IJMMS 31:10 (2002) 577–601
PII. S016117120210603X
http://ijmms.hindawi.com


# AMENABILITY AND COAMENABILITY OF ALGEBRAIC QUANTUM GROUPS

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Received 5 June 2001 and in revised form 25 January 2002

We define concepts of amenability and coamenability for algebraic quantum groups in the sense of Van Daele (1998). We show that coamenability of an algebraic quantum group always implies amenability of its dual. Various necessary and/or sufficient conditions for amenability or coamenability are obtained. Coamenability is shown to have interesting consequences for the modular theory in the case that the algebraic quantum group is of compact type.

2000 Mathematics Subject Classification: 46L05, 46L65, 16W30, 22D25, 58B32.

**1. Introduction.** The concept of amenability was first introduced into the realm of quantum groups by Voiculescu in [25] for Kac algebras, and studied further by Enock and Schwartz in [7] and by Ruan in [20] (which also deals with Hopf von Neumann algebras). On the other hand, amenability and coamenability of (regular) multiplicative unitaries have been defined by Baaj and Skandalis in [1], and studied in [2, 3, 6]. Amenability and coamenability for Hopf $C^*$-algebras have been considered by Ng in [17, 18].

The present paper is a continuation of the earlier paper [5] of the authors, in which we studied the concept of coamenability for compact quantum groups as defined by Woronowicz [15, 28]. We showed there that the quantum group $\mathrm{SU}_q(2)$ is coamenable and that a coamenable compact quantum group has a faithful Haar integral. Combining these results gives a new proof of Nagy's theorem that the Haar integral of $\mathrm{SU}_q(2)$ is faithful [16].

In this paper, we extend the class of quantum groups for which we study the concept of coamenability, and we also initiate a study of the *dual* notion of amenability. The quantum groups we consider are the algebraic quantum groups introduced by Van Daele in [24]. This class is sufficiently large to include compact quantum groups and discrete quantum groups (for a more precise statement of what is meant here, see Proposition 3.2). An algebraic quantum group admits a dual that is also an algebraic quantum group; moreover, there is a Pontryagin-type duality theorem to the effect that the double dual is canonically isomorphic to the original algebraic quantum group (see Theorem 3.1).

If $\Gamma$ is a discrete group, there are associated to it in a natural way two algebraic quantum groups, namely $(A,\Delta) = (\mathbf{C}(\Gamma),\Delta)$, and its dual $(\hat{A},\hat{\Delta})$, where $\mathbf{C}(\Gamma)$ is the group algebra of $\Gamma$ and $\Delta$ is the comultiplication on $\mathbf{C}(\Gamma)$ given by $\Delta(x) = x\otimes x$, for all $x\in\Gamma$. Then $(A,\Delta)$ is coamenable if and only if $(\hat{A},\hat{\Delta})$ is amenable; moreover, each of these conditions is in turn equivalent to amenability of $\Gamma$ (see Examples 3.4 and 4.6).

We first relate coamenability of an algebraic quantum group $(A,\Delta)$ to a property of the multiplicative unitary $W$ naturally associated to it. This provides a link between our theory and that of Baaj and Skandalis [1], although we make no use of their results. Then we also obtain several other equivalent formulations of coamenability (in Theorem 4.2) that generalize well-known results in the group algebra case.

One basic question in the theory is whether coamenability of an algebraic quantum group $(A,\Delta)$ is always equivalent to amenability of its dual $(\hat{A},\hat{\Delta})$. In fact, we show that coamenability of $(A,\Delta)$ always implies amenability of $(\hat{A},\hat{\Delta})$, but it is conceivable that the converse is not always true. When $(A,\Delta)$ is of compact type (i.e., the algebra $A$ is unital) and its Haar state is tracial, the converse is known to hold, as may be deduced from a deep result of Ruan [20, Theorem 4.5].

In another direction, we prove below that if $M$ is the von Neumann algebra associated to an algebraic quantum group $(A,\Delta)$, then coamenability of $(A,\Delta)$ implies injectivity of $M$ (Theorem 4.8). It may also be deduced from [20, Theorem 4.5] that the converse is true in the compact tracial case. Related to this, we give a direct proof in Theorem 4.9, that if $(A,\Delta)$ is of compact type and its Haar state is tracial, then injectivity of $M$ implies amenability of the dual $(\hat{A},\hat{\Delta})$.

In the final section of this paper, we investigate the modular properties of a coamenable algebraic quantum group of compact type. The unital Haar functional $\varphi$ of such an algebraic quantum group $(A,\Delta)$ is a KMS-state when extended to the analytic extension $A_{\mathrm{r}}$ of $A$. We show, in the case that $(A,\Delta)$ is coamenable, the modular group can be given a description in terms of the multiplicative unitary of $(A,\Delta)$.

We will continue our investigations of the concepts studied in this paper in a subsequent paper [4]. Extensions of these results to the general case of locally compact quantum groups will also be considered.

We now give a brief summary of how the paper is organized. Section 2 establishes some preliminaries on multiplier algebras, $C^*$-algebra and von Neumann algebras. We give a careful treatment of slice maps in connection with multiplier $C^*$-algebras (Theorem 2.1). This is a material that is often assumed in the literature, but does not appear to have been anywhere explicitly formulated and established in terms of known results. Section 3 sets out the background material on algebraic quantum groups that will be needed in the sequel. The most important results of the paper are to be found in Section 4, where most of the results mentioned in the earlier part of this introduction are proved. The final section, Section 5, discusses some consequences of coamenability for the modular properties of an algebraic quantum group of compact type.

We end this introductory section by noting some conventions of terminology that will be used throughout the paper.

Every algebra will be an (not necessarily unital) associative algebra over the complex field $\mathbb{C}$. The identity map on a set $V$ will be denoted by $\iota_V$, or simply by $\iota$, if no ambiguity is involved.

If $V$ and $W$ are linear spaces, $V'$ denotes the linear space of linear functionals on $V$ and $V \otimes W$ denotes the linear space tensor product of $V$ and $W$. The *flip map* $\chi$ from $V \otimes W$ to $W \otimes V$ is the linear map sending $v \otimes w$ onto $w \otimes v$, for all $v \in V$ and $w \in W$. If $V$ and $W$ are Hilbert spaces, $V \otimes W$ denotes their Hilbert space tensor product; we denote by $B(V)$ and $B_0(V)$ the $C^*$-algebras of bounded linear operators and compact operators on $V$, respectively. If $v \in V$ and $w \in W$, $\omega_{v,w}$ denotes the weakly continuous bounded linear functional on $B(V)$ that maps $x$ onto $(x(v), w)$. We set $\omega_v = \omega_{v,v}$. We will often also use the notation $\omega_v$ to denote a restriction to a $C^*$-subalgebra of $B(V)$ (the domain of $\omega_v$ will be determined by the context).

If $V$ and $W$ are algebras, $V \otimes W$ denotes their algebra tensor product. If $V$ and $W$ are $C^*$-algebras, then $V \otimes W$ denotes their $C^*$-tensor product with respect to the minimal (spatial) $C^*$-norm.

**2. $C^*$-algebra preliminaries.** In this section we review some results related to multiplier algebras, especially multiplier algebras of $C^*$-algebras, and also we review elements of the theory of (multiplier) slice maps. These topics are fundamental to $C^*$-algebraic quantum group theory but those parts of their theory that are most relevant are scattered throughout the literature and are often presented only in a very sketchy form. Therefore, for the convenience of the reader and in order to establish notation and terminology, we present a brief, but sufficiently detailed, background account.

First, we introduce the multiplier algebra of a nondegenerate $*$-algebra. This generalizes the usual idea of the multiplier algebra of a $C^*$-algebra. Recall that a nonzero algebra $A$ is *nondegenerate* if, for every $a \in A$, $a = 0$ if $ab = 0$, for all $b \in A$ or $ba = 0$, for all $b \in A$. Obviously, all unital algebras are nondegenerate. If $A$ and $B$ are nondegenerate algebras, so is $A \otimes B$.

Denote by $\mathrm{End}(A)$ the unital algebra of linear maps from a nondegenerate $*$-algebra $A$ to itself. Let $M(A)$ denote the set of elements $x \in \mathrm{End}(A)$ such that there exists an element $y \in \mathrm{End}(A)$ satisfying $x(a)^*b = a^*y(b)$, for all $a, b \in A$. Then $M(A)$ is a unital subalgebra of $\mathrm{End}(A)$. The linear map $y$ associated to a given $x \in M(A)$ is uniquely determined by nondegeneracy and we denote it by $x^*$. The unital algebra $M(A)$ becomes a $*$-algebra when endowed with the involution $x \mapsto x^*$.

When $A$ is a $C^*$-algebra, the closed graph theorem implies that $M(A)$ consists of bounded operators. If we endow $M(A)$ with the operator norm, it becomes a $C^*$-algebra. It is then the usual multiplier algebra in the sense of $C^*$-algebra theory.

Suppose now that $A$ is simply a nondegenerate $*$-algebra and that $A$ is a selfadjoint ideal in a $*$-algebra $B$. For $b \in B$, define $L_b \in M(A)$ by $L_b(a) = ba$, for all $a \in A$. Then the map $L : B \to M(A)$, $b \mapsto L_b$, is a $*$-homomorphism. If $A$ is an *essential* ideal in $B$ in the sense that an element $b$ of $B$ is necessarily equal to zero if $ba = 0$, for all $a \in A$, or $ab = 0$, for all $a \in A$, then $L$ is injective. In particular, $A$ is an essential selfadjoint ideal in itself (by nondegeneracy) and therefore we have an injective $*$-homomorphism $L : A \to M(A)$. We identify the image of $A$ under $L$ with $A$. Then $A$ is an essential selfadjoint ideal of $M(A)$. Obviously, $M(A) = A$ if and only if $A$ is unital.

If $T$ is an arbitrary nonempty set, denote by $\mathbf{F}(T)$ and $\mathbf{K}(T)$ the nondegenerate $*$-algebras of all complex-valued functions on $T$ and of all finitely supported such functions, respectively, where the operations are the pointwise-defined ones. Clearly, $\mathbf{K}(T)$ is an essential ideal in $\mathbf{F}(T)$, and therefore we have a canonical injective $*$-homomorphism from $\mathbf{F}(T)$ to $M(\mathbf{K}(T))$; a moment's reflection shows that this homomorphism is surjective and we therefore can, and do henceforth, use this to identify $M(\mathbf{K}(T))$ with $\mathbf{F}(T)$.

If $A$ and $B$ are nondegenerate $*$-algebras, then it is easily verified that $A \otimes B$ is an essential selfadjoint ideal in $M(A) \otimes M(B)$. Hence, by the preceding remarks, there exists a canonical injective $*$-homomorphism from $M(A) \otimes M(B)$ into $M(A \otimes B)$. We use this to identify $M(A) \otimes M(B)$ as a unital $*$-subalgebra of $M(A \otimes B)$. In general, these algebras are not equal.

If $\pi : A \to B$ is a homomorphism, it is said to be *nondegenerate* if the linear span of the set $\pi(A)B = \{\pi(a)b \mid a \in A,\ b \in B\}$ and the linear span of the set $B\pi(A)$ are both equal to $B$. In this case, there exists a unique extension to a homomorphism $\overline{\pi} : M(A) \to M(B)$ (see [23]), which is determined by $\overline{\pi}(x)(\pi(a)b) = \pi(xa)b$, for every $x \in M(A)$, $a \in A$ and $b \in B$. Note that $\overline{\pi}$ is a $*$-homomorphism whenever $\pi$ is a $*$-homomorphism. We will henceforth use the same symbol $\pi$ to denote the original map and its extension $\overline{\pi}$.

If $\pi : A \to B$ is a $*$-homomorphism between $C^*$-algebras, we will use the term nondegenerate only in its usual sense in $C^*$-theory. Thus, in this case, $\pi$ is nondegenerate if the closed linear span of the set $\pi(A)B = \{\pi(a)b \mid a \in A,\ b \in B\}$ is equal to $B$.

If $\omega$ is a linear functional on $A$ and $x \in M(A)$, we define the linear functionals $x\omega$ and $\omega x$ on $A$ by setting $(x\omega)(a) = \omega(ax)$ and $(\omega x)(a) = \omega(xa)$, for all $a \in A$.

We say $\omega$ is *positive* if $\omega(a^*a) \geq 0$, for all $a \in A$; if $\omega$ is positive, we say it is *faithful* if, for all $a \in A$, $\omega(a^*a) = 0 \Rightarrow a = 0$.

Suppose given $C^*$-algebras $A$ and $B$. If $\omega \in A^*$, the linear map defined by the assignment $a \otimes b \mapsto \omega(a)b$ extends to a norm-bounded linear map $\omega \otimes \iota$ from $A \otimes B$ to $B$. We call $\omega \otimes \iota$ a *slice map*. Obviously, if $\tau \in B^*$, we can define the slice map $\iota \otimes \tau : A \otimes B \to A$ in a similar manner. The next result shows how we can extend these maps to $M(A \otimes B)$. This result is frequently used in the literature, usually without explicit explanation of how $\omega \otimes \iota$ is to be understood or how it is constructed. Similar remarks apply to the corollary.

**Theorem 2.1.** *Let $A$ and $B$ be $C^*$-algebras and let $\omega \in A^*$. Then the slice map $\omega \otimes \iota : A \otimes B \to B$ admits a unique extension to a norm-bounded linear map $\omega \otimes \iota : M(A \otimes B) \to M(B)$ that is strictly continuous on the unit ball of $M(A \otimes B)$.*

*If $x \in M(A \otimes B)$ and $b \in B$, then*

$$b(\omega \otimes \iota)(x) = (\omega \otimes \iota)\big((1 \otimes b)x\big), \qquad (\omega \otimes \iota)(x)b = (\omega \otimes \iota)\big(x(1 \otimes b)\big). \tag{2.1}$$

**Proof.** To prove this, we may assume that $\omega$ is positive, since the set of positive elements of $A^*$ linearly spans $A^*$. In this case, the slice map $\omega \otimes \iota$ is completely positive [26, page 4] and is easily seen to be strict in the sense defined by Lance in [14, page 49]. Hence, by [14, Corollary 5.7], $\omega \otimes \iota$ admits an extension to a norm-bounded linear map $\omega \otimes \iota : M(A \otimes B) \to M(B)$ that is strictly continuous on the unit ball of $M(A \otimes B)$. Uniqueness of $\omega \otimes \iota$ and the properties in the second paragraph of the statement of the theorem now follow immediately from the strict continuity condition. □

Of course, an analogous result holds in the preceding theorem for an element $\tau \in B^*$.

Recall that a norm-bounded linear functional on a $C^*$-algebra $A$ has a unique extension to a norm-bounded strictly continuous functional on $M(A)$. We will usually denote the original functional and its extension by the same symbol. This result, which is pointed out in the appendix of [11], follows easily from a result of Taylor [22] that asserts that, for each $\omega \in A^*$, there exist an element $a \in A$ and a functional $\theta \in A^*$ such that $\omega(b) = \theta(ab)$, for all $b \in A$. We then define $\omega$ on $M(A)$ by setting $\omega(x) = \theta(ax)$, for all $x \in M(A)$. If $\omega \in A^*$ and $\tau \in B^*$, it follows that the norm-bounded linear functional $\omega \otimes \tau : A \otimes B \to \mathbb{C}$ admits a unique extension to a strictly continuous norm-bounded linear functional on $M(A \otimes B)$. In agreement with our standing convention, we will denote the extension by the same symbol $\omega \otimes \tau$. Using these observations, we have the following corollary.

**Corollary 2.2.** *Let $A$ and $B$ be $C^*$-algebras and let $\omega \in A^*$ and $\tau \in B^*$. Let $x \in M(A \otimes B)$. Then*

$$(\omega \otimes \tau)(x) = \omega\big((\iota \otimes \tau)(x)\big) = \tau\big((\omega \otimes \iota)(x)\big). \tag{2.2}$$

We will also need to consider slice maps in the context of von Neumann algebras. Let $M$, $N$ be von Neumann algebras on Hilbert spaces $H$ and $K$, respectively. We denote the von Neumann algebra tensor product by $M \bar{\otimes} N$ (this is the weak closure of the $C^*$-tensor product $M \otimes N$ in $B(H \otimes K)$). We denote by $M_*$ the predual of $M$ consisting of the normal elements of $M^*$. Recall that for any $\omega \in M_*$ and $\tau \in N_*$, we can define a unique functional $\omega \bar{\otimes} \tau \in (M \bar{\otimes} N)_*$ such that $\|\omega \bar{\otimes} \tau\| = \|\omega\| \|\tau\|$ and $(\omega \bar{\otimes} \tau)(x \otimes y) = \omega(x)\tau(y)$, for all $x \in M$ and $y \in N$. If $\omega \in M_*$, we show now how we can define a slice map $\omega \bar{\otimes} \iota$ from $M \bar{\otimes} N$ to $N$. For any $x \in M \bar{\otimes} N$, the assignment $\tau \mapsto (\omega \bar{\otimes} \tau)(x)$ defines a bounded functional on $N_*$. Since $N_*^* = N$, there exists a unique element $z \in N$ such that $(\omega \bar{\otimes} \tau)(x) = \tau(z)$, for all $\tau \in N_*$. We define $(\omega \bar{\otimes} \iota)(x)$ to be equal to $z$. Thus, $(\omega \bar{\otimes} \tau)(x) = \tau((\omega \bar{\otimes} \iota)(x))$, as one would expect of a slice map. Clearly, $\|(\omega \bar{\otimes} \iota)(x)\| \le \|\omega\| \|x\|$. The map $\omega \bar{\otimes} \iota$ which sends $x \in M \bar{\otimes} N$ to $(\omega \bar{\otimes} \iota)(x) \in N$ is obviously linear and norm-bounded. Finally, it is evident that $\omega \bar{\otimes} \iota$ is an extension of the usual slice map $\omega \otimes \iota : M \otimes N \to N$. In a similar fashion, for each $\tau \in N_*$, one can define a slice map $\iota \bar{\otimes} \tau : M \bar{\otimes} N \to M$.

We finish this section on $C^*$-algebra preliminaries by recalling briefly a useful fact concerning completely positive maps that will be needed in the sequel. Suppose that $\pi : A \to B$ is a completely positive unital linear map between unital $C^*$-algebras $A$ and $B$. If $a \in A$ and $\pi(a^*)\pi(a) = \pi(a^*a)$, then $\pi(xa) = \pi(x)\pi(a)$, for all $x \in A$ [26, page 5]. In particular, if $u$ is a unitary in $A$ for which $\pi(u) = 1$, it follows easily that $\pi(u^*xu) = \pi(x)$, for all $x \in A$.

**3. Algebraic quantum groups.** We begin this section by defining a multiplier Hopf $*$-algebra. References for this section are [13, 23, 24].

Let $A$ be a nondegenerate $*$-algebra and let $\Delta$ be a nondegenerate $*$-homomorphism from $A$ into $M(A \otimes A)$. Suppose that the following conditions hold:

(1) $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$,

(2) the linear mappings defined by the assignments $a \otimes b \mapsto \Delta(a)(b \otimes 1)$ and $a \otimes b \mapsto \Delta(a)(1 \otimes b)$ are bijections from $A \otimes A$ onto itself.

Then the pair $(A, \Delta)$ is called a *multiplier Hopf $*$-algebra*.

In condition (1), we are regarding both maps as maps into $M(A \otimes A \otimes A)$, so that their equality makes sense. It follows from condition (2), by taking adjoints, that the maps defined by the assignments $a\otimes b\mapsto(b\otimes 1)\Delta(a)$ and $a\otimes b\mapsto(1\otimes b)\Delta(a)$ are also bijections from $A\otimes A$ onto itself.

Let $(A,\Delta)$ be a multiplier Hopf $*$-algebra and let $\omega$ be a linear functional on $A$ and $a$ an element in $A$. There is a unique element $(\omega\otimes\iota)\Delta(a)$ in $M(A)$ for which

$$\begin{aligned}(\omega\otimes\iota)\big(\Delta(a)\big)b&=(\omega\otimes\iota)\big(\Delta(a)(1\otimes b)\big),\\ b(\omega\otimes\iota)\big(\Delta(a)\big)&=(\omega\otimes\iota)\big((1\otimes b)\Delta(a)\big),\end{aligned}\tag{3.1}$$

for all $b\in A$. The element $(\iota\otimes\omega)\Delta(a)$ in $M(A)$ is determined similarly. Thus, $\omega$ induces linear maps $(\omega\otimes\iota)\Delta$ and $(\iota\otimes\omega)\Delta$ from $A$ to $M(A)$.

There exists a unique nonzero $*$-homomorphism $\varepsilon$ from $A$ to $\mathbf{C}$ such that, for all $a\in A$,

$$(\varepsilon\otimes\iota)\Delta(a)=(\iota\otimes\varepsilon)\Delta(a)=a.\tag{3.2}$$

The map $\varepsilon$ is called the *counit* of $(A,\Delta)$. Also, there exists a unique antimultiplicative linear isomorphism $S$ on $A$ that satisfies the conditions

$$\begin{aligned}m(S\otimes\iota)\big(\Delta(a)(1\otimes b)\big)&=\varepsilon(a)b,\\ m(\iota\otimes S)\big((b\otimes 1)\Delta(a)\big)&=\varepsilon(a)b,\end{aligned}\tag{3.3}$$

for all $a,b\in A$. Here $m:A\otimes A\to A$ denotes the linearization of the multiplication map $A\times A\to A$. The map $S$ is called the *antipode* of $(A,\Delta)$. Note that $S(S(a^*)^*)=a$, for all $a\in A$.

Let $\pi_1$ and $\pi_2$ be nondegenerate homomorphisms from $A$ into some algebras $B$ and $C$, respectively. Clearly, the homomorphism $\pi_1\otimes\pi_2:A\otimes A\to B\otimes C$ is then nondegenerate. Hence, we may form the product $\pi_1\pi_2:A\to M(B\otimes C)$ defined by $\pi_1\pi_2=(\pi_1\otimes\pi_2)\Delta$, where $\pi_1\otimes\pi_2$ is extended to $M(A\otimes A)$ by nondegeneracy. Obviously, $\pi_1\pi_2$ is a nondegenerate homomorphism and it is $*$-preserving whenever both $\pi_1$ and $\pi_2$ are $*$-preserving. This product is easily seen to be associative, with $\varepsilon$ as a unit.

For later use, we remark that the set of nonzero multiplicative linear functionals $\omega$ on $A$ is a group under this product, with inverse operation given by $\omega^{-1}=\omega S$. To see this, note that multiplicativity implies $\omega\otimes\omega=\omega\circ m$. Therefore, if $a,b\in A$, we have $\omega(b)(\omega(\omega S))(a)=(\omega\otimes\omega)((b\otimes 1)(\iota\otimes S)(\Delta(a)))=\omega\circ m(\iota\otimes S)((b\otimes 1)\Delta(a))=\omega(\varepsilon(a)b)=\omega(b)\varepsilon(a)$. Hence, $(\omega(\omega S)(a))=\varepsilon(a)$, for all $a\in A$, as required.

If $\omega\in A'$, we say $\omega$ is *left invariant* if $(\iota\otimes\omega)\Delta(a)=\omega(a)1$, for all $a\in A$. Right invariance is defined similarly. If a nonzero left-invariant linear functional on $A$ exists, it is unique, up to multiplication by a nonzero scalar. Similarly, for a nonzero right-invariant linear functional. If $\varphi$ is a left-invariant functional on $A$, the functional $\psi=\varphi S$ is right invariant.

If $A$ admits a nonzero, left-invariant, positive linear functional $\varphi$, we call $(A,\Delta)$ an *algebraic quantum group* and we call $\varphi$ a *left Haar integral* on $(A,\Delta)$. Faithfulness of $\varphi$ is automatic.

Note that although $\psi=\varphi S$ is right invariant, it may not be positive. On the other hand, it is proved in [13] that a nonzero, right-invariant, positive linear functional on $A$—a *right Haar integral*—necessarily exists. As for a left Haar integral, a right Haar integral is necessarily faithful.

There is a unique bijective homomorphism $\rho : A \to A$ such that $\varphi(ab) = \varphi(b\rho(a))$, for all $a, b \in A$. Moreover, $\rho(\rho(a^*)^*) = a$.

We now discuss duality of algebraic quantum groups. If $(A, \Delta)$ is an algebraic quantum group, denote by $\hat{A}$ the linear subspace of $A'$ consisting of all functionals $\varphi a$, where $a \in A$. Since $\varphi a = \rho(a)\varphi$, we have $\hat{A} = \{a\varphi \mid a \in A\}$. If $\omega_1, \omega_2 \in \hat{A}$, one can define a linear functional $(\omega_1 \otimes \omega_2)\Delta$ on $A$ by setting

$$(\omega_1 \otimes \omega_2)\Delta(a) = (\varphi \otimes \varphi)\big((a_1 \otimes a_2)\Delta(a)\big), \tag{3.4}$$

where $\omega_1 = \varphi a_1$ and $\omega_2 = \varphi a_2$. Using this, the space $\hat{A}$ can be made into a nondegenerate $*$-algebra. The multiplication is given by $\omega_1\omega_2 = (\omega_1 \otimes \omega_2)\Delta$ and the involution is given by setting $\omega^*(a) = \omega(S(a)^*)^-$, for all $a \in A$ and $\omega_1, \omega_2, \omega \in \hat{A}$; it is clear that $\omega_1\omega_2, \omega^* \in A'$ but we can show that, in fact, $\omega_1\omega_2, \omega^* \in \hat{A}$.

One can realize $M(\hat{A})$ as a linear space by identifying it as the linear subspace of $A'$ consisting of all $\omega \in A'$ for which $(\omega \otimes \iota)\Delta(a)$ and $(\iota \otimes \omega)\Delta(a)$ belong to $A$. (It is clear that $\hat{A}$ belongs to this subspace.) In this identification of $M(\hat{A})$, the multiplication and involution are determined by

$$\begin{gathered}(\omega_1\omega_2)(a) = \omega_1\big((\iota \otimes \omega_2)\Delta(a)\big) = \omega_2\big((\omega_1 \otimes \iota)\Delta(a)\big),\\ \omega^*(a) = \omega\big(S(a)^*\big)^-,\end{gathered} \tag{3.5}$$

for all $a \in A$ and $\omega_1, \omega_2, \omega \in M(\hat{A})$.

Note that the counit $\varepsilon$ of $A$ is the unit of the $*$-algebra $M(\hat{A})$.

There is a unique $*$-homomorphism $\hat{\Delta}$ from $\hat{A}$ to $M(\hat{A} \otimes \hat{A})$ such that for all $\omega_1, \omega_2 \in \hat{A}$ and $a, b \in A$,

$$\begin{gathered}\big((\omega_1 \otimes 1)\hat{\Delta}(\omega_2)\big)(a \otimes b) = (\omega_1 \otimes \omega_2)\big(\Delta(a)(1 \otimes b)\big),\\ \big(\hat{\Delta}(\omega_1)(1 \otimes \omega_2)\big)(a \otimes b) = (\omega_1 \otimes \omega_2)\big((a \otimes 1)\Delta(b)\big).\end{gathered} \tag{3.6}$$

Of course, we are here identifying $A' \otimes A'$ as a linear subspace of $(A \otimes A)'$ in the usual way, so that elements of $\hat{A} \otimes \hat{A}$ can be regarded as linear functionals on $A \otimes A$.

The pair $(\hat{A}, \hat{\Delta})$ is an algebraic quantum group, called the *dual* of $(A, \Delta)$. Its counit $\hat{\varepsilon}$ and antipode $\hat{S}$ are given by $\hat{\varepsilon}(a\varphi) = \varphi(a)$ and $\hat{S}(a\varphi) = (a\varphi) \circ S$, for all $a \in A$.

There is an algebraic quantum group version of Pontryagin's duality theorem for locally compact Abelian groups which asserts that $(A, \Delta)$ is canonically isomorphic to the dual of $(\hat{A}, \hat{\Delta})$; that is, $(A, \Delta)$ is isomorphic to its double dual $(\hat{\hat{A}}, \hat{\hat{\Delta}})$. This is stated more precisely in the following result.

**Theorem 3.1.** *Suppose that $(A, \Delta)$ is an algebraic quantum group with double dual $(\hat{\hat{A}}, \hat{\hat{\Delta}})$. Let $\pi : A \to \hat{\hat{A}}$ be the canonical map defined by $\pi(a)(\omega) = \omega(a)$, for all $a \in A$ and $\omega \in \hat{A}$. Then $\pi$ is an isomorphism of the algebraic quantum groups $(A, \Delta)$ and $(\hat{\hat{A}}, \hat{\hat{\Delta}})$; that is, $\pi$ is a $*$-algebra isomorphism of $A$ onto $\hat{\hat{A}}$ for which $(\pi \otimes \pi)\Delta = \hat{\hat{\Delta}}\pi$.*

We will need to consider an object associated to an algebraic quantum group called its analytic extension. (See [13] for full details.) We need first to recall the concept of a GNS pair. Suppose given a positive linear functional $\omega$ on a $*$-algebra $A$. Let $H$ be a Hilbert space, and let $\Lambda : A \to H$ be a linear map with dense range for which

$(\Lambda(a),\Lambda(b)) = \omega(b^*a)$, for all $a,b \in A$. Then we call $(H,\Lambda)$ a *GNS pair* associated to $\omega$. As is well known, such a pair always exists and is essentially unique. For, if $(H',\Lambda')$ is another GNS pair associated to $\omega$, the map $\Lambda(a) \mapsto \Lambda'(a)$ extends to a unitary $U : H \to H'$.

If $\varphi$ is a left Haar integral on an algebraic quantum group $(A,\Delta)$, and $(H,\Lambda)$ is an associated GNS pair, then it can be shown that there is a unique $*$-homomorphism $\pi : A \to B(H)$ such that $\pi(a)\Lambda(b) = \Lambda(ab)$, for all $a,b \in A$. Moreover, $\pi$ is faithful and nondegenerate. We let $A_{\mathrm{r}}$ denote the norm closure of $\pi(A)$ in $B(H)$. Thus, $A_{\mathrm{r}}$ is a nondegenerate $C^*$-subalgebra of $B(H)$. The representation $\pi : A \to B(H)$ is essentially unique, for if $(H',\Lambda')$ is another GNS pair associated to $\varphi$, and $\pi' : A \to B(H')$ is the corresponding representation, then, as we observed above, there exists a unitary $U : H \to H'$ such that $U\Lambda(a) = \Lambda'(a)$, for all $a \in A$, and consequently, $\pi'(a) = U\pi(a)U^*$.

Now observe that there exists a unique nondegenerate $*$-homomorphism $\Delta_{\mathrm{r}} : A_{\mathrm{r}} \to M(A_{\mathrm{r}} \otimes A_{\mathrm{r}})$ such that, for all $a \in A$ and all $x \in A \otimes A$, we have

$$\begin{aligned} \Delta_{\mathrm{r}}(\pi(a))(\pi\otimes\pi)(x) &= (\pi\otimes\pi)(\Delta(a)x), \\ (\pi\otimes\pi)(x)\Delta_{\mathrm{r}}(\pi(a)) &= (\pi\otimes\pi)(x\Delta(a)). \end{aligned} \tag{3.7}$$

We observe also that if $\omega \in A_{\mathrm{r}}^*$ and $x \in A_{\mathrm{r}}$, then the elements $(\omega\otimes\iota)(\Delta_{\mathrm{r}}(x))$ and $(\iota\otimes\omega)(\Delta_{\mathrm{r}}(x))$ both belong to $A_{\mathrm{r}}$.

First, suppose that $\omega$ is given as $\omega = \tau(\pi(a)\cdot)$, for some element $a \in A$ and functional $\tau \in A_{\mathrm{r}}^*$. For $x = \pi(b)$, where $b \in A$, we have

$$(\omega\otimes\iota)(\Delta_{\mathrm{r}}(x)) = \pi((\tau\pi\otimes\iota)((a\otimes 1)\Delta(b))) \in \pi(A). \tag{3.8}$$

By continuity, we get $(\omega\otimes\iota)(\Delta_{\mathrm{r}}(x)) \in A_{\mathrm{r}}$ for all $x \in A_{\mathrm{r}}$. It now follows that $(\omega\otimes\iota)(\Delta_{\mathrm{r}}(x)) \in A_{\mathrm{r}}$, for arbitrary $\omega \in A_{\mathrm{r}}^*$ and $x \in A_{\mathrm{r}}$, by Taylor's result on linear functionals mentioned earlier and a continuity argument. That $(\iota\otimes\omega)(\Delta_{\mathrm{r}}(x)) \in A_{\mathrm{r}}$ is proved in a similar way.

We also recall that the Banach space $A_{\mathrm{r}}^*$ becomes a Banach algebra under the product induced from $\Delta_{\mathrm{r}}$, that is, defined by $\tau\omega = (\tau\otimes\omega)\Delta_{\mathrm{r}}$, for all $\tau,\omega \in A_{\mathrm{r}}^*$.

Since the sets $\Delta(A)(1\otimes A)$ and $\Delta(A)(A\otimes 1)$ span $A\otimes A$, $\Delta_{\mathrm{r}}(A_{\mathrm{r}})(1\otimes A_{\mathrm{r}})$ and $\Delta_{\mathrm{r}}(A_{\mathrm{r}})(A_{\mathrm{r}}\otimes 1)$ have dense linear span in $A_{\mathrm{r}}\otimes A_{\mathrm{r}}$. We get from this the following cancellation laws, for a given functional $\omega \in A_{\mathrm{r}}^*$:

(1) if $\tau\omega = 0$, for all $\tau \in A_{\mathrm{r}}^*$, then $\omega = 0$;
(2) if $\omega\tau = 0$, for all $\tau \in A_{\mathrm{r}}^*$, then $\omega = 0$.

Using these cancellation properties, it follows easily that

$$A_{\mathrm{r}} = \left[(\omega\otimes\iota)(\Delta_{\mathrm{r}}(x)) \mid x \in A_{\mathrm{r}},\ \omega \in A_{\mathrm{r}}^*\right] = \left[(\iota\otimes\omega)(\Delta_{\mathrm{r}}(x)) \mid x \in A_{\mathrm{r}},\ \omega \in A_{\mathrm{r}}^*\right]. \tag{3.9}$$

Note that we use $[\cdot]$ to denote the closed linear span.

We also need to recall that there is a unique unitary operator $W$ on $H\otimes H$ such that

$$W((\Lambda\otimes\Lambda)(\Delta(b)(a\otimes 1))) = \Lambda(a)\otimes\Lambda(b), \tag{3.10}$$

for all $a, b \in A$. This unitary satisfies the equation

$$W_{12}W_{13}W_{23} = W_{23}W_{12}; \tag{3.11}$$

thus, it is a multiplicative unitary, said to be *associated to* $(H, \Lambda)$. Here we have used the leg numbering notation of [1].

We can show that $W \in M(A_r \otimes B_0(H))$, so especially $W \in (A_r \otimes B_0(H))'' = M \bar{\otimes} B(H)$, where $M$ denotes the von Neumann algebra generated by $A_r$. Further, $A_r$ is the norm closure of the linear space $\{(\iota \otimes \omega)(W) \mid \omega \in B_0(H)^*\}$. Also, $\Delta_r(a) = W^*(1 \otimes a)W$, for all $a \in A_r$.

The pair $(A_r, \Delta_r)$ is a reduced locally compact quantum group in the sense of [12, Definition 4.1]; we call it the *analytic extension* of $(A, \Delta)$ associated to $\varphi$.

Consider now the algebraic dual $(\hat{A}, \hat{\Delta})$ of $(A, \Delta)$. A right-invariant linear functional $\hat{\psi}$ is defined on $\hat{A}$ by setting $\hat{\psi}(\hat{a}) = \varepsilon(a)$, for all $a \in A$. Here $\hat{a} = a\varphi$ and $\varepsilon$ is the counit of $(A, \Delta)$. Since the linear map, $A \to \hat{A}$, $a \mapsto \hat{a}$, is a bijection (by faithfulness of $\varphi$), the functional $\hat{\psi}$ is well defined. Now define a linear map $\hat{\Lambda} : \hat{A} \to H$ by setting $\hat{\Lambda}(\hat{a}) = \Lambda(a)$, for all $a \in \mathcal{A}$. Since $\hat{\psi}(\hat{b}^*\hat{a}) = \varphi(b^*a) = (\Lambda(a), \Lambda(b))$, for all $a, b \in A$, it follows that $(H, \hat{\Lambda})$ is a GNS pair associated to $\hat{\psi}$. It can be shown that it is unitarily equivalent to the GNS pair for a left Haar integral $\hat{\varphi}$ of $(\hat{A}, \hat{\Delta})$. Hence, we can use $(H, \hat{\Lambda})$ to define a representation of the analytic extension $(\hat{A}_r, \hat{\Delta}_r)$ of $(\hat{A}, \hat{\Delta})$ on the space $H$. There is a unique $*$-homomorphism $\hat{\pi} : \hat{A} \to B(H)$ such that $\hat{\pi}(a)\hat{\Lambda}(b) = \hat{\Lambda}(ab)$, for all $a, b \in \hat{A}$. Moreover, $\hat{\pi}$ is faithful and nondegenerate. Let $\hat{A}_r$ be the norm closure of $\hat{\pi}(A)$ in $B(H)$, so $\hat{A}_r$ is a nondegenerate $C^*$-subalgebra of $B(H)$. The von Neumann algebra generated by $\hat{A}_r$ will be denoted by $\hat{M}$. We can show that $W \in M(B_0(H) \otimes \hat{A}_r)$ and that $\hat{A}_r$ is the norm closure of the linear space $\{(\omega \otimes \iota)(W) \mid \omega \in B_0(H)^*\}$. Define a linear map $\hat{\Delta}_r : \hat{A}_r \to M(\hat{A}_r \otimes \hat{A}_r)$ by setting $\hat{\Delta}_r(a) = W(a \otimes 1)W^*$, for all $a \in \hat{A}_r$. Then $\hat{\Delta}_r$ is the unique $*$-homomorphism $\hat{\Delta}_r : \hat{A}_r \to M(\hat{A}_r \otimes \hat{A}_r)$ such that, for all $a \in \hat{A}$ and $x \in \hat{A} \otimes \hat{A}$,

$$\begin{aligned} \hat{\Delta}_r(\hat{\pi}(a))(\hat{\pi} \otimes \hat{\pi})(x) &= (\hat{\pi} \otimes \hat{\pi})(\hat{\Delta}(a)x), \\ (\hat{\pi} \otimes \hat{\pi})(x)\hat{\Delta}_r(\hat{\pi}(a)) &= (\hat{\pi} \otimes \hat{\pi})(x\hat{\Delta}(a)). \end{aligned} \tag{3.12}$$

Note that we can show that $W \in M(A_r \otimes \hat{A}_r)$ and $(\Delta_r \otimes \iota)(W) = W_{13}W_{23}$.

An algebraic quantum group $(A, \Delta)$ is of *compact type* if $A$ is unital, and of *discrete type* if there exists a nonzero element $h \in A$ satisfying $ah = ha = \varepsilon(a)h$, for all $a \in A$.

**Proposition 3.2.** *Let $(A, \Delta)$ be an algebraic quantum group. If it is of compact type, its analytical extension $(A_r, \Delta_r)$ is a compact quantum group in the sense of Woronowicz. If it is of discrete type, its analytical extension $(A_r, \Delta_r)$ is a discrete quantum group in the sense of Woronowicz and Van Daele.*

The duality of discrete and compact quantum groups is stated precisely in the following result.

**Proposition 3.3.** *An algebraic quantum group $(A, \Delta)$ is of compact type (resp., of discrete type) if and only if its dual $(\hat{A}, \hat{\Delta})$ is of discrete type (resp., of compact type).*

**Example 3.4.** We finish this section with a brief discussion of the algebraic quantum groups associated to a discrete group $\Gamma$. This illustrates the ideas outlined above and provides the motivation for concepts we introduce later.

First, consider the $*$-algebra $\mathbf{K}(\Gamma)$. This is provided with a comultiplication $\hat{\Delta}$ making it an algebraic quantum group by setting $\hat{\Delta}(f)(x,y) = f(xy)$, for all $f \in \mathbf{K}(\Gamma)$. Here we are identifying $\mathbf{K}(\Gamma) \otimes \mathbf{K}(\Gamma)$ with $\mathbf{K}(\Gamma \times \Gamma)$ by identifying the tensor product $g \otimes h$ of two elements $g,h \in \mathbf{K}(\Gamma)$ with the function in $\mathbf{K}(\Gamma \times \Gamma)$ defined by $(x,y) \mapsto g(x)h(y)$. We then identify $M(\mathbf{K}(\Gamma) \otimes \mathbf{K}(\Gamma))$ with $\mathbf{F}(\Gamma \times \Gamma)$. The reason for using the notation $\hat{\Delta}$ will be apparent shortly.

Now, let $A = \mathbf{C}(\Gamma)$ be the group algebra of $\Gamma$. Recall that, as a linear space, $A$ has canonical linear basis the elements of $\Gamma$ and that the multiplication on $A$ extends that of $\Gamma$ and the adjoint operation is determined by $x^* = x^{-1}$, for all $x \in \Gamma$. We can make $A$ into an algebraic quantum group by providing it with the comultiplication $\Delta : A \to A \otimes A$ determined on the elements of $\Gamma$ by setting $\Delta(x) = x \otimes x$. We will now sketch the proof that the dual $(\hat{A}, \hat{\Delta})$ is the algebraic quantum group $(\mathbf{K}(\Gamma), \hat{\Delta})$.

First, observe that a left Haar integral for $(A, \Delta)$ is given by the unique linear functional $\varphi$ on $A$ for which $\varphi(x) = \delta_{x1}$, for all $x \in \Gamma$, where 1 is the unit of $\Gamma$ and $\delta$ is the usual Kronecker delta function. If $x, y \in \Gamma$, then $(x\varphi)(y) = \varphi(yx) = \delta_{x^{-1},y}$. It follows that the functionals $x\varphi$ $(x \in \Gamma)$ provide a linear basis for $\hat{A}$. Hence, if $e_x$ $(x \in \Gamma)$ is the canonical linear basis for $\mathbf{K}(\Gamma)$ given by $e_x(y) = \delta_{xy}$, we have a linear isomorphism from $\hat{A}$ to $\mathbf{K}(\Gamma)$ given by mapping $x\varphi$ onto $e_{x^{-1}}$. Using this isomorphism as an identification, it is straightforward to check that the multiplications, adjoint operations and comultiplications on $\hat{A}$ and $\mathbf{K}(\Gamma)$ are the same; thus, $(\hat{A}, \hat{\Delta}) = (\mathbf{K}(\Gamma), \hat{\Delta})$, as claimed.

A GNS pair $(H, \Lambda)$ associated to $\varphi$ is given by taking $H = \ell^2(\Gamma)$ and $\Lambda(x) = e_{x^{-1}}$, for all $x \in \Gamma$. We choose $e_{x^{-1}}$ rather than $e_x$ in this formula so as to ensure $\hat{\Lambda}(e_x) = e_x$. We need this to get the correct form for $\hat{\pi}$: it follows easily now that the representation, $\hat{\pi} : \hat{A} \to B(H)$, is the one obtained by left multiplication by elements of $\hat{A}$; it therefore extends from $\hat{A} = \mathbf{K}(\Gamma)$ to a $*$-isomorphism $\hat{\pi}$ from $\ell^\infty(\Gamma)$ onto a von Neumann subalgebra of $B(H)$. It is trivially verified that this von Neumann is the one generated by $\hat{\pi}(\hat{A})$; hence, $\hat{\pi}(\ell^\infty(\Gamma)) = \hat{M}$.

Of course, the representation $\pi : A \to B(H)$ is the one associated to the (right) regular representation of $\Gamma$ on $\ell^2(\Gamma)$. Hence, the analytic extension $A_{\mathrm{r}}$ associated to $(A, \Delta)$ is the reduced group $C^*$-algebra $C^*_r(\Gamma)$ and the corresponding von Neumann algebra $M$ is simply the group von Neumann algebra of $\Gamma$.

We will return to this motivating setup in the sequel.

## 4. Amenability and coamenability.

We will retain all the notation from Section 3. If $(A, \Delta)$ is an algebraic quantum group, recall that we use the symbol $M$ to denote the von Neumann algebra generated by $A_{\mathrm{r}}$. Of course, $A_{\mathrm{r}}$ and $\pi(A)$ are weakly dense in $M$. Since the map $\Delta_{\mathrm{r}}$ is unitarily implemented, it has a unique weakly continuous extension to a unital $*$-homomorphism $\Delta_{\mathrm{r}} : M \to M \bar{\otimes} M$, given explicitly by $\Delta_{\mathrm{r}}(a) = W^*(1 \otimes a)W$, for all $a \in M$. The Banach space $M_*$ may then be regarded as a Banach algebra when equipped with the canonical multiplication induced by $\Delta_{\mathrm{r}}$; thus, the product of two elements $\omega$ and $\sigma$ is given by $\omega\sigma = (\omega \bar{\otimes} \sigma) \circ \Delta_{\mathrm{r}}$.

We use the same symbol $R$ to denote the anti-unitary antipode of $A_{\mathrm{r}}$ and of $M$, and we denote by $\tau$ the scaling group of $(A_{\mathrm{r}},\Delta_{\mathrm{r}})$ (see [12, 13]).

Recall also that we use the symbol $\hat{M}$ to denote the von Neumann algebra generated by $\hat{A}_{\mathrm{r}}$, so that $\hat{A}_{\mathrm{r}}$ and $\hat{\pi}(A)$ are weakly dense in $\hat{M}$. As with $\Delta_{\mathrm{r}}$, since $\hat{\Delta}_{\mathrm{r}}$ is unitarily implemented, it has a unique extension to a weakly continuous unital $*$-homomorphism $\hat{\Delta}_{\mathrm{r}}:\hat{M}\to\hat{M}\bar{\otimes}\hat{M}$, given explicitly by $\hat{\Delta}_{\mathrm{r}}(a)=W(a\otimes 1)W^*$, for all $a\in\hat{M}$.

It should be noted that both $M$ and $\hat{M}$ are in the standard representation. This follows easily from [13] and standard von Neumann algebra theory (see [21], for example). As a consequence the normal states on these algebras are vector states.

In this section, we introduce the concepts of amenability and coamenability for an algebraic quantum group. We begin with the latter concept. Our definition is an adaptation of one we gave in [5] for a compact quantum group. Suppose then $(A,\Delta)$ is an algebraic quantum group and let $(H,\Lambda)$ be a GNS pair associated to a left Haar integral. Since the representation $\pi:A\to B(H)$ is injective, we can use it to endow $A$ with a $C^*$-norm by setting $\|a\|=\|\pi(a)\|$, for $a\in A$. We say that $(A,\Delta)$ is *coamenable* if its counit $\varepsilon$ is norm-bounded with respect to this norm.

It follows readily from the remarks in the introduction [5] that the group algebra of a discrete group $\Gamma$ is coamenable according to this definition if and only if $\Gamma$ is amenable.

On the other hand, coamenability is automatic in the case of a discrete-type algebraic quantum group.

**Proposition 4.1.** *An algebraic quantum group of discrete type is coamenable.*

**Proof.** We may suppose our algebraic quantum group of discrete type is the dual $(\hat{A},\hat{\Delta})$ of an algebraic quantum group $(A,\Delta)$ of compact type. If $\varphi$ is a left Haar integral for $(A,\Delta)$ and $a\in A$, then $\hat{\varepsilon}(\hat{a})=\varphi(a)$, for all $a\in A$, where, as usual, $\hat{a}=a\varphi$. Now $\|\hat{\pi}(\hat{a})\|\geq\|\hat{\Lambda}(\hat{a}\hat{1})\|_2$, since $\|\hat{\Lambda}(\hat{1})\|_2=\|\Lambda(1)\|_2=\varphi(1^*1)^{1/2}=1$. Let $c\in A$ and write $\Delta(c)=\sum_i b_i\otimes c_i$. Then, by definition,

$$\begin{aligned}(\hat{a}\hat{1})(c)&=(a\varphi\odot\varphi)\Delta(c)=\sum_i(a\varphi)(b_i)(\varphi)(c_i)=\sum_i\varphi(b_ia)\varphi(c_i)\\&=\varphi\Big(\sum_i b_i\varphi(c_i)a\Big)=\varphi(\varphi(c)a)=\varphi(a)\varphi(c)=\varphi(a)\hat{1}(c).\end{aligned}\tag{4.1}$$

Hence, $\hat{a}\hat{1}=\varphi(a)\hat{1}$ and therefore $\hat{\Lambda}(\hat{a}\hat{1})=\varphi(a)\hat{\Lambda}(\hat{1})=\varphi(a)\Lambda(1)$. This implies that

$$||\hat{\pi}(\hat{a})||\geq||\hat{\Lambda}(\hat{a}\hat{1})||_2=|\varphi(a)|\,||\Lambda(1)||_2=|\varphi(a)|=|\hat{\varepsilon}(\hat{a})|.\tag{4.2}$$

Since the map, $a\mapsto\hat{a}$, is a bijection from $A$ onto $\hat{A}$, it follows that the counit $\hat{\varepsilon}$ is norm-decreasing and therefore $(\hat{A},\hat{\Delta})$ is coamenable. □

Coamenability may be characterized in several ways. All of the following are essentially well known in the case of the group algebra of a discrete group. Most of these characterizations are related to results obtained by various authors in different settings (see [5, 6, 7, 17, 18, 20, 25]).

**Theorem 4.2.** *Let $(H,\Lambda)$ be a GNS pair, and $W$ the corresponding multiplicative unitary, associated to an algebraic quantum group $(A,\Delta)$. Then the following are equivalent conditions:*

(1) *$(A,\Delta)$ is coamenable;*

(2) *there exists a net $(v_i)$ of unit vectors in $H$ such that*

$$\lim_i \|W(v_i \otimes v) - v_i \otimes v\|_2 = 0, \tag{4.3}$$

*for all $v \in H$;*

(3) *there exists a state $\varepsilon_{\mathrm{r}}$ on $A_{\mathrm{r}}$ such that $(\varepsilon_{\mathrm{r}} \otimes \iota)(W) = 1$;*

(4) *there exists a nonzero multiplicative linear functional on $A_{\mathrm{r}}$;*

(5) *the Banach algebra $A_{\mathrm{r}}^*$ is unital;*

(6) *the Banach algebra $M_*$ has a bounded left approximate unit;*

(7) *$M_*$ has a bounded right approximate unit;*

(8) *$M_*$ has a bounded two-sided approximate unit.*

**Proof.** Note first that condition (3) makes sense—that is, $(\varepsilon_{\mathrm{r}} \otimes \iota)(W)$ is defined—since $W \in M(A_{\mathrm{r}} \otimes B_0(H))$.

Now suppose condition (1) holds and we will show that (3) follows. Since $\varepsilon$ is norm-bounded, there is clearly a unique norm-bounded multiplicative linear functional $\varepsilon_{\mathrm{r}}$ on $A_{\mathrm{r}}$ such that $\varepsilon_{\mathrm{r}} \circ \pi = \varepsilon$. Obviously, $(\varepsilon_{\mathrm{r}} \otimes \iota)\Delta_{\mathrm{r}} = \iota$. Using the argument of [5, Theorem 2.5], we show now that $(\varepsilon_{\mathrm{r}} \otimes \iota)(W) = 1$. We have

$$\begin{aligned} W &= ((\varepsilon_{\mathrm{r}} \otimes \iota)\Delta_{\mathrm{r}} \otimes \iota)(W) = (\varepsilon_{\mathrm{r}} \otimes \iota \otimes \iota)(\Delta_{\mathrm{r}} \otimes \iota)(W) \\ &= (\varepsilon_{\mathrm{r}} \otimes \iota \otimes \iota)(W_{13}W_{23}) = (1 \otimes (\varepsilon_{\mathrm{r}} \otimes \iota)(W))W. \end{aligned} \tag{4.4}$$

As $W$ is invertible it follows that $(\varepsilon_{\mathrm{r}} \otimes \iota)(W) = 1$, hence condition (3) holds.

Conversely, suppose condition (3) holds and we will show that (1) follows. Let $\varepsilon_{\mathrm{r}}$ be as in (3). Since $\varepsilon_{\mathrm{r}}$ is a positive linear functional on $A_{\mathrm{r}}$, the slice map $\varepsilon_{\mathrm{r}} \otimes \iota$ from $M(A_{\mathrm{r}} \otimes B_0(H))$ to $M(B_0(H)) = B(H)$ is completely positive. Note that for $a \in A_{\mathrm{r}}$, $1 \otimes a \in M(A_{\mathrm{r}} \otimes B_0(H))$, so $\Delta_{\mathrm{r}}(a) = W^*(1 \otimes a)W$ belongs to $M(A_{\mathrm{r}} \otimes B_0(H))$. Hence, using (3) and the fact recalled at the end of Section 2, we get, for all $a \in A_{\mathrm{r}}$,

$$\begin{aligned} (\varepsilon_{\mathrm{r}} \otimes \iota)\Delta_{\mathrm{r}}(a) &= (\varepsilon_{\mathrm{r}} \otimes \iota)(W^*(1 \otimes a)W) \\ &= (\varepsilon_{\mathrm{r}} \otimes \iota)(W^*)(\varepsilon_{\mathrm{r}} \otimes \iota)(1 \otimes a)(\varepsilon_{\mathrm{r}} \otimes \iota)(W) \\ &= 1^* a 1 \\ &= a. \end{aligned} \tag{4.5}$$

Now, set $\delta = \varepsilon_{\mathrm{r}} \circ \pi$. Then, for all $a, b \in A$,

$$\begin{aligned} \pi(((\delta \otimes \iota)\Delta(a))b) &= ((\varepsilon_{\mathrm{r}} \otimes \iota)(\pi \otimes \pi)\Delta(a))\pi(b) \\ &= ((\varepsilon_{\mathrm{r}} \otimes \iota)\Delta_{\mathrm{r}}(\pi(a)))\pi(b) \\ &= \pi(a)\pi(b) \\ &= \pi(ab). \end{aligned} \tag{4.6}$$

Hence, $((\delta \otimes \iota)\Delta(a))b = ab$, by injectivity of $\pi$, and therefore $(\delta \otimes \iota)\Delta(a) = a$, by nondegeneracy of $A$. It follows that $\delta = \varepsilon$. Hence, for $a \in A$, $\|\varepsilon(a)\| = \|\varepsilon_{\mathrm{r}}(\pi(a))\| \leq \|\pi(a)\| = \|a\|$. Therefore, $\varepsilon$ is norm-bounded; that is, $(A,\Delta)$ is coamenable.

To prove the implication (2)⇒(3), suppose there exists a net of unit vectors $(v_i)$ such that $\lim_i \|W(v_i \otimes v) - v_i \otimes v\|_2 = 0$, for all $v \in H$. By weak$^*$ compactness of the state space of $B(H)$, the net $(\omega_{v_i})$ of vector states on $B(H)$ has a state $\varepsilon'$ on $B(H)$ as an accumulation point. By going to a subnet of $(v_i)$, if necessary we may suppose that $\varepsilon'(x) = \lim_i (xv_i, v_i)$, for all $x \in B(H)$. Let $\varepsilon_{\mathrm{r}}$ denote the restriction of $\varepsilon'$ to $A_{\mathrm{r}}$. In the following, the slice maps $\varepsilon_{\mathrm{r}} \otimes \iota$ and $\iota \otimes \omega_v$ for $v \in H$ are defined on $M(A_{\mathrm{r}} \otimes B_0(H))$. Using the assumption, we get

$$\begin{aligned}\omega_v(\varepsilon_{\mathrm{r}} \otimes \iota)(W) &= \varepsilon_{\mathrm{r}}(\iota \otimes \omega_v)(W) = \lim_i ((\iota \otimes \omega_v)(W)v_i, v_i) \\ &= \lim_i (W(v_i \otimes v), v_i \otimes v) = \lim_i (v_i \otimes v, v_i \otimes v) = \omega_v(1)\end{aligned} \tag{4.7}$$

for all $v \in H$. It follows that $(\varepsilon_{\mathrm{r}} \otimes \iota)(W) = 1$, hence $\varepsilon_{\mathrm{r}}$ satisfies (3).

Suppose now condition (3) holds, so that there exists a state $\varepsilon_{\mathrm{r}}$ on $A_{\mathrm{r}}$ such that $(\varepsilon_{\mathrm{r}} \otimes \iota)(W) = 1$. Let $\varepsilon'$ be a state extension of $\varepsilon_{\mathrm{r}}$ to $M$. Using the well known fact that the set of normal states on $M$ is weak$^*$ dense in the set of states on $M$ in combination with the fact that every normal state on $M$ is a vector state (as $M$ is in standard form), we deduce that there exists a net $(v_i)$ of unit vectors in $H$ such that $\varepsilon'(x) = \lim_i (xv_i, v_i)$, for all $x \in M$. Then, for all $v \in H$,

$$\begin{aligned}\lim_i (W(v_i \otimes v), v_i \otimes v) &= \lim_i ((\iota \otimes \omega_v)(W)v_i, v_i) = \varepsilon'((\iota \otimes \omega_v)(W)) \\ &= \varepsilon_{\mathrm{r}}((\iota \otimes \omega_v)(W)) = \omega_v((\varepsilon_{\mathrm{r}} \otimes \iota)(W)) \\ &= \omega_v(1) = \lim_i (v_i \otimes v, v_i \otimes v).\end{aligned} \tag{4.8}$$

It is now straightforward to check that $\lim_i \|W(v_i \otimes v) - v_i \otimes v\|_2 = 0$. This proves that condition (2) holds.

If condition (1) holds, then the norm-bounded linear functional $\varepsilon_{\mathrm{r}}$ defined on $A_{\mathrm{r}}$ mentioned in the proof of (1)⇒(3) is obviously nonzero and multiplicative, and it is easily seen to be a unit for $A_{\mathrm{r}}^*$. Hence conditions (4) and (5) follow from (1).

Suppose condition (4) holds and let $\eta$ be nonzero multiplicative linear functional on $A_{\mathrm{r}}$. It is well known that such a functional is norm bounded. Using this and the normboundedness of the anti-unitary antipode $R$, it is then clearly enough to show that $(\eta \otimes \eta R)\Delta_{\mathrm{r}}(\pi(a)) = \varepsilon(a)$, for all $a \in A$, in order to show that condition (1) holds. First, we show that any multiplicative linear functional $\omega$ on $A$ is invariant under $S^2$. As pointed out in Section 3, the set of nonzero multiplicative linear functionals on $A$ has a group structure such that $\omega^{-1} = \omega S$. Therefore we get $\omega S^2 = (\omega^{-1})^{-1} = \omega$, as required. Now set $\omega = \eta\pi$. If $a \in A$, we infer from [13, Proposition 5.5] that $\pi(a)$ is an analytic element of the scaling group $\tau$ on $A_{\mathrm{r}}$ and $\tau_{ni}(\pi(a)) = \pi(S^{-2n}(a))$, for every integer $n$. This implies that

$$\eta\tau_{ni}(\pi(a)) = \omega(S^{-2n}(a)) = \omega(a) = \eta\pi(a). \tag{4.9}$$

By analyticity of the group $\tau$, it follows that $\eta\tau_{i/2} = \eta$ on $\pi(A)$. This may be seen as follows. It is known [9, Proposition 4.23] that $\tau_t$ leaves $\pi(A)$ invariant for each $t \in \mathbf{R}$. As $\pi(A)$ is dense in $A_{\mathrm{r}}$, [10, Corollary 1.22] implies that $\pi(A)$ is a core for $\tau_z$ for any $z \in \mathbf{C}$. Hence $\eta(\tau_{ni}(x)) = \eta(x)$ for all $n \in \mathbf{Z}$ and $x$ in the domain of $\tau_{ni}$. Thus, for an element $x \in A_{\rm r}$ that is analytic of exponential type with respect to $\tau$ in the sense of [8, Definition 4.1], it follows from complex function theory (see, e.g., [27, Lemma 5.5]) that $\eta(\tau_z(x)) = \eta(x)$ for all $z \in \mathbf{C}$. Now, the set of such elements in $A_{\rm r}$ is easily seen to be invariant under each $\tau_t$, $t \in \mathbf{R}$, and dense in $A_{\rm r}$ (by the proof of [8, Lemma 4.2]). Hence [10, Corollary 1.22] says that this set is a core for any $\tau_z$, $z \in \mathbf{C}$. Thus, for any $z \in \mathbf{C}$, we have $\eta(\tau_z(x)) = \eta(x)$ for all $x$ in the domain of $\tau_z$. In particular, choosing $z = i/2$, we get $\eta\tau_{i/2} = \eta$ on $\pi(A)$ as asserted. Using [13, Theorem 5.6], we then get

$$\eta R\pi(a) = \eta\tau_{i/2}\pi(S(a)) = \eta\pi(S(a)), \tag{4.10}$$

for all $a \in A$. This gives

$$\begin{aligned}(\eta\otimes\eta R)\Delta_{\rm r}(\pi(a)) &= (\eta\pi\otimes\eta R\pi)(\Delta(a)) = (\eta\pi\otimes\eta\pi S)(\Delta(a))\\ &= \eta\pi(m(\iota\otimes S)\Delta(a)) = \varepsilon(a),\end{aligned} \tag{4.11}$$

where $m : A\otimes A \to A$ is the linearization of the multiplication of $A$. Thus, $(\eta \otimes \eta R)\Delta_{\rm r}(\pi(a)) = \varepsilon(a)$, for all $a \in A$, as required, and condition (1) holds.

Now suppose condition (5) holds and let $\eta$ be a unit for $A_{\rm r}^*$. For $a \in A$ and $\rho \in A_{\rm r}^*$ we then have

$$\rho(\pi(a)) = (\eta\rho)(\pi(a)) = \rho((\eta\otimes\iota)\Delta_{\rm r}(\pi(a))). \tag{4.12}$$

Since $A_{\rm r}^*$ separates $A_{\rm r}$ we get $(\eta\otimes\iota)\Delta_{\rm r}(\pi(a)) = \pi(a)$, for all $a \in A$. In the same way we also get $(\iota\otimes\eta)\Delta_{\rm r}(\pi(a)) = \pi(a)$, for all $a \in A$. From the uniqueness property of the counit we can then conclude that $\eta\pi = \varepsilon$. Since $\eta$ is bounded by assumption, it follows that $\varepsilon$ is bounded (with respect to the norm on $A$ inherited from the one on $\pi(A)$). Hence condition (1) holds.

Suppose condition (2) holds, so that there exists a net $(v_i)$ of unit vectors in $H$ such that $\lim_i \|W(v_i\otimes v) - v_i\otimes v\|_2 = 0$, for all $v \in H$. Define $\omega_i \in M_*$ to be the restriction of $\omega_{v_i}$ to $M$. Then, for all $v \in H$ and all $x$ in the unit ball of $M$, we have

$$\begin{aligned}|(\omega_i\omega_v)(x) - \omega_v(x)| &= |(\omega_i\otimes\omega_v)(W^*(1\otimes x)W) - \omega_v(x)|\\ &= |(W^*(1\otimes x)W(v_i\otimes v) - (1\otimes x)(v_i\otimes v), v_i\otimes v)|\\ &= |((1\otimes x)W(v_i\otimes v), W(v_i\otimes v)) - ((1\otimes x)(v_i\otimes v), v_i\otimes v)|\\ &= |((1\otimes x)(W(v_i\otimes v) - v_i\otimes v), W(v_i\otimes v))\\ &\quad + ((1\otimes x)(v_i\otimes v), W(v_i\otimes v) - v_i\otimes v)|\\ &\le 2\|v\|\|W(v_i\otimes v) - v_i\otimes v\|_2.\end{aligned} \tag{4.13}$$

Hence,

$$\|\omega_i\omega_v - \omega_v\| \le 2\|v\|\|W(v_i\otimes v) - v_i\otimes v\|_2 \longrightarrow 0. \tag{4.14}$$

Since $M$ is in standard form, every normal state on $M$ is equal to (the restriction of) $\omega_v$, for some unit vector $v \in H$. It follows therefore from our calculations that $(\omega_i)$ is a bounded left approximate unit for $M_*$. Hence, condition (2) implies condition (6).

To see that (6)⇒(7) and (7)⇒(8), we just remark that if $(\omega_i)$ is a bounded left approximate unit for $M_*$, and we set $\omega_i^o = \omega_i R \in M_*$, then, using the fact that

$\chi(R\bar{\otimes}R)\Delta_{\mathrm{r}} = \Delta_{\mathrm{r}}R$, it is straightforward to check that $(\omega_i^o)$ is a bounded right approximate unit for $M_*$. The map $\chi$ is, of course, the flip map on $M\bar{\otimes}M$. It is then easily seen that $(\omega_i + \omega_i^o - \omega_i^o\omega_i)$ is a bounded two-sided approximate unit for $M_*$.

Finally, assume condition (8) holds and that $(\omega_i)$ is bounded two-sided approximate unit for $M_*$. By going to a subnet of $(\omega_i)$, if necessary, we may suppose that $(\omega_i)$ converges in the weak$^*$-topology in $M^*$ to an element $\omega$. We use the same symbol to denote an element in $M^*$ and its restriction to $A_{\mathrm{r}}$. Let $x \in \pi(A)$. Then, for all $\omega' \in M_*$, we have

$$\begin{aligned}\omega'((\omega\otimes\iota)\Delta_{\mathrm{r}}(x)) &= \omega((\iota\otimes\omega')\Delta_{\mathrm{r}}(x)) = \lim_i \omega_i((\iota\otimes\omega')\Delta_{\mathrm{r}}(x)) \\ &= \lim_i \omega_i((\iota\bar{\otimes}\omega')\Delta_{\mathrm{r}}(x)) = \lim_i \omega_i\omega'(x) = \omega'(x).\end{aligned} \tag{4.15}$$

Since the set $M_*$ separates the elements of $M$, it follows that $(\omega\otimes\iota)\Delta_{\mathrm{r}}(x) = x$. Similarly, we get $(\iota\otimes\omega)\Delta_{\mathrm{r}}(x) = x$. Hence, for all $a \in A$, $(\omega\otimes\iota)\Delta_{\mathrm{r}}(\pi(a)) = (\iota\otimes\omega)\Delta_{\mathrm{r}}(\pi(a)) = \pi(a)$. From the uniqueness property of the counit, we conclude that $\omega\pi = \varepsilon$. Since $\omega$ is norm-bounded, it follows that $\varepsilon$ is norm-bounded also. Hence, condition (8) implies (1). This completes the proof of the theorem. □

To each algebraic quantum group $(A,\Delta)$ one may construct a unique universal C$^*$-algebraic quantum group $(A_u,\Delta_u)$ (see [11]). Coamenability of $(A,\Delta)$ may be seen to be equivalent to the fact that the canonical homomorphism from $A_u$ onto $A_{\mathrm{r}}$ is injective (see [5] for the compact case).

In the case that the algebraic quantum group $(A,\Delta)$ is of compact type, we can prove some results that make explicit use of the existence of the unit in $A$. In this case we can choose a unique left-invariant unital linear functional $\varphi$ on $A$. This is the left Haar integral of $A$ and it is also right invariant. We refer to $\varphi$ as the *Haar state* of $A$. If $(H,\Lambda)$ is a GNS pair associated to $\varphi$, then the restriction $\varphi_{\mathrm{r}}$ of the state $\omega_{\Lambda(1)}$ to $A_{\mathrm{r}}$ is a left and right invariant state for the comultiplication $\Delta_{\mathrm{r}} : A_{\mathrm{r}} \to A_{\mathrm{r}}\otimes A_{\mathrm{r}}$.

The following result generalizes [15, Lemma 10.2].

**Lemma 4.3.** *Let $(A,\Delta)$ be an algebraic quantum group of compact type and let $B$ be a $C^*$-algebra admitting a faithful state (e.g., let $B$ be separable). Let $\theta : A \to B$ be a unital $*$-linear map that is either multiplicative or antimultiplicative. Then the linear map, $\theta' : A \to A\otimes B$, $a \mapsto (\iota\otimes\theta)\Delta(a)$, is isometric.*

**Proof.** We will prove the result in the multiplicative case only—the proof in the antimultiplicative case is similar. We identify $A$ as a $*$-subalgebra of $A_{\mathrm{r}}$. Let $\tau$ be a faithful state on $B$. Since the Haar state $\varphi_{\mathrm{r}}$ on $A_{\mathrm{r}}$ is faithful, the state $\varphi_{\mathrm{r}}\otimes\tau$ on $A_{\mathrm{r}}\otimes B$ is faithful. Hence, by two applications of [15, Theorem 10.1] (to $\varphi_{\mathrm{r}}\otimes\tau$ and then to $\varphi_{\mathrm{r}}$), if $a \in A$, we get $\|\theta'(a)\|^2 = \|\theta'(a)^*\theta'(a)\| = \lim[(\varphi_{\mathrm{r}}\otimes\tau)(\theta'(a^*a)^n)]^{1/n} = \lim[(\varphi_{\mathrm{r}}\otimes\tau\theta)\Delta((a^*a)^n)]^{1/n} = \lim[\tau\theta(1)\varphi_{\mathrm{r}}((a^*a)^n)]^{1/n} = \|a^*a\| = \|a\|^2$. Thus, $\theta'$ is isometric, as required. □

**Theorem 4.4.** *Let $(A,\Delta)$ be an algebraic quantum group of compact type and suppose there exists a nonzero continuous $*$-homomorphism $\theta$ from $A$ onto a finite-dimensional $C^*$-algebra $B$. Suppose also that for the antipode $S$ of $A$ we have $\theta S^2 = \theta$. Then $(A,\Delta)$ is coamenable.*

**Proof.** Let $1_B$ denote the unit of $B$ and let the map $\varepsilon_B : A \to B$ be defined by setting $\varepsilon_B(a) = \varepsilon(a)1_B$. Let $m_B : B \otimes B \to B$ and $m : A \otimes A \to A$ be the multiplications. Since $\theta$ is multiplicative, $m_B \circ (\theta \otimes \theta) = \theta \circ m$. Using this and the fact that $m(\iota \otimes S)\Delta(a) = \varepsilon(a)1$, for all $a \in A$, we get

$$\varepsilon_B = m_B \circ (\theta \otimes \iota) \circ [(\iota \otimes \theta S) \circ \Delta]. \tag{4.16}$$

This is a product of three continuous linear maps. The map $m_B$ is continuous, since $B$ is finite-dimensional; $\theta \otimes \iota$ is continuous, since $\theta$ is; finally, the map $(\iota \otimes \theta S) \circ \Delta$ is continuous by the preceding lemma, since $\theta S$ is obviously unital, $*$-linear and antimultiplicative. Hence, $\varepsilon_B$ is continuous. It follows immediately that $\varepsilon$ is continuous and therefore that $(A,\Delta)$ is coamenable. □

The assertion, (4) implies (1) in Theorem 4.2, may be rephrased as saying that $(A,\Delta)$ is coamenable whenever there exists a nonzero continuous complex valued homomorphism on $A$. Note that we established in our proof of this fact that such a homomorphism is always $S^2$-invariant. On the other hand, we don't know whether the $S^2$-invariance assumption in Theorem 4.4 is redundant.

**Corollary 4.5.** *Let $(A,\Delta)$ be an algebraic quantum group of compact type and suppose that its analytic extension $A_{\mathrm{r}}$ is of type I, as a $C^*$-algebra. Suppose also that for the antipode $S$ of $A$ we have $S^2 = \iota_A$. Then $(A,\Delta)$ is coamenable.*

**Proof.** Since $A_{\mathrm{r}}$ is unital, it admits a maximal ideal $I$. The quotient algebra $B = A_{\mathrm{r}}/I$ is a $C^*$-algebra of type I and is both unital and simple. Therefore, it is finite dimensional. Hence, the restriction of the quotient map is a nonzero continuous $*$-homomorphism $\theta$ from $A$ onto a finite-dimensional $C^*$-algebra, namely $B$. From the assumption that $S^2 = \iota_A$, the existence of $\theta$ implies coamenability of $(A,\Delta)$, by Theorem 4.4. □

It would be interesting to know whether Corollary 4.5 remains true in the case that $S^2 \neq \iota_A$; that is, where the Haar state of $(A,\Delta)$ is not tracial [1, 27].

We now define a notion *dual* to coamenability, namely amenability. As pointed out in the introduction, this notion is due to Voiculescu in the Kac algebra case [25] (see also [7, 20]).

Let $(A,\Delta)$ be an algebraic quantum group with von Neumann algebra $M$. A *right-invariant mean* for $(A,\Delta)$ is a state $m$ on $M$ such that

$$m((\iota\bar{\otimes}\omega)\Delta_{\mathrm{r}}(x)) = \omega(1)m(x), \tag{4.17}$$

for all $x \in M$ and $\omega \in M_*$. A *left-invariant mean* is defined analogously, but we will have no need for this concept in this paper. We say that $(A,\Delta)$ is *amenable* if $(A,\Delta)$ admits a right-invariant mean. Using the existence of the anti-unitary antipode $R$ on $(M,\Delta_{\mathrm{r}})$ [12, 13], this is easily seen to be equivalent to requiring that $(A,\Delta)$ admits a left-invariant mean.

**Example 4.6.** The relation to amenability in the classical case of a discrete group $\Gamma$ is worth considering in some little detail. Recall from Example 3.4 that the group algebra $(A,\Delta) = (\mathbf{C}(\Gamma),\Delta)$ is an algebraic quantum group with dual $(\hat{A},\hat{\Delta}) = (\mathbf{K}(\Gamma),\hat{\Delta})$ and that the map, $\hat{\pi} : \hat{A} \to B(\ell^2(\Gamma))$, extends to a $*$-isomorphism from $\ell^\infty(\Gamma)$ onto

$\hat{M}$ and that this $*$-isomorphism is just the usual representation by multiplication operators. For $x \in \Gamma$, let $R_x : \ell^\infty(\Gamma) \to \ell^\infty(\Gamma)$ be the right translation operator given by $R_x f(y) = f(yx)$, for all $y \in \Gamma$. We claim that, for all $f \in \ell^\infty(\Gamma)$,

$$\hat{\pi} R_x(f) = (\iota \bar{\otimes} \omega_{e_x}) \hat{\Delta}_{\mathrm{r}} \hat{\pi}(f), \tag{4.18}$$

where $e_x$ is defined as in Example 3.4. Since both sides of this equation belong to the algebra $\hat{M} = \hat{\pi}(\ell^\infty(\Gamma))$ and this is diagonal with respect to the orthonormal basis $(e_x)_{x\in\Gamma}$, to see the equality holds we need only show, for all $y \in \Gamma$,

$$\omega_{e_y}(\hat{\pi} R_x(f)) = \omega_{e_y}((\iota \bar{\otimes} \omega_{e_x}) \hat{\Delta}_{\mathrm{r}} \hat{\pi}(f)). \tag{4.19}$$

Now the functional $\omega_{e_y} \hat{\pi}$ on $\ell^\infty(\Gamma)$ is easily verified to be just the operation of evaluation at $y$, so we need only to show that

$$f(yx) = (W(\hat{\pi}(f) \otimes 1) W^*(e_y \otimes e_x), (e_y \otimes e_x)). \tag{4.20}$$

However, direct computation shows that $W^*(e_y \otimes e_x) = e_{yx} \otimes e_x$. Hence,

$$\begin{aligned} (W(\hat{\pi}(f) \otimes 1) W^*(e_y \otimes e_x), e_y \otimes e_x) &= ((f \otimes 1)(e_{yx} \otimes e_x), e_{yx} \otimes e_x) \\ &= (f e_{yx}, e_{yx}) \\ &= f(yx). \end{aligned} \tag{4.21}$$

This shows that (4.20) holds and therefore (4.18) also holds.

Suppose now $m$ is a state on $\hat{M}$ and let $\tilde{m}$ be the state on $\ell^\infty(\Gamma)$ determined by $\tilde{m} \circ \hat{\pi} = m$. Using the fact that $\hat{M}$ is diagonal with respect to the orthonormal basis $(e_x)_{x\in\Gamma}$, it is easily checked that $m$ is a right-invariant mean on $\hat{M}$ if and only if $m((\iota \bar{\otimes} \omega_{e_x}) \hat{\Delta}_{\mathrm{r}} \hat{\pi}(f)) = m\hat{\pi}(f)$, for all $x \in \Gamma$ and $f \in \ell^\infty(\Gamma)$. Using (4.18) this translates into the condition $m\hat{\pi}(R_x f) = m\hat{\pi}(f)$; that is, $\tilde{m}(R_x f) = \tilde{m}(f)$. Thus, $(\mathbf{K}(\Gamma), \hat{\Delta}) = (\hat{A}, \hat{\Delta})$ is amenable if and only if the group $\Gamma$ is amenable.

Note that $(\mathbf{C}(\Gamma), \Delta) = (A, \Delta)$ is always amenable, since $\omega_{\Lambda(1)}$ provides a right-invariant mean on $M$, as is easily verified.

Amenability is automatic for an algebraic quantum group $(A, \Delta)$ of compact type. If $\varphi$ is Haar state of $A$ and $(H, \Lambda)$ is a GNS pair for $\varphi$, then the vector state $m = \omega_\Lambda(1)$ is easily seen to define a right-invariant mean on $M$.

**Theorem 4.7.** *If the algebraic quantum group $(A, \Delta)$ is coamenable, then the dual algebraic quantum group $(\hat{A}, \hat{\Delta})$ is amenable.*

**Proof.** Let $(H, \Lambda)$ be a GNS pair, and $W$ the corresponding multiplicative unitary, associated to $(A, \Delta)$. Assume that $(A, \Delta)$ is coamenable and let $\varepsilon_{\mathrm{r}}$ be a state on $A_{\mathrm{r}}$ such that $(\varepsilon_{\mathrm{r}} \otimes \iota)(W) = 1$, according to Theorem 4.2. Let $\varepsilon'$ denote a state extension of $\varepsilon_{\mathrm{r}}$ to $B(H)$. Then the restriction $m$ of $\varepsilon'$ to $\hat{M}$ is a right-invariant mean for $(\hat{A}, \hat{\Delta})$.

Indeed let $v$ be a unit vector in $H$. To see that the restriction is right-invariant, we need only show that

$$m(\iota \bar{\otimes} \omega_v)(\hat{\Delta}_{\mathrm{r}}(x)) = m(\iota \bar{\otimes} \omega_v)(W(x \otimes 1) W^*) = m(x), \quad x \in \hat{M}. \tag{4.22}$$

Observe that $\varepsilon'(\iota\bar{\otimes}\omega_v)$ is a state on $B(H\otimes H) = B(H)\bar{\otimes}B(H)$ and

$$\varepsilon'(\iota\bar{\otimes}\omega_v)(W) = \varepsilon_{\rm r}(\iota\otimes\omega_v)(W) = \omega_v(\varepsilon_{\rm r}\otimes\iota)(W) = \omega_v(1) = 1. \tag{4.23}$$

Hence $\varepsilon'(\iota\bar{\otimes}\omega_v)$ is multiplicative at $W$ and $W^*$ (see using the fact recalled at the end of Section 2). It follows that

$$\begin{aligned} m(\iota\bar{\otimes}\omega_v)(W(x\otimes 1)W^*) &= \varepsilon'(\iota\bar{\otimes}\omega_v)(W(x\otimes 1)W^*) \\ &= \varepsilon'(\iota\bar{\otimes}\omega_v)(x\otimes 1) = \varepsilon'(x)\omega_v(1) = m(x) \end{aligned} \tag{4.24}$$

for all $x\in\hat{M}$, as required. □

Theorem 4.7 raises the question as to whether its converse holds; that is, if the algebraic quantum group $(A,\Delta)$ is such that its dual $(\hat{A},\hat{\Delta})$ is amenable, is $(A,\Delta)$ coamenable? Recall from Example 3.4, that if $(A,\Delta)$ is the algebraic quantum group $(\mathbf{C}(\Gamma),\Delta)$, where $\Gamma$ is a discrete group, its dual is $(\hat{A},\hat{\Delta}) = (\mathbf{K}(\Gamma),\hat{\Delta})$. Hence, $(\hat{A},\hat{\Delta})$ is amenable if and only if $\Gamma$ is amenable, as we saw in Example 4.6. On the other hand, as mentioned before, it is well known that $(A,\Delta)$ is coamenable if and only if $\Gamma$ is amenable. Thus, in this case, the converse of the preceding theorem holds. In the more general case that $(A,\Delta)$ is of compact type and the Haar functional is tracial, the above question may also be answered positively as may be deduced from Ruan's main result [20, Theorem 4.5].

Recall that amenability of C*-algebras and of von Neumann algebras may be described by several equivalent formulations (see [19] for an overview of these and for references to the literature). The most commonly used terminology is nuclearity for $C^*$-algebras and injectivity for von Neumann algebras. The following result may be seen as a quantum group counterpart of the well known result that the group von Neumann algebra of a (locally compact) group is injective whenever the group is amenable.

We will need some easy preliminaries on generalised limits for our next theorem. Let $(I,\le)$ be a directed pair; that is, $I$ is a nonempty set and $\le$ is a reflexive, transitive relation on $I$ that is directed upwards in the sense that for all $i,j\in I$, there exists $k\in I$ such that $i,j\le k$. Then there is a norm-decreasing linear functional $\omega$ on $\ell^\infty(I)$ such that $\omega(x) = \lim_i x_i$, for each convergent sequence $x$ in $\ell^\infty(I)$. We call any such functional $\omega$ a *generalised limit functional* for $(I,\le)$. Obviously, $\omega$ is a state of $\ell^\infty(I)$. It is elementary to see that a generalised limit functional $\omega$ exists. First, define $\omega$ as a norm-decreasing linear functional in the obvious way on the linear subspace of $\ell^\infty(I)$ consisting of elements $x$ that are eventually constant in the sense that there is a scalar $\lambda$ and an element $i\in I$ for which $x(j) = \lambda$, for all $j\ge i$. Then use the Hahn-Banach theorem to extend $\omega$ to a norm-decreasing linear functional on $\ell^\infty(I)$. (This is a weakening of the usual concept of a Banach limit on the positive integers.)

**Theorem 4.8.** *Let $(A,\Delta)$ be a coamenable algebraic quantum group. Then its von Neumann algebra $M$ is injective.*

**Proof.** Of course, we have to show that there is a norm-bounded idempotent operator $E$ on $B(H)$ with range $M$. As usual, $(H,\Lambda)$ is a GNS pair associated to a left Haar integral $\varphi$ on $A$ and $W$ is the corresponding multiplicative unitary. By the proof of

Theorem 4.2(7) and coamenability of $(A,\Delta)$, there is a net $(\omega_i)_{i\in I}$ of normal states on $M$ which is a right approximate unit for $M_*$. As $M$ is in the standard representation, we may, and do, choose a net $(v_i)_{i\in I}$ of unit vectors in $H$ such that, $\omega_i$ is the restriction of $\omega_{v_i}$ to $M$. Now choose a generalised limit functional on $\ell^\infty(I)$ and denote its value at a bounded net $(x_i)_i$ in $\mathbf{C}$ by $\mathrm{Lim}_i\, x_i$.

Let $x\in B(H)$. If $v,w\in H$, then $((\omega_{v,w}\bar{\otimes}\omega_{v_i})(W^*(1\otimes x)W))_i$ is a bounded net. Set $\eta_x(v,w)=\mathrm{Lim}_i\,(\omega_{v,w}\bar{\otimes}\omega_{v_i})(W^*(1\otimes x)W)$. It is easily verified that this defines a sesquilinear form and that $|\eta_x(v,w)|\le\|x\|\|v\|\|w\|$. Hence, there is a unique linear operator $E(x)$ such that $(E(x)v,w)=\eta_x(v,w)$, for all $v,w\in H$. It follows that $\|E(x)\|\le\|x\|$. The map, $E:B(H)\to B(H)$, $x\mapsto E(x)$, is obviously linear.

We will show next that $E(B(H))\subseteq M$. Suppose that $x\in B(H)$ and $E(x)\notin M$. Then there exists a unitary $U\in M'$ such that $U^*E(x)U\neq E(x)$. Hence, there exists an element $v\in H$ such that $(U^*E(x)Uv,v)\neq(E(x)v,v)$. That is, $\omega_{U(v)}(E(x))\neq\omega_v(E(x))$. Set $\tau=\omega_{U(v)}-\omega_v$. Then $\tau(E(x))\neq 0$. However, we clearly have $\tau(M)=0$. Hence, for all $i\in I$, $(\tau\bar{\otimes}\omega_{v_i})(W^*(1\otimes x)W)=0$, since $W^*(1\otimes x)W\in M\bar{\otimes}B(H)$. It follows that $(\omega_{U(v)}\bar{\otimes}\omega_{v_i})(W^*(1\otimes x)W)=(\omega_v\bar{\otimes}\omega_{v_i})(W^*(1\otimes x)W)$. From this we get

$$\mathrm{Lim}_i\,(\omega_{U(v)}\bar{\otimes}\omega_{v_i})(W^*(1\otimes x)W)=\mathrm{Lim}_i\,(\omega_v\bar{\otimes}\omega_{v_i})(W^*(1\otimes x)W); \tag{4.25}$$

that is, $(E(x)U(v),U(v))=(E(x)v,v)$ and therefore $\tau(E(x))=0$. This is a contradiction and to avoid it we must have $E(x)\in M$.

To complete the proof we need only show now that $E(x)=x$, for all $x\in M$. We have, for each element $v$ in $H$,

$$(E(x)v,v)=\mathrm{Lim}_i\,(\omega_v\bar{\otimes}\omega_{v_i})(W^*(1\otimes x)W)=\mathrm{Lim}_i\,(\omega_v\omega_i)(x)=\omega_v(x). \tag{4.26}$$

Consequently, $(E(x)v,v)=(xv,v)$, for all $v\in H$, and therefore $E(x)=x$, as required. □

After some work, one may see that the above result can be deduced from [17, Proposition 3.10], which itself may be seen as a generalization of [6, Proposition 5.6]. Both of these results deals with nuclearity (of crossed products in the context of Hopf $C^*$-algebras and in the context of regular multiplicative unitaries, respectively). Injectivity of crossed products in the Kac algebra case has been considered in [7, Section 3]. On the other hand, the converse to Theorem 4.8 is known to hold in the compact tracial case, as may be obtained from [20, Theorem 4.5]. To illustrate the concepts we include here a direct proof of a related result on *dual* amenability.

**Theorem 4.9.** *Suppose that $(A,\Delta)$ is an algebraic quantum group $(A,\Delta)$ of compact type and that its Haar state $\varphi$ is tracial. If $M$ is injective, then the dual algebraic quantum group $(\hat{A},\hat{\Delta})$ is amenable.*

**Proof.** It is clear that $\omega_w$, where $w=\Lambda(1)$, is a normal state on $M$ and that $\omega_w\circ\pi=\varphi$, so that $\omega_w$ is the *analytic extension* of $\varphi$ to $M$. The hypothesis gives the existence of a unital norm-decreasing positive idempotent linear operator $E:B(H)\to B(H)$ such that $E(B(H))=M$. Define a norm-decreasing linear functional $m$ on $\hat{M}$ by setting $m(x)=\omega_w(E(x))$, for all $x\in\hat{M}$. Since $\omega_w$ and $E$ are unital, so is $m$ and therefore $1=m(1)=\|m\|$. Hence, $m$ is a state. We are going to prove now that $m$ is a right-invariant mean on $\hat{M}$.

To show this, we need only show, for all unit vectors $v \in H$ and all $x \in \hat{M}$,

$$m((\iota\bar{\otimes}\omega_v)(\hat{\Delta}_{\mathrm{r}}(x))) = m(x). \tag{4.27}$$

Using the fact that $\Lambda(A)$ is dense in $H$, we may suppose that $v = \Lambda(a)$; hence, $\varphi(a^*a) = 1$. Since $\hat{A}_r$ acts nondegenerately on $H$ and $\hat{\pi}(\hat{A})$ is dense in $\hat{A}_r$, there is a norm-bounded net $(e_i)$ in $\hat{A}$ such that $(\hat{\pi}(e_i)(v))$ converges to $v$. Hence, the net of vector states $(\omega_{\hat{\pi}(e_i)v})$ on $\hat{M}$ converges to $\omega_v$ in norm, and therefore

$$||(\iota\bar{\otimes}\omega_v)\hat{\Delta}_{\mathrm{r}}(x) - (\iota\bar{\otimes}\omega_{\hat{\pi}(e_i)v})\hat{\Delta}_{\mathrm{r}}(x)|| \leq ||\omega_v - \omega_{\hat{\pi}(e_i)v}||\|x\| \longrightarrow 0. \tag{4.28}$$

Recall that $W \in M(A_{\mathrm{r}} \otimes \hat{A}_{\mathrm{r}})$, from which it follows that $Z_i = (\pi(1)\otimes\hat{\pi}(e_i^*))W$ and $Z_i^* = W^*(\pi(1)\otimes\hat{\pi}(e_i))$ both belong to $A_{\mathrm{r}} \otimes \hat{A}_{\mathrm{r}}$, for all indices $i$. Now write $\Delta a = \sum_j a_j \otimes b_j$, for some elements $a_j, b_j \in A$. Then

$$\begin{aligned} m((\iota\bar{\otimes}\omega_v)\hat{\Delta}_{\mathrm{r}}(x)) &= \lim_i \omega_w \circ E((\iota\bar{\otimes}\omega_{\hat{\pi}(e_i)v})\hat{\Delta}_{\mathrm{r}}(x)) \\ &= \lim_i \omega_w \circ E \circ (\iota\bar{\otimes}\omega_v)((\pi(1)\otimes\hat{\pi}(e_i^*))W(x\otimes 1)W^*(\pi(1)\otimes\hat{\pi}(e_i))) \\ &= \lim_i (\omega_w E \otimes \omega_v)(Z_i(x\otimes 1)Z_i^*) \\ &= \lim_i (\omega_w \otimes \omega_v)(Z_i(E(x)\otimes 1)Z_i^*). \end{aligned} \tag{4.29}$$

For the last equality we are using the fact that $E(cyd) = cE(y)d$, for all $c, d \in M$ and $y \in B(H)$. Hence,

$$\begin{aligned} &m((\iota\bar{\otimes}\omega_v)\hat{\Delta}_{\mathrm{r}}(x)) \\ &\quad= \lim_i (W(E(x)\otimes 1)W^*(\Lambda(1)\otimes\hat{\pi}(e_i)v), \Lambda(1)\otimes\hat{\pi}(e_i)v) \\ &\quad= (W(E(x)\otimes 1)W^*(\Lambda(1)\otimes\Lambda(a)), \Lambda(1)\otimes\Lambda(a)) \\ &\quad= ((E(x)\otimes 1)W^*(\Lambda(1)\otimes\Lambda(a)), W^*(\Lambda(1)\otimes\Lambda(a))) \\ &\quad= (E(x)\otimes 1)(\Lambda\otimes\Lambda)(\Delta(a)), (\Lambda\otimes\Lambda)(\Delta(a)) \\ &\quad= \sum_{jk} ((E(x)\otimes 1)(\pi(a_j)\Lambda(1)\otimes\pi(b_j)\Lambda(1)), \pi(a_k)\Lambda(1)\otimes\pi(b_k)\Lambda(1)) \\ &\quad= \sum_{jk} \omega_w(\pi(a_k)^* E(x)\pi(a_j))\varphi(b_k^* b_j). \end{aligned} \tag{4.30}$$

Thus, to show $m$ is right-invariant, it is sufficient to show that

$$\sum_{jk} \omega_w(\pi(a_k)^* E(x)\pi(a_j))\varphi(b_k^* b_j) = m(x) = \omega_w(E(x)), \tag{4.31}$$

for all $x \in \hat{M}$. Using the fact that $\omega_w$ is weakly continuous, it therefore suffices to show that

$$\sum_{jk} \omega_w(\pi(a_k)^* \pi(b)\pi(a_j))\varphi(b_k^* b_j) = \omega_w(\pi(b)), \tag{4.32}$$

for all $b \in A$; that is,

$$\sum_{jk} \varphi(a_k^* b a_j)\varphi(b_k^* b_j) = \varphi(b). \tag{4.33}$$

However, the left side of this equation is $(\varphi \otimes \varphi)(\Delta(a)^*(b \otimes 1)\Delta(a))$ and this is equal to the right side of the equation, since $\varphi(a^*a) = 1$ and $\varphi$ is tracial. Consequently, we have shown $m((\iota \bar{\otimes} \omega_v)\hat{\Delta}_{\mathrm{r}}(x)) = m(x)$, for all $x \in \hat{M}$. Therefore, $(\hat{A}, \hat{\Delta})$ is amenable. $\square$

Suppose that $(A, \Delta)$ is an algebraic quantum group of compact type and its Haar state is tracial. Since nuclearity of $A_{\mathrm{r}}$ implies injectivity of $M$, it follows from Theorem 4.9 that nuclearity of $A_{\mathrm{r}}$ implies amenability of $(\hat{A}, \hat{\Delta})$.

Note also that the tracial assumption in Theorem 4.9 is equivalent to assuming that $S^2 = \iota$ [1, 27].

**5. Coamenability and modular properties.** In this section, we investigate the modular properties of a coamenable algebraic quantum group $(A, \Delta)$ of compact type. The unital Haar functional $\varphi$ of $(A, \Delta)$ is a KMS-state when extended to $A_{\mathrm{r}}$. In the case that $(A, \Delta)$ is coamenable, we show that the modular group can be given a description in terms of the multiplicative unitary of $(A, \Delta)$.

We will make use of the existence of a certain family $(f_z)_{z \in \mathbf{C}}$ of functionals on $A$; these functionals are quite particular to the compact quantum group case. Observe here that the $C^*$-algebraic compact quantum group $(A_{\mathrm{r}}, \Delta_{\mathrm{r}})$ has clearly $(A, \Delta)$ as its canonical dense Hopf $*$-algebra (see [5, Appendix]), and that we may therefore use Woronowicz's results from [27, 28]. Before stating the properties of these functionals, let us recall that an entire function $g : \mathbf{C} \to \mathbf{C}$ has *exponential growth on the right half plane* if there exist real numbers $M$ and $r$, with $M > 0$, such that $|g(z)| \leq M e^{r \operatorname{Re}(z)}$, for all $z \in \mathbf{C}$ for which $\operatorname{Re}(z) \geq 0$.

There exists a unique family $(f_z)_{z \in \mathbf{C}}$ of unital multiplicative linear functionals on $A$, that we will call the *modular functionals*, satisfying the following conditions:

(1) for each element $a \in A$, the map $z \mapsto f_z(a)$ is a entire function of exponential growth on the right half-plane;
(2) $f_0 = \varepsilon$ and $f_{z+w} = (f_z \otimes f_w)\Delta$, for all $z, w \in \mathbf{C}$;
(3) $f_z(S(a)) = f_{-z}(a)$ and $\overline{f_z(a^*)} = f_{-\bar{z}}(a)$, for all $a \in A$ and $z \in \mathbf{C}$;
(4) $S^2(a) = f_{-1} * a * f_1$, for all $a \in A$;
(5) The unique homomorphism $\rho : A \to A$ for which $\varphi(ab) = \varphi(b\rho(a))$, for all $a, b \in A$, is given by $\rho(a) = f_1 * a * f_1$.

Here $\tau * a * \tau' = (\tau' \otimes \iota \otimes \tau)(\Delta \otimes \iota)\Delta(a)$, for all $a \in A$ and all linear functionals $\tau, \tau'$ on $A$. Note in particular that it follows from condition (3) that $f_{it}$ is a unital $*$-homomorphism from $A$ to $\mathbf{C}$, for all real numbers $t$. Hence, for each element $t \in \mathbf{R}$, we can define a unital $*$-automorphism $\sigma_t$ of $\pi(A)$ by setting

$$\sigma_t(\pi(a)) = \pi(f_{-it} * a * f_{-it}), \tag{5.1}$$

for all $a \in A$. Since $\varphi_{\mathrm{r}}\sigma_t = \varphi_{\mathrm{r}}$ and $\|a\|^2 = \lim_n \varphi_{\mathrm{r}}((a^*a)^n)^{1/n}$, for all $a \in A_{\mathrm{r}}$, the $*$-homomorphisms $(\sigma_t)_{t \in \mathbf{R}}$ are isometric and can therefore be extended to a one-parameter group of $*$-automorphisms of $A_{\mathrm{r}}$, also denoted by $(\sigma_t)_{t \in \mathbf{R}}$. Moreover, we have a unitary implementation. This is easily seen by noting first that the linear map from $\Lambda(A)$ to itself defined by sending $\Lambda(a)$ onto $\Lambda(f_{-it} * a * f_{-it})$ is an isometric bijection. Its extension to $H$ is a unitary $V_t$. We easily verify then that $\sigma_t(x) = V_t x V_t^*$, for all $x \in \pi(A)$ and therefore for all $x \in A_r$.

However, if we now suppose that $(A,\Delta)$ is coamenable, more can be said about the unitaries $\{V_t\}_{t\in\mathbf{R}}$. In this case it follows from [5, Corollary 3.7] that all positive functionals on $A$ are bounded with respect to the norm on $A$ (the one induced from the norm on $A_r$). Hence, there is an extension $\tilde{f}_{it}$ of $f_{it}$ to $A_r$; more precisely, there is a unique $*$-homomorphism $\tilde{f}_{it}$ on $A_r$ such that $\tilde{f}_{it}\pi(a) = f_{it}(a)$, for all $a \in A$. The map $\tilde{f}_{it} \otimes \iota : A_r \otimes B_0(H) \to B_0(H)$ is clearly a nondegenerate $*$-homomorphism and therefore has a unique extension to a $*$-homomorphism $\tilde{f}_{it} \otimes \iota : M(A_r \otimes B_0(H)) \to M(B_0(H)) = B(H)$. Since $W \in M(A_r \otimes B_0(H))$, we may define an element $F_t \in B(H)$ by setting $F_t = (\tilde{f}_{it} \otimes \iota)(W)$. Now observe that $\tilde{f}_{i(s+t)} = (\tilde{f}_{is} \otimes \tilde{f}_{it})\Delta_r$, for all $s,t \in \mathbf{R}$. This result follows by extending the formula $f_{is+it} = (f_{is} \otimes f_{it})\Delta$ from $A$ to $A_r$. Using the equation $(\Delta_r \otimes \iota)W = W_{13}W_{23}$ we get

$$\begin{aligned} F_{s+t} &= (\tilde{f}_{i(s+t)} \otimes \iota)(W) = ((\tilde{f}_{is} \otimes \tilde{f}_{it})\Delta_r \otimes \iota)(W) = (\tilde{f}_{is} \otimes \tilde{f}_{it} \otimes \iota)(\Delta_r \otimes \iota)(W) \\ &= (\tilde{f}_{is} \otimes \tilde{f}_{it} \otimes \iota)(W_{13}W_{23}) = (\tilde{f}_{is} \otimes \iota)(W)(\tilde{f}_{it} \otimes \iota)(W) = F_s F_t. \end{aligned} \tag{5.2}$$

Since we have seen that $F_0 = (\tilde{\varepsilon} \otimes \iota)(W) = 1$, where the functional $\tilde{\varepsilon} : A_r \to \mathbf{C}$ is the extension of the counit of $(A,\Delta)$, it is clear that the operators $F_t$ are invertible in $B(H)$. In fact, $F_t$ is a unitary, since

$$\begin{aligned} F_t^* F_t &= ((\tilde{f}_{it} \otimes \iota)(W))^* (\tilde{f}_{it} \otimes \iota)(W) = (\tilde{f}_{it} \otimes \iota)(W^*)(\tilde{f}_{it} \otimes \iota)(W) \\ &= (\tilde{f}_{it} \otimes \iota)(W^*W) = (\tilde{f}_{it} \otimes \iota)(1) = 1. \end{aligned} \tag{5.3}$$

Hence, the map, $t \mapsto F_t$, is a unitary representation of $\mathbf{R}$ on $H$. We will use Stone's theorem to produce a (densely-defined unbounded) selfadjoint generator $F$. To this end we must show that the representation, $t \mapsto F_t$, is strongly continuous. In fact, we will prove that it is weakly continuous, which amounts to the same thing within the set of unitaries. For $a,b \in A$ and $t \in \mathbf{R}$, we have

$$\begin{aligned} (F_t\Lambda(a),\Lambda(b)) &= (\Lambda(a),F_t^*\Lambda(b)) = (\Lambda(a),(\tilde{f}_{it} \otimes \iota)(W^*)\Lambda(b)) \\ &= \tilde{f}_{it}((\iota \otimes \omega_{\Lambda(a),\Lambda(b)})(W^*)) = \tilde{f}_{it}\pi((\iota \otimes \varphi)((1 \otimes b^*)\Delta(a))) \\ &= f_{it}((\iota \otimes \varphi)((1 \otimes b^*)\Delta(a))). \end{aligned} \tag{5.4}$$

Hence, setting $c = (\iota \otimes \varphi)((1 \otimes b^*)\Delta(a))$, we get $(F_t\Lambda(a),\Lambda(b)) = f_{it}(c)$. The important point here is that $c$ is independent of $t$. Since $t \mapsto f_{it}(c)$ is continuous, the function $t \mapsto (F_t\Lambda(a),\Lambda(b))$ is continuous. That $t \mapsto (F_t u,v)$ is continuous, for all $u,v \in H$, follows now by density of $\Lambda(A)$ in $H$.

That $(\tilde{f}_{it}\otimes\iota)\Delta_{\mathrm{r}}(x) = F_t^* x F_t$, for all $x\in A_{\mathrm{r}}$, follows from the fact that $\Delta_{\mathrm{r}}(x) = W^*(1\otimes x)W$ and the extension $\tilde{f}_{it}\otimes\iota : M(A_{\mathrm{r}}\otimes B_0(H))\to B(H)$ is a $*$-homomorphism and $1\otimes x\in A_{\mathrm{r}}\otimes M(B_0(H))\subseteq M(A_{\mathrm{r}}\otimes B_0(H))$. We showed in (5.4) that $(F_t\Lambda(a),\Lambda(b)) = f_{it}((\iota\otimes\varphi)((1\otimes b^*)\Delta(a)))$. Taking $a=1$, it follows that $(F_t\Lambda(1),\Lambda(b)) = f_{it}((\iota\otimes\varphi)(1\otimes b^*)) = \varphi(b^*) = (\Lambda(1),\Lambda(b))$, for all $b\in A$. Hence, $F_t\Lambda(1)=\Lambda(1)$. Consequently, $F_t\Lambda(a) = F_t\pi(a)\Lambda(1) = F_t\pi(a)F_t^*F_t\Lambda(1) = (\tilde{f}_{-it}\otimes\iota)\Delta_{\mathrm{r}}(\pi(a))\Lambda(1) = \pi(a * f_{-it})\Lambda(1) = \Lambda(a*f_{-it})$.

Let $\chi: A\otimes A\to A\otimes A$ be the flip map. Set $\Delta_{\mathrm{op}} = \chi\Delta$. Then we easily verify that $(A,\Delta_{\mathrm{op}})$ is an algebraic quantum group of compact type, called the *opposite* of $(A,\Delta)$. The Haar state of $(A,\Delta_{\mathrm{op}})$ is the Haar state $\varphi$ of $(A,\Delta)$. Likewise, these quantum groups have the same counit. However, the coinverse of $(A,\Delta_{\mathrm{op}})$ is $S^{-1}$, where $S$ is the coinverse of $(A,\Delta)$. It is not hard to check that the family of modular functionals associated to $(A,\Delta_{\mathrm{op}})$ is the same as that associated to $(A,\Delta)$.

Now let $W_{\mathrm{op}}$ denote the multiplicative unitary associated to $(A,\Delta_{\mathrm{op}})$ in the GNS construction $(H,\pi,\Lambda)$ for the Haar state $\varphi$. If the comultiplication on $A_{\mathrm{r}}$ associated to $\Delta_{\mathrm{op}}$ on $A$ is denoted by $\Delta_{\mathrm{op,r}}$, then it is easily verified that $\Delta_{\mathrm{op,r}}(x) = \Sigma\Delta_{\mathrm{r}}(x)\Sigma$, for all $x\in A_{\mathrm{r}}$, where $\Sigma$ is the flip operator on $H\otimes H$.

As before, define a strongly-continuous one-parameter group $(E_t)_t$ of unitaries by setting $E_t = (\tilde{f}_{it}\otimes\iota)(W_{\mathrm{op}})$, for all $t\in\mathbf{R}$. Then $(\tilde{f}_{it}\otimes\iota)\Delta_{\mathrm{op,r}}(x) = E_t^* x E_t$, for all $x\in A_{\mathrm{r}}$ and $E_t\Lambda(a) = \Lambda(f_{-it}*a)$, for all $a\in A$. Also, there is a selfadjoint operator $E$ in $H$ such that $E_t = \exp(itE)$, for all $t$. Clearly, then $(\iota\otimes\tilde{f}_{it})\Delta_{\mathrm{r}}(x) = (\tilde{f}_{it}\otimes\iota)\Delta_{\mathrm{op,r}}(x) = E_t^* x E_t$.

Let $a\in A$. Then $\sigma_t(\pi(a)) = \pi(f_{-it}*a*f_{-it}) = (\tilde{f}_{-it}\otimes\iota)\Delta_{\mathrm{r}}\pi(f_{-it}*a) = F_t\pi(f_{-it}*a)F_t^* = F_t(\iota\otimes\tilde{f}_{-it})\Delta_{\mathrm{r}}\pi(a)F_t^* = F_tE_t\pi(a)E_t^*F_t^*$. It follows from density of $\pi(A)$ in $A_{\mathrm{r}}$ that $\sigma_t(x) = F_tE_t x E_t^* F_t^*$, for all $x\in A_{\mathrm{r}}$.

We defined the unitary $V_t$ on $H$ by setting $V_t\Lambda(a) = \Lambda(f_{-it}*a*f_{-it})$, for all $a\in A$. Clearly, then, $V_t = F_tE_t = E_tF_t$.

Recall that if $a\in A$, then $z\mapsto f_z(a)$ is an analytic function on $\mathbf{C}$. Set $\sigma_z(\pi(a)) = \pi(f_{-iz}*a*f_{-iz})$. Then it is easily verified that the map $z\mapsto\sigma_z\pi(a)$ is analytic, in the sense that if $\tau\in A_{\mathrm{r}}^*$, then $z\mapsto\tau\sigma_z\pi(a)$ is analytic. Hence, the map $z\mapsto\sigma_z\pi(a)$ provides an analytic extension to the plane of the function $t\mapsto\sigma_t\pi(a)$ on $\mathbf{R}$. This shows that $\pi(A)$ is contained in the set of analytic elements for the $C^*$-dynamical system $(A_{\mathrm{r}},\sigma)$. Moreover, it follows from condition (5) in the list of properties associated to the family $(f_z)_z$ that we stated at the beginning of this section that $\varphi_{\mathrm{r}}(\pi(a)\pi(b)) = \varphi_{\mathrm{r}}(\pi(b)\sigma_i\pi(a))$. Now, as $\pi(A)$ is dense in $A_{\mathrm{r}}$ and invariant under each $\sigma_t$ for $t\in\mathbf{R}$, $\pi(A)$ is a core for $\sigma_i$ (using [10, Corollary 1.22]). Hence, it follows that the state $\varphi_{\mathrm{r}}$ satisfies the KMS condition for the automorphism group $(\sigma_t)_t$ at inverse temperature $\beta=1$.

We summarize some of the previous discussion in the following result.

**Theorem 5.1.** *For all $t\in\mathbf{R}$, set $F_t = (\tilde{f}_{it}\otimes\iota)(W)$ and $E_t = (\tilde{f}_{it}\otimes\iota)(W_{\mathrm{op}})$. Then $F_t\Lambda(a) = \Lambda(a*f_{-it})$ and $E_t\Lambda(a) = \Lambda(f_{-it}*a)$, for all $a\in A$, from which it follows that $F_tE_t = E_tF_t$, for all $t\in\mathbf{R}$. Moreover, there exists selfadjoint operators $F$ and $E$ in $H$ such that $F_t = \exp(itF)$ and $E_t = \exp(itE)$, for all $t\in\mathbf{R}$.*

*For all $t\in\mathbf{R}$, set $V_t = F_tE_t$, so that $V_t = \exp(it(E+F))$. Then $\sigma_t(x) = V_t x V_t^*$, for all $x\in A_{\mathrm{r}}$. Here $\sigma_t$ is the unique automorphism of $A_{\mathrm{r}}$ for which $\sigma_t(\pi(a)) =$*

*$\pi(f_{-it} * a * f_{-it})$, for all $a \in A$. The Haar state $\varphi_{\mathrm{r}}$ on $A_{\mathrm{r}}$ satisfies the KMS condition for $(\sigma_t)_t$ at inverse temperature $\beta = 1$.*

**Acknowledgments.** The authors would like to thank the referee for his careful reading of the first version of this paper and for his many helpful comments. The first and the third authors were partially supported by the Norwegian Research Council.

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