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ARGUMENT ESTIMATES OF CERTAIN MULTIVALENT FUNCTIONS INVOLVING A LINEAR OPERATOR

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The purpose of this paper is to derive some argument properties of certain multivalent functions in the open unit disk involving a linear operator. We also investigate their integral preserving property in a sector.

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1. Introduction. Let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1,2,\dots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f \in \mathcal{A}_p$ is said to be p-valently starlike of order α in $\mathcal{U}$, if it satisfies

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \quad (0 \le \alpha < p;\ z \in \mathcal{U}). \tag{1.2}$$

We denote this class by $\mathcal{S}_p^*(\alpha)$. A function $f \in \mathcal{A}_p$ is said to be p-valently convex of order α in $\mathcal{U}$, if it satisfies

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \quad (0 \le \alpha < p;\ z \in \mathcal{U}). \tag{1.3}$$

The class of p-valently convex functions of order α is denoted by $\mathcal{K}_p(\alpha)$. It follows from (1.2) and (1.3) that

$$f \in \mathcal{K}_p(\alpha) \Longleftrightarrow \frac{zf'}{p} \in \mathcal{S}_p(\alpha). \tag{1.4}$$

Further, a function $f \in \mathcal{A}_p$ is said to be p-valently close-to-convex of order β and type α, if there exists a function $g \in \mathcal{S}_p^*(\alpha)$ such that

$$\operatorname{Re}\left\{\frac{zf'(z)}{g(z)}\right\} > \beta \quad (0 \le \alpha, \beta < p;\ z \in \mathcal{U}). \tag{1.5}$$

It is well known (see [10]) that every p-valently close-to-convex function is p-valent in $\mathcal{U}$.

For arbitrary fixed real numbers A and B $(-1 \le B < A \le 1)$, let $\mathcal{P}(A,B)$ denote the class of functions of the form

$$\phi(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{1.6}$$

which are analytic in $\mathcal{U}$ and satisfies the condition

$$\phi(z) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}), \tag{1.7}$$

where the symbol $\prec$ stands for subordination. The class $\mathcal{P}(A,B)$ was introduced and studied by Janowski [8].

We note that a function $\phi \in \mathcal{P}(A,B)$, if and only if

$$\begin{aligned} \left| \phi(z) - \frac{1-AB}{1-B^2} \right| &< \frac{A-B}{1-B^2} \quad (B \neq -1,\ z \in \mathcal{U}), \\ \operatorname{Re}\{\phi(z)\} &> \frac{1-A}{2} \quad (B = -1,\ z \in \mathcal{U}). \end{aligned} \tag{1.8}$$

For a function $f \in \mathcal{A}$, given by (1.1), the generalized Bernardi-Libera-Livingston integral operator F [1] is defined by

$$\begin{aligned} F(z) &= \frac{\gamma+p}{z^{\gamma}} \int_0^z t^{\gamma-1} f(t)\,dt \\ &= z^p + \sum_{n=1}^{\infty} \frac{\gamma+p}{\gamma+p+n} a_{n+p} z^{n+p} \quad (\gamma > -p;\ z \in \mathcal{U}). \end{aligned} \tag{1.9}$$

It readily follows from (1.9) that

$$f \in \mathcal{A}_p \Longrightarrow F \in \mathcal{A}_p. \tag{1.10}$$

Let

$$\phi_p(a,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+p} \quad (c \neq 0,-1,-2,\ldots;\ z \in \mathcal{U}), \tag{1.11}$$

and we define a linear operator $L_p(a,c)$ on $\mathcal{A}_p$ by

$$L_p(a,c)f(z) = \phi_p(a,c;z) * f(z) \quad (z \in \mathcal{U}), \tag{1.12}$$

where $(x)_n = \Gamma(n+x)/\Gamma(x)$ and the symbol $*$ is the Hadamard product or convolution. Clearly, $L_p(a,c)$ maps $\mathcal{A}_p$ into itself. Further, $L_p(a,a)$ is the identity operator and

$$L_p(a,c) = L_p(a,b)L_p(b,c) = L_p(b,c)L_p(a,b) \quad (b,c \neq 0,-1,-2,\ldots). \tag{1.13}$$

Thus, if $a \neq 0,-1,-2,\ldots$, then $L_p(a,c)$ has an inverse $L_p(c,a)$. We also observe that for $f \in \mathcal{A}_p$,

$$L_p(p+1,p)f(z) = \frac{zf'(z)}{p}, \qquad L_p(\mu+p,1)f(z) = D^{\mu+p-1}f(z), \tag{1.14}$$

where μ $(\mu > -p)$ is any real number. In case of $p = 1$ and $\mu \in \mathbb{N}$, $D^{\mu}f(z)$ is the Ruscheweyh derivative [14]. The operator $L_p(a,c)$ was introduced and studied by Saitoh and Nunokawa [15]. This operator is a generalization of the linear operator

$L(a,c)$ introduced by Carlson and Shaffer [3] in their systemic investigation of certain classes of starlike, convex, and prestarlike hypergeometric functions.

In the present paper, we give some argument properties of certain class of analytic functions in $\mathcal{A}_p$ involving the linear operator $L_p(a,c)$. An application of a certain integral operator is also considered. The results obtained here, besides extending the works of Bulboacă [2], Chichra [4], Cho et al. [5], Fukui et al. [6], Libera [9], Nunokawa [13], and Sakaguchi [16], it yields a number of new results.

2. Main results.

To establish our main results, we need the following lemmas.

Lemma 2.1 [11]. *Let $h(z)$ be convex (univalent) in $\mathcal{U}$ and let $\psi(z)$ be analytic in $\mathcal{U}$ with $\operatorname{Re}\{\psi(z)\} \geq 0$. If $\phi(z)$ is analytic in $\mathcal{U}$ and $\phi(0) = \psi(0)$, then*

$$\phi(z) + \psi(z) z\phi'(z) \prec h(z) \quad (z \in \mathcal{U}) \tag{2.1}$$

implies

$$\phi(z) \prec h(z) \quad (z \in \mathcal{U}). \tag{2.2}$$

Lemma 2.2 [12]. *Let $\phi(z)$ be analytic in $\mathcal{U}$, $\phi(0) = 1$, $\phi(z) \neq 0$ in $\mathcal{U}$ and suppose that there exists a point $z_0 \in \mathcal{U}$ such that*

$$\begin{gathered} |\arg \phi(z)| < \frac{\pi}{2}\eta \quad (|z| < |z_0|), \\ |\arg \phi(z_0)| = \frac{\pi}{2}\eta, \end{gathered} \tag{2.3}$$

where $\eta > 0$. Then

$$\frac{z_0 \phi'(z_0)}{\phi(z_0)} = ik\eta, \tag{2.4}$$

where

$$\begin{gathered} k \geq \frac{1}{2}\left(d + \frac{1}{d}\right) \quad \text{when } \arg \phi(z_0) = \frac{\pi}{2}\eta, \\ k \leq -\frac{1}{2}\left(d + \frac{1}{d}\right) \quad \text{when } \arg \phi(z_0) = -\frac{\pi}{2}\eta, \end{gathered} \tag{2.5}$$

where

$$\phi(z_0)^{1/\eta} = \pm id \quad (d > 0). \tag{2.6}$$

We now derive the following theorem.

Theorem 2.3. *Let $a > 0$, $-1 \leq B < A \leq 1$, $f \in \mathcal{A}_p$, and suppose that $g \in \mathcal{A}_p$ satisfies*

$$\frac{L_p(a+1,c)g(z)}{L_p(a,c)g(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}). \tag{2.7}$$

If

$$\begin{gathered} \left| \arg\left\{ (1-\lambda)\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} + \lambda \frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)} - \beta \right\} \right| \\ < \frac{\pi}{2}\delta \quad (\lambda \geq 0;\ 0 \leq \beta < 1;\ 0 < \delta \leq 1;\ z \in \mathcal{U}), \end{gathered} \tag{2.8}$$

then

$$\left|\arg\left\{\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}-\beta\right\}\right|<\frac{\pi}{2}\eta \quad (z\in\mathcal{U}), \tag{2.9}$$

where η $(0<\eta\le 1)$ *is the solution of the equation*

$$\delta=\begin{cases}\eta+\dfrac{2}{\pi}\tan^{-1}\left\{\dfrac{\lambda\eta\sin(\pi/2)(1-t(A,B))}{a(1+A)/(1+B)+\lambda\eta\cos(\pi/2)(1-t(A,B))}\right\}, & \text{for } B\ne -1,\\ \eta, & \text{for } B=-1,\end{cases} \tag{2.10}$$

when

$$t(A,B)=\frac{2}{\pi}\sin^{-1}\left(\frac{A-B}{1-AB}\right). \tag{2.11}$$

Proof. Let

$$\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}=\beta+(1-\beta)\phi(z). \tag{2.12}$$

Then $\phi(z)$ is analytic in $\mathcal{U}$ with $\phi(0)=1$. On differentiating both sides of (2.12) and using the identity

$$z(L_p(a,c)f(z))'=aL_p(a+1,c)f(z)-(a-p)L_p(a,c)f(z) \tag{2.13}$$

in the resulting equation, we deduce that

$$(1-\lambda)\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}+\lambda\frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)}-\beta=(1-\beta)\left\{\phi(z)+\frac{\lambda z\phi'(z)}{ar(z)}\right\}, \tag{2.14}$$

where

$$r(z)=\frac{L_p(a+1,c)g(z)}{L_p(a,c)g(z)}. \tag{2.15}$$

If we let

$$r(z)=\rho e^{(\pi\theta/2)i}, \tag{2.16}$$

then from (2.7) followed by (1.8), it follows that

$$\begin{gathered}\frac{1-A}{1-B}<\rho<\frac{1+A}{1+B},\\ -t(A,B)<\theta<t(A,B)\quad \text{for } B\ne -1,\end{gathered} \tag{2.17}$$

when $t(A,B)$ is given by (2.11), and

$$\begin{gathered}\frac{1-A}{2}<\rho<\infty,\\ -1<\theta<1\quad \text{for } B=-1.\end{gathered} \tag{2.18}$$

Let $h(z)$ be the function which maps onto the angular domain $\{w : |\arg\{w\}| < (\pi/2)\delta\}$ with $h(0) = 1$. Applying Lemma 2.1 for this $h(z)$ with $\psi(z) = \lambda/(ar(z))$, we see that $\operatorname{Re}\phi(z) > 0$ in $\mathcal{U}$ and hence $\phi(z) \neq 0$ in $\mathcal{U}$.

If there exists a point $z_0 \in \mathcal{U}$ such that conditions (2.3) are satisfied, then by Lemma 2.2 we obtain (2.4) under restrictions (2.5) and (2.6).

At first, suppose that $p(z_0)^{1/\eta} = id$ $(d > 0)$. For the case $B \neq -1$, we obtain

$$\begin{aligned}
\arg&\left\{(1-\lambda)\frac{L_p(a,c)f(z_0)}{L_p(a,c)g(z_0)}+\lambda\frac{L_p(a+1,c)f(z_0)}{L_p(a+1,c)g(z_0)}-\beta\right\}\\
&=\arg\phi(z_0)+\arg\left\{1+\frac{\lambda}{ar(z_0)}\frac{z_0\phi'(z_0)}{\phi(z_0)}\right\}\\
&=\frac{\pi}{2}\eta+\arg\left\{1+i\eta k\lambda\frac{e^{-(\pi\theta/2)i}}{\rho a}\right\}\\
&=\frac{\pi}{2}\eta+\tan^{-1}\left\{\frac{\lambda\eta k\sin(\pi/2)(1-\theta)}{\rho a+\lambda\eta k\cos(\pi/2)(1-\theta)}\right\}\\
&\geq\frac{\pi}{2}\eta+\tan^{-1}\left\{\frac{\lambda\eta\sin(\pi/2)(1-t(A,B))}{a(1+A)/(1+B)+\lambda\eta\cos(\pi/2)(1-t(A,B))}\right\}\\
&\geq\frac{\pi}{2}\delta,
\end{aligned}\tag{2.19}$$

where δ and $t(A,B)$ are given by (2.10) and (2.11), respectively. Similarly, for the case $B = -1$, we have

$$\arg\left\{(1-\lambda)\frac{L_p(a,c)f(z_0)}{L_p(a,c)g(z_0)}+\lambda\frac{L_p(a+1,c)f(z_0)}{L_p(a+1,c)g(z_0)}-\beta\right\}\geq\frac{\pi}{2}\eta.\tag{2.20}$$

This is a contradiction to the assumption of our theorem.

Next, suppose that $\phi(z_0)^{1/\eta} = -id$ $(d > 0)$. For the case $B \neq -1$, applying the same method as above, we have

$$\begin{aligned}
\arg&\left\{(1-\lambda)\frac{L_p(a,c)f(z_0)}{L_p(a,c)g(z_0)}+\lambda\frac{L_p(a+1,c)f(z_0)}{L_p(a+1,c)g(z_0)}-\beta\right\}\\
&\leq-\frac{\pi}{2}\eta-\tan^{-1}\left\{\frac{\lambda\eta\sin(\pi/2)(1-t(A,B))}{a(1+A)/(1+B)+\lambda\eta\cos(\pi/2)(1-t(A,B))}\right\}\\
&\leq-\frac{\pi}{2}\delta,
\end{aligned}\tag{2.21}$$

where δ and $t(A,B)$ are given by (2.10) and (2.11), respectively and for the case $B = -1$, we have

$$\arg\left((1-\lambda)\frac{L_p(a,c)f(z_0)}{L_p(a,c)g(z_0)}+\lambda\frac{L_p(a+1,c)f(z_0)}{L_p(a+1,c)g(z_0)}-\beta\right)\leq-\frac{\pi}{2}\eta\tag{2.22}$$

which contradicts the assumption. Therefore we complete the proof of the theorem. □

Remark 2.4. For $a = c = p$, $A = 1$, $B = -1$, and $\lambda = 1$, Theorem 2.3 is the recent result obtained by Nunokawa [13].

Taking $a = \mu + p$ $(\mu > -p)$, $c = 1$, $A = 1$, and $B = 0$ in Theorem 2.3, we have the following corollary.

Corollary 2.5. *If $f \in \mathcal{A}_p$ satisfies*

$$\left| \arg\left\{(1-\lambda)\frac{D^{\mu+p-1}f(z)}{D^{\mu+p-1}g(z)} + \lambda\frac{D^{\mu+p}f(z)}{D^{\mu+p}g(z)} - \beta\right\}\right| < \frac{\pi}{2}\delta \quad (\lambda \geq 0;\ 0 < \delta \leq 1;\ 0 \leq \beta < 1;\ z \in \mathcal{U}) \tag{2.23}$$

for some $g \in \mathcal{A}_p$ satisfying the condition

$$\left| \frac{D^{\mu+p}g(z)}{D^{\mu+p-1}g(z)} - 1 \right| < \alpha \quad (0 < \alpha \leq 1;\ z \in \mathcal{U}), \tag{2.24}$$

then

$$\left| \arg\left\{\frac{D^{\mu+p-1}f(z)}{D^{\mu+p-1}g(z)}\right\}\right| < \frac{\pi}{2}\eta \quad (z \in \mathcal{U}), \tag{2.25}$$

where η $(0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left\{\frac{\lambda\eta\sin(\pi/2 - \sin^{-1}\alpha)}{(\mu+p)(1+\alpha) + \lambda\eta\cos(\pi/2 - \sin^{-1}\alpha)}\right\}. \tag{2.26}$$

Letting $B \to A$ $(A < 1)$ and $g(z) = z^p$ in Theorem 2.3, we get the following corollary.

Corollary 2.6. *If $f \in \mathcal{A}_p$ satisfies*

$$\left| \arg\left\{(1-\lambda)\frac{L_p(a,c)f(z)}{z^p} + \lambda\frac{L_p(a+1,c)f(z)}{z^p} - \beta\right\}\right| < \frac{\pi}{2}\delta \quad (a > 0;\ \lambda \geq 0;\ 0 \leq \beta < 1;\ 0 < \delta \leq 1;\ z \in \mathcal{U}), \tag{2.27}$$

then

$$\left| \arg\left\{\frac{L_p(a,c)f(z)}{z^p} - \beta\right\}\right| < \frac{\pi}{2}\eta \quad (z \in \mathcal{U}), \tag{2.28}$$

where η $(0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left\{\frac{\lambda\eta}{a}\right\}. \tag{2.29}$$

Corollary 2.7. *Under the hypothesis of Corollary 2.6, we have*

$$|\arg\{H'(z) - \beta\}| < \frac{\pi}{2}\eta \quad (z \in \mathcal{U}), \tag{2.30}$$

where the function $H(z)$ is defined in $\mathcal{U}$ by

$$H(z) = \int_0^z \frac{L_p(a,c)f(t)}{t^p}\,dt \tag{2.31}$$

and η $(0 < \eta \le 1)$ is the solution of (2.29).

Remark 2.8. Taking $a = c = p$, $\lambda = 1$, and $\beta = 0$ in Corollary 2.6, $a = c = p$ and $\beta = 0$ in Corollary 2.7, we get the corresponding results obtained by Cho et al. [5].

Setting $A = 1-(2\alpha/p)$ $(0 \le \alpha < p)$, $B = -1$, and $\delta = 1$ in Theorem 2.3, we have the following corollary.

Corollary 2.9. *Let $a > 0$, $f \in \mathcal{A}_p$, and $g \in \mathcal{S}_p^*(\alpha)$. If*

$$\operatorname{Re}\left\{(1-\lambda)\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} + \lambda\frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)}\right\} > \beta \quad (\lambda \ge 0;\ 0 \le \beta < 1;\ z \in \mathcal{U}), \tag{2.32}$$

then

$$\operatorname{Re}\left\{\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)}\right\} > \beta \quad (z \in \mathcal{U}). \tag{2.33}$$

Remark 2.10. For $a = c = p = 1$ and $\alpha = 0$, Corollary 2.9 is the result by Bulboacă [2]. If we put $a = c = p = 1$, $\beta = 0$, and $g(z) = z$ in Corollary 2.9, then we have the result due to Chichra [4]. Further, taking $a = c = p$, $\lambda = 1$, and $\alpha = \beta = 0$ in Corollary 2.9, we get the corresponding results of Libera [9] and Sakaguchi [16].

Theorem 2.11. *If $f \in \mathcal{A}_p$ satisfies*

$$\left|\arg\left\{\frac{L_p(a,c)f(z)}{z^p} - \beta\right\}\right| < \frac{\pi}{2}\delta \quad (0 \le \beta < 1;\ 0 < \delta \le 1;\ z \in \mathcal{U}), \tag{2.34}$$

then

$$\left|\arg\left\{\frac{(\gamma+p)\int_0^z t^{\gamma-1}L_p(a,c)f(t)\,dt}{z^{\gamma+p}} - \beta\right\}\right| < \frac{\pi}{2}\eta \quad (0 < \gamma + p;\ z \in \mathcal{U}), \tag{2.35}$$

where η $(0 < \eta \le 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left\{\frac{\eta}{\gamma+p}\right\}. \tag{2.36}$$

Proof. Consider the function $\phi(z)$ defined in $\mathcal{U}$ by

$$\frac{(\gamma+p)\int_0^z t^{\gamma-1}L_p(a,c)f(t)\,dt}{z^{\gamma+p}} = \beta + (1-\beta)\phi(z). \tag{2.37}$$

Then $\phi(z)$ is analytic in $\mathcal{U}$ with $\phi(0) = 1$. Differentiating both sides of (2.37) and simplifying, we get

$$\frac{L_p(a,c)f(z)}{z^p} - \beta = (1-\beta)\left\{\phi(z) + \frac{z\phi'(z)}{\gamma+p}\right\}. \tag{2.38}$$

Now, by using Lemma 2.1 and a similar method in the proof of Theorem 2.3, we get (2.35). □

Taking $a = p+1$, $c = p$, $\beta = \rho/p$, and $\delta = 1$ in Theorem 2.11, we have the following corollary.

Corollary 2.12. *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{\frac{f'(z)}{z^{p-1}}\right\} > \rho \quad (0 \le \rho < p;\ z \in \mathcal{U}), \tag{2.39}$$

then

$$\left|\arg\left\{\frac{(\gamma+p)\int_0^z t^{\gamma-1} f'(t)\,dt}{z^{\gamma+p}} - \rho\right\}\right| < \frac{\pi}{2}\eta \quad (z \in \mathcal{U}), \tag{2.40}$$

where η $(0 < \eta \le 1)$ is the solution of the equation

$$\eta + \frac{2}{\pi}\tan^{-1}\left\{\frac{\eta}{\gamma+p}\right\} = 1. \tag{2.41}$$

Theorem 2.13. *If $f \in \mathcal{A}_p$ satisfies*

$$\left|\arg\left\{\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)} - \frac{a-p-\gamma}{a}\right\}\right| < \frac{\pi}{2}\delta \quad (a > 0;\ p+\gamma > 0;\ 0 < \delta \le 1;\ z \in \mathcal{U}), \tag{2.42}$$

then

$$\left|\arg\left\{\frac{z^{\gamma} L_p(a,c)f(z)}{\int_0^z t^{\gamma-1} L_p(a,c)f(t)\,dt}\right\}\right| < \frac{\pi}{2}\eta \quad (z \in \mathcal{U}), \tag{2.43}$$

where η $(0 < \eta \le 1)$ is the solution of (2.36).

Proof. Our proof of Theorem 2.13 is much akin to that of Theorem 2.3. Indeed, in place of (2.37), we define the function $\phi(z)$ by

$$\phi(z) = \frac{z^{\gamma} L_p(a,c)f(z)}{(\gamma+p)\int_0^z t^{\gamma-1} L_p(a,c)f(t)\,dt} \quad (z \in \mathcal{U}), \tag{2.44}$$

and apply Lemma 2.1 (with $\psi(z) = 1/(\gamma+p)$) as before. We choose to skip the details involved. □

Setting $a = c = p$ and $\delta = 1$ in Theorem 2.13, we obtain the following corollary.

Corollary 2.14. *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > -\gamma \quad (\gamma+p > 0;\ z \in \mathcal{U}), \tag{2.45}$$

then

$$\left|\arg\left\{\frac{z^{\gamma} f(z)}{\int_0^z t^{\gamma-1} f(t)\,dt}\right\}\right| < \frac{\pi}{2}\eta \quad (z \in \mathcal{U}), \tag{2.46}$$

where η $(0 < \eta \le 1)$ is the solution of (2.41).

Replacing $f(z)$ by $zf'(z)/p$ in Corollary 2.14, we deduce the following corollary.

Corollary 2.15. *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > -\gamma \quad (\gamma+p>0;\ z\in\mathcal{U}), \tag{2.47}$$

then

$$\left|\arg\left\{\frac{zf'(z)}{f(z)-(\gamma/z^{\gamma})\int_0^z t^{\gamma-1}f(t)\,dt}\right\}\right| < \frac{\pi}{2}\eta \quad (z\in\mathcal{U}), \tag{2.48}$$

where η $(0<\eta\le 1)$ is the solution of (2.41).

By setting $\gamma = 0$ in Corollary 2.15, we have the following corollary.

Corollary 2.16. *If $f \in \mathcal{K}_p(0)$, then*

$$\left|\arg\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}\eta \quad (z\in\mathcal{U}), \tag{2.49}$$

where η $(0<\eta\le 1)$ is the solution of the equation:

$$\eta+\frac{2}{\pi}\tan^{-1}\left\{\frac{\eta}{p}\right\}=1. \tag{2.50}$$

Similarly, we have the following theorem.

Theorem 2.17. *If $f \in \mathcal{A}_p$ satisfies*

$$\left|\arg\left\{\frac{L_p(a+1,c)f(z)}{L_p(a,c)f(z)}-\beta\right\}\right| < \frac{\pi}{2}\delta \quad (a>0;\ 0\le\beta<1;\ 0<\delta\le 1;\ z\in\mathcal{U}), \tag{2.51}$$

then

$$\left|\arg\left\{\frac{L_p(a,c)f(z)}{z^p}\right\}\right| < \frac{\pi}{2}\eta \quad (z\in\mathcal{U}), \tag{2.52}$$

where η $(0<\eta\le 1)$ is the solution of the equation

$$\delta=\frac{2}{\pi}\tan^{-1}\left\{\frac{\eta}{(1-\beta)a}\right\}. \tag{2.53}$$

Theorem 2.18. *Let $f \in \mathcal{A}_p$ and suppose that*

$$B<A\le B+\frac{p(1-B)}{a} \quad (a>0;\ -1\le B<A\le 1). \tag{2.54}$$

If

$$\begin{aligned}&\left|\arg\left\{(1-\lambda)\frac{L_p(a+1,c)f(z)}{L_p(a,c)g(z)}+\lambda\frac{(L_p(a+1,c)f(z))'}{(L_p(a,c)g(z))'}-\beta\right\}\right|\\&\qquad<\frac{\pi}{2}\delta \quad (\lambda\ge 0;\ 0\le\beta<1;\ 0<\delta\le 1;\ z\in\mathcal{U}),\end{aligned} \tag{2.55}$$

for some $g \in \mathcal{A}_p$ *satisfying*

$$\frac{L_p(a+1,c)g(z)}{L_p(a,c)g(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}), \tag{2.56}$$

then

$$\left| \arg\left\{ \frac{L_p(a+1,c)f(z)}{L_p(a,c)g(z)} - \beta \right\} \right| < \frac{\pi}{2}\eta \quad (z \in \mathcal{U}), \tag{2.57}$$

where η $(0 < \eta \le 1)$ *is the solution of the equation*

$$\delta = \begin{cases} \eta + \dfrac{2}{\pi}\tan^{-1}\left\{\dfrac{\lambda\eta\sin(\pi/2)(1-t(A,B))}{(p(1+B)+a(A-B))/(1+B)+\lambda\eta\cos(\pi/2)(1-t(A,B))}\right\}, & \textit{for } B \ne -1, \\ \eta, & \textit{for } B = -1, \end{cases} \tag{2.58}$$

when

$$t(A,B) = \frac{2}{\pi}\sin^{-1}\left(\frac{a(A-B)}{p(1-B^2)-aB(A-B)}\right). \tag{2.59}$$

Proof. Let

$$\frac{L_p(a+1,c)f(z)}{L_p(a,c)g(z)} = \beta + (1-\beta)\phi(z), \qquad r(z) = \frac{L_p(a+1,c)g(z)}{L_p(a,c)g(z)}, \tag{2.60}$$

we have

$$(1-\lambda)\frac{L_p(a+1,c)f(z)}{L_p(a,c)g(z)} + \lambda\frac{(L_p(a+1,c)f(z))'}{(L_p(a+1,c)g(z))'} - \beta = (1-\beta)\left\{\phi(z) + \frac{\lambda z\phi'(z)}{ar(z)+p-a}\right\}. \tag{2.61}$$

The remaining part of the proof of Theorem 2.18 is similar to that of Theorem 2.3. So we omit the details. □

Put $a = c = p$, $\lambda = 1$, $A = \alpha/p$, and $B = 0$ in Theorem 2.18, we have the following corollary.

Corollary 2.19. *If* $f \in \mathcal{A}_p$ *satisfies*

$$\left| \arg\left\{ \frac{(zf'(z))'}{g'(z)} - \beta \right\} \right| < \frac{\pi}{2}\delta \quad (0 \le \beta < p;\ 0 < \delta \le 1;\ z \in \mathcal{U}), \tag{2.62}$$

for some $g \in \mathcal{A}_p$ *satisfying the condition*

$$\left| \frac{zg'(z)}{g(z)} - p \right| < \alpha \quad (0 < \alpha \le p;\ z \in \mathcal{U}), \tag{2.63}$$

then

$$\left|\arg\left\{\frac{zf'(z)}{g(z)}-\beta\right\}\right| < \frac{\pi}{2}\eta \quad (z\in\mathcal{U}), \tag{2.64}$$

where η $(0<\eta\le 1)$ *is the solution of the equation*

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left\{\frac{\eta\sin(\pi/2-\sin^{-1}(\alpha/p))}{p+\alpha+\eta\cos(\pi/2-\sin^{-1}(\alpha/p))}\right\}. \tag{2.65}$$

Lemma 2.20. *Let*

$$\alpha = \xi + \frac{\xi}{\gamma+p+a\xi} \quad (0\le (a-1)/a<\xi<\alpha<1) \tag{2.66}$$

and the function $G(z)$ *be defined by*

$$G(z) = \frac{\gamma+p}{z^{\gamma}}\int_0^z t^{\gamma-1}g(t)\,dt \quad (g\in\mathcal{A}_p) \tag{2.67}$$

for $\gamma > (a\xi^2+(p+1-a)\xi-p)/(1-\xi)$. *If* $g\in\mathcal{A}_p$ *satisfies*

$$\left|\frac{L_p(a+1,c)g(z)}{L_p(a,c)g(z)}-1\right| < \alpha \quad (z\in\mathcal{U}), \tag{2.68}$$

then

$$\left|\frac{L_p(a+1,c)G(z)}{L_p(a,c)G(z)}-1\right| < \xi \quad (z\in\mathcal{U}). \tag{2.69}$$

Proof. Defining the function $w(z)$ by

$$\frac{L_p(a+1,c)G(z)}{L_p(a,c)G(z)} = 1+\xi w(z), \tag{2.70}$$

we see that $w(z)$ is analytic in $\mathcal{U}$ with $w(0)=0$. Now, using the identities

$$z\big(L_p(a,c)G(z)\big)' = aL_p(a+1,c)G(z)-(a-p)L_p(a,c)G(z), \tag{2.71}$$

$$z\big(L_p(a,c)G(z)\big)' = (\gamma+p)L_p(a,c)g(z)-\gamma L_p(a,c)G(z) \tag{2.72}$$

in (2.70), we get

$$\frac{L_p(a,c)G(z)}{L_p(a,c)g(z)} = \frac{\gamma+p}{\gamma+p+a\xi w(z)}. \tag{2.73}$$

Making use of the logarithmic differentiation of both sides of (2.73) and using identity (2.71) for both $g(z)$ and $f(z)$ in the resulting equation, we deduce that

$$\left|\frac{L_p(a+1,c)g(z)}{L_p(a,c)g(z)}-1\right| = \xi\left|w(z)+\frac{zw'(z)}{\gamma+p+a\xi w(z)}\right|. \tag{2.74}$$

We assume that there exists a point $z_0 \in \mathcal{U}$ such that $\max_{|z|<|z_0|} |w(z)| = |w(z_0)| = 1$. Then by Jack's lemma [7], we have $z_0 w'(z_0) = kw(z_0)$ $(k \geq 1)$. Let $w(z_0) = e^{i\theta}$, and apply this result to $w(z)$ at $z_0 \in \mathcal{U}$, we get

$$\begin{aligned} \left| \frac{L_p(a+1,c)g(z_0)}{L_p(a,c)g(z_0)} - 1 \right| &= \xi \left| 1 + \frac{k}{\gamma + p + a\xi e^{i\theta}} \right| \\ &= \xi \left[\frac{(\gamma+p+k)^2 + 2a\xi(\gamma+p+k)\cos\theta + (a\xi)^2}{(\gamma+p)^2 + 2a\xi(\gamma+p)\cos\theta + (a\xi)^2} \right]^{1/2}. \end{aligned} \tag{2.75}$$

Since the right side of (2.75) is decreasing for $0 \leq \theta < 2\pi$ and $\gamma > \{a\xi^2 + (p+1-a)\xi - p\}/(1-\xi)$, we obtain

$$\left| \frac{L_p(a+1,c)g(z_0)}{L_p(a,c)g(z_0)} - 1 \right| \leq \frac{\xi(\gamma+p+1+a\xi)}{\gamma+p+a\xi}, \tag{2.76}$$

which contradicts our hypothesis and hence we get

$$|w(z)| = \frac{1}{\xi} \left| \frac{L_p(a+1,c)G(z)}{L_p(a,c)G(z)} - 1 \right| < 1 \quad (z \in \mathcal{U}). \tag{2.77}$$

This completes the proof of Lemma 2.20. □

Remark 2.21. We note that for $a = c = p = 1$, Lemma 2.20 yields the corresponding result obtained by Fukui et al. [6].

Theorem 2.22. *Let α be as given in (2.66) and $\gamma^* > \max\{(a\xi^2 + (p+1-a)\xi - p)/(1-\xi),\ a\xi - p\}$. If $f \in \mathcal{A}_p$ satisfies*

$$\left| \arg \left\{ \frac{L_p(a+1,c)f(z)}{L_p(a,c)g(z)} - \beta \right\} \right| < \frac{\pi}{2}\delta \quad (0 \leq \beta < 1;\ 0 < \delta \leq 1;\ z \in \mathcal{U}), \tag{2.78}$$

for some $f \in \mathcal{A}_p$ satisfying condition (2.68), then

$$\left| \arg \left\{ \frac{L_p(a+1,c)F(z)}{L_p(a,c)G(z)} - \beta \right\} \right| < \frac{\pi}{2}\eta \quad (z \in \mathcal{U}), \tag{2.79}$$

where the function $F(z)$ and $G(z)$ are defined for γ^* by (1.9) and (2.67), respectively and η $(0 < \eta \leq 1)$ is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\eta \sin(\pi/2 - \sin^{-1}(a\xi/(\gamma^*+p)))}{\gamma^* + p + a\xi + \eta\cos(\pi/2 - \sin^{-1}(a\xi/(\gamma^*+p)))} \right\}. \tag{2.80}$$

Proof. Consider the function $\phi(z)$ defined in $\mathcal{U}$ by

$$\frac{L_p(a+1,c)F(z)}{L_p(a,c)G(z)} = \beta + (1-\beta)\phi(z). \tag{2.81}$$

Then $\phi(z)$ is analytic in $\mathcal{U}$ with $\phi(0) = 1$. Taking logarithmic differentiation on both sides of (2.81) and using identity (2.71) in the resulting equation, we get

$$\frac{z(L_p(a+1,c)F(z))'}{L_p(a+1,c)F(z)} = p-a+a\frac{L_p(a+1,c)G(z)}{L_p(a,c)G(z)}+(1-\beta)\frac{z\phi'(z)}{\beta+(1-\beta)\phi(z)}. \tag{2.82}$$

From the definition of $F(z)$, we have

$$(\gamma^*+p)L_p(a,c)f(z) = a(L_p(a+1,c)F(z))'+\gamma^*L_p(a+1,c)F(z). \tag{2.83}$$

Again, from (2.71) and (2.72), it follows that

$$(\gamma^*+p)L_p(a+1,c)g(z) = zL_p(a+1,c)G(z)+(p+\gamma^*-a)L_p(a,c)G(z). \tag{2.84}$$

Thus, by using (2.83) and (2.84) followed by (2.82), we obtain

$$\frac{L_p(a+1,c)f(z)}{L_p(a,c)g(z)}-\beta = (1-\beta)\left\{\phi(z)+\frac{z\phi'(z)}{ar(z)+\gamma^*+p-a}\right\}, \tag{2.85}$$

where $r(z) = L_p(a+1,c)G(z)/L_p(a,c)G(z)$. By using Lemma 2.20, we have

$$r(z) \prec 1+\xi z \quad (z\in\mathcal{U}), \tag{2.86}$$

where ξ is given by (2.66). Letting

$$ar(z)+\gamma^*+p-a = \rho e^{i\pi\theta/2} \tag{2.87}$$

and using the techniques of Theorem 2.3, the remaining part of the proof of Theorem 2.22 follows. □

Remark 2.23. We easily find the following:

$$\gamma > \begin{cases} a\xi-p, & \text{if } \dfrac{a-1}{a}<\xi<\dfrac{2a-1}{2a}, \\ \dfrac{2(a-p)-1}{2}, & \text{if } \xi=\dfrac{2a-1}{2a}, \\ \dfrac{a\xi^2+(p+1-a)\xi-p}{1-\xi}, & \text{if } \dfrac{2a-1}{2a}<\xi<1. \end{cases} \tag{2.88}$$

Taking $a = c = p$ in Theorem 2.22, we get the following corollary.

Corollary 2.24. *Let*

$$\alpha = \xi+\frac{\xi}{\gamma^*+p(1+\xi)} \quad ((p-1)/p<\xi<\alpha<1), \tag{2.89}$$

where $\gamma^* > \max\{(p\xi^2+\xi-p)/(1-\xi), p(\xi-1)\}$*. If* $f \in \mathcal{A}_p$ *satisfies*

$$\left|\arg\left\{\frac{zf'(z)}{g(z)}-\beta\right\}\right| < \frac{\pi}{2}\delta \quad (0 \le \beta < p;\ 0 < \delta \le 1;\ z \in \mathcal{U}) \tag{2.90}$$

for some $g \in \mathcal{A}_p$ *satisfying the condition*

$$\left|\frac{zg'(z)}{g(z)}-p\right| < p\alpha \quad (z \in \mathcal{U}), \tag{2.91}$$

then

$$\left|\arg\left\{\frac{zF'(z)}{G(z)}-\beta\right\}\right| < \frac{\pi}{2} \quad (z \in \mathcal{U}), \tag{2.92}$$

where η $(0 < \eta \le 1)$ *is the solution of the equation*

$$\delta = \eta + \frac{2}{\pi}\tan^{-1}\left\{\frac{\eta\sin\left(\pi/2-\sin^{-1}\left(p\xi/(\gamma^*+p)\right)\right)}{\gamma^*+p(1+\xi)+\eta\cos\left(\pi/2-\sin^{-1}\left(p\xi/(\gamma^*+p)\right)\right)}\right\}. \tag{2.93}$$

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