t) &\sim 2t^{-1/2}(d+O(t^{-1/2}))\sin\left(\frac{t}{2}-\frac{1}{2}d^2\log(2t)+\theta\right), \quad \text{as } t \longrightarrow \infty. \end{aligned} \tag{3.14}$$

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