IJMMS 31:2 (2002) 103–108
PII. S0161171202012899
http://ijmms.hindawi.com

ON CONSTRAINED UNIFORM APPROXIMATION

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Received 22 April 2001 and in revised form 8 October 2001

The problem of uniform approximants subject to Hermite interpolatory constraints is considered with an alternate approach. The uniqueness and the convergence aspects of this problem are also discussed. Our approach is based on the work of P. Kirchberger (1903) and a generalization of Weierstrass approximation theorem.

2000 Mathematics Subject Classification: 41A05, 41A10, 41A29, 41A52.

1. Introduction. Let π_m denote the set of all polynomials of degree less than or equal to m. Let $[a,b]$ be a closed finite interval and $C[a,b]$ the space of real continuous functions on $[a,b]$ with uniform norm

$$\|h\|_\infty = \max_{x\in[a,b]} |h(x)|. \tag{1.1}$$

Here we discuss uniform approximation of a prescribed $f \in C[a,b]$ by the polynomials that are also Hermite interpolants to a set of given data at a finite number of preassigned points in the interval $[a,b]$. More precisely, we consider the following problem due to Loeb et al. [6].

Problem 1.1. Suppose that k, m, and n_i, where $i = 1,2,\dots,k$, are positive integers with $m \ge (\sum_{i=1}^{k} n_i) - 1$ and that $\{u_i\}_{i=1}^{k}$ is a subset of $[a,b]$ satisfying $a \le u_1 < u_2 < \cdots < u_k \le b$. Then the problem is to find a best uniform approximant to a given $f \in C[a,b]$ from the class

$$\Phi_{m,\beta} = \{\phi \in \pi_m : \phi^{(j)}(u_i) = \beta_{ij},\ 1 \le i \le k,\ 0 \le j \le n_i - 1\}, \tag{1.2}$$

where

$$S_\beta := \{\beta_{ij} : 1 \le i \le k,\ 0 \le j \le n_i - 1\} \tag{1.3}$$

is a subset of the real numbers with $\beta_{i0} = f(u_i)$.

This problem originates from the work of Paszkowski [8, 9] who studied it by imposing the interpolatory conditions only on the values of interpolating polynomials at k distinct points of $[a,b]$ in the sense of Lagrange interpolation. To do this, he followed the classical Tchebycheff approach of approximating a continuous function by elements of an n-dimensional Haar subspace. In [2] Deutsch discussed Paszkowski's results [8, Theorems 2 and 5] with a different method. Deutsch's work is based on a characterization theorem of best approximation that involves extreme points of the closed unit ball in the dual of the underlying space [3, Corollary 2.6]. Later, Loeb et al. [6] extended the work of Deutsch by constraining the uniform approximants with

Hermite interpolatory conditions. In fact, they discussed Problem 1.1 through the notions of the n-dimensional extended Haar subspace of order ν of $C[a,b]$ (see [4]) and the generalized weight functions (see [7]). They also established a convergence result by giving a generalization of a theorem of de La Vallée Poussin [1, page 77].

In the present paper we continue the study of Problem 1.1. Our approach for its solution is based on the work of Kirchberger [5] that deals with extreme values of the error function. Uniqueness and convergence problems are also addressed in our work taking into account an extension of Weierstrass approximation theorem [10].

2. Notations and reformulation of Problem 1.1. For the sake of convenience, we set $I_k := \{1,2,\ldots,k\}$, $N_i := \{0,1,\ldots,n_i-1\}$, $s := \sum_{i\in I_k} n_i$, and

$$W(x) := \prod_{i\in I_k} (x-u_i)^{n_i}. \tag{2.1}$$

The notation $H_{s-1}(x,S_\beta)$, where S_β is given in (1.3), stands for the polynomial of degree less than or equal to $s-1$ that satisfies the following conditions:

$$H_{s-1}^{(j)}(u_i,S_\beta) = \beta_{ij}, \quad \forall i \in I_k, \forall j \in N_i. \tag{2.2}$$

For every $f \in C[a,b]$, we define

$$f_{H,\beta}(x) := f(x) - H_{s-1}(x,S_\beta). \tag{2.3}$$

The error function corresponding to $f,g \in C[a,b]$ and the set of its extreme points in $[a,b]$ will be denoted, respectively, as follows:

$$\begin{gathered} e_{f,g}(x) := |f(x)-g(x)|, \\ \operatorname{crit}(e_{f,g}) := \{x \in [a,b] : |e_{f,g}(x)| = \|e_{f,g}\|_\infty\}. \end{gathered} \tag{2.4}$$

Let π_m^* denote the $(m-s+1)$-dimensional subspace of π_m generated by the polynomials $x^j W(x)$, $j = 0,1,2,\ldots,m-s$, where W is given by (2.1). The following remark gives an explicit representation of the elements of $\Phi_{m,\beta}$ (see (1.2)).

Remark 2.1. A typical element ϕ^* in the approximating class $\Phi_{m,\beta}$ is of the form

$$\phi^*(x) = H_{s-1}(x,S_\beta) + q^*(x), \tag{2.5}$$

where $q^* \in \pi_m^*$. This shows that $\phi^* \in \Phi_{m,\beta}$ is a best approximant to f from the class $\Phi_{m,\beta}$ if and only if $q^* \in \pi_m^*$ is a best approximant to $f_{H,\beta}$ from the class π_m^*. In particular, if $m = s-1$ then $H_{s-1}(x,S_\beta)$ will be the best approximation to f. For this obvious reason we assume that $m \geq s$ in the rest of the paper.

In view of the above remark, Problem 1.1 can be reformulated as follows.

Problem 2.2. For a given function $f \in C[a,b]$, find a best approximation to $f_{H,\beta}$ (see (2.3)) in the uniform norm from the class π_m^*.

3. Characterization of best approximation. This section deals with a necessary and sufficient condition for a solution of Problem 2.2. We note that every $q \in \pi_m^*$ can be expressed as

$$q(x) = W(x)R_q(x), \tag{3.1}$$

where $R_q(x)$ is a polynomial of degree at most $m-s$. For an $f \in C[a,b]$ and a $q \in \pi_m^*$, we set

$$E_{f_{H,\beta},q}(x) := \frac{f_{H,\beta}(x)}{W(x)} - R_q(x). \tag{3.2}$$

An alternate form of the characterization theorem [6, Theorem 3.1] that solves Problem 1.1 may be stated as follows.

Theorem 3.1. *Let $f \in C[a,b]$ such that $f_{H,\beta} \notin \pi_m^*$. Then q^* is a best uniform approximant to $f_{H,\beta}$ from the class π^* if and only if there exist N points $\alpha_i \in \operatorname{crit}(e_{f_{H,\beta},q^*})$ satisfying the following conditions:*

(a) *$N = m-s+2$;*

(b) *$a \le \alpha_1 < \alpha_2 < \cdots < \alpha_N \le b$;*

(c) *$\operatorname{sgn}(E_{f_{H,\beta},q^*}(\alpha_i)) = (-1)^{i+1}\operatorname{sgn}(E_{f_{H,\beta},q^*}(\alpha_1))$, for all $i = 2,3,\ldots,N$.*

Our method of proof is based on the following lemma which may be found in the standard texts of approximation theory, for example, [9, Lemma 7.1].

Lemma 3.2. *Let Y be a linear subspace of $C[a,b]$ and let $h \in C[a,b]$. Then $g^* \in Y$ is a best uniform approximant to f in Y if and only if there does not exist any $g \in Y$ such that*

$$\{h(x) - g^*(x)\}g(x) > 0 \tag{3.3}$$

for all $x \in \operatorname{crit}(e_{f,g^*})$ (see (2.3)).

Remark 3.3. If we set $Y = \pi_m^*$ and $h = f_{H,\beta}$ in the above lemma, then the necessary and sufficient condition for $q^* \in \pi_m^*$ to be a best approximation to $f_{H,\beta}$ is that there does not exist any $p \in \pi_m^*$ such that

$$E_{f_{H,\beta},q^*}(x)R_p(x) > 0 \tag{3.4}$$

for all $x \in \operatorname{crit}(e_{f_{H,\beta},q^*})$ where $R_p(x)$ and $E_{f_{H,\beta},q^*}$ are, respectively, given in (3.1) and (3.2). To justify this, it is enough to note that

$$\{f_{H,\beta}(x) - q^*(x)\}p(x) = E_{f_{H,\beta},q^*}(x)R_p(x)W^2(x). \tag{3.5}$$

Remark 3.4. If $f_{H,\beta} \notin \pi_m^*$, then $W(x)$ is a nonvanishing function on the compact set $\operatorname{crit}(e_{f_{H,\beta},p})$ regardless of the choice of $p \in \pi_m^*$. Consequently, $f_{H,\beta}$ is continuous as well as nowhere zero on $\operatorname{crit}(e_{f_{H,\beta},p})$.

4. Proof of Theorem 3.1. If q^* is not a best approximation to $f_{H,\beta}$, then by Remark 3.3, there exists $p \in \pi_m^*$ such that

$$E_{f_{H,\beta},q^*}(x)R_p(x) > 0 \tag{4.1}$$

as x ranges over $\operatorname{crit}(e_{f_{H,\beta},q^*})$. We note that R_p being a polynomial of degree less than or equal to $m-s$ (see (3.1)) changes sign at most at $m-s$ places. Therefore, it follows from (4.1) that $E_{f_{H,\beta},q^*}(x)$ cannot change sign more than $(m-s)$ times as x ranges over $\operatorname{crit}(e_{f_{H,\beta},q^*})$. This contradicts Theorem 3.1(c) as $N = m-s+2$.

Conversely, assume that there are N points $\alpha_i \in \operatorname{crit}(e_{f_{H,\beta},q^*})$, $i = 1,2,\dots,N$, satisfying Theorem 3.1(b) and (c) but $N \le m-s+1$. For each $i = 1,2,3,\dots,N-1$, fix a point $w_i \in (\alpha_i,\alpha_{i+1})$ such that

$$\begin{gathered} u_j \notin [w_i,\alpha_{i+1}], \quad j = 1,2,\dots,k, \\ (w_i,\alpha_{i+1}], \ \operatorname{crit}(e_{f_{H,\beta},q^*}) \text{ are disjoint}, \\ \operatorname{sgn}(E_{f_{H,\beta},q^*}(w_i)) = \operatorname{sgn}(E_{f_{H,\beta},q^*}(\alpha_{i+1})). \end{gathered} \tag{4.2}$$

The choice of w_i, as required above, directly follows from Remark 3.4. Now we set

$$\tilde{p}(x) := \operatorname{sgn}(E_{f_{H,\beta},q^*}(\alpha_1))W(x)\prod_{i=1}^{N-1}(w_i - x). \tag{4.3}$$

Then $\tilde{p} \in \pi_m^*$ with $R_{\tilde{p}}(x) = \operatorname{sgn}(E_{f_{H,\beta},q^*}(\alpha_1))\prod_{i=1}^{N-1}(w_i - x)$. We claim that

$$E_{f_{H,\beta},q^*}(\alpha)R_{\tilde{p}}(\alpha) > 0 \quad \forall \alpha \in \operatorname{crit}(e_{f_{H,\beta},q^*}). \tag{4.4}$$

This can be seen by restricting α to each set $(\alpha_i,\alpha_{i+1}] \cap \operatorname{crit}(e_{f_{H,\beta},q^*})$ for $i = 0,1,\dots,N-1$, where $\alpha_0 = a$, and then using Theorem 3.1(c) along with (4.2). Hence by Remark 3.3, we note that q^* cannot be a best uniform approximant to $f_{H,\beta}$ from the class π_m^*. This completes the proof. □

Remark 4.1. An immediate consequence of Theorem 3.1 is that $H_{s-1}(x,S_\beta) + q^*(x)$ is a best uniform approximant to $f(x)$ from the class $\Phi_{m,\beta}$.

5. Uniqueness. We retain the setting of the previous sections in order to establish the uniqueness of the solution of Problem 2.2. More precisely, we prove the following theorem.

Theorem 5.1. *There is exactly one best uniform approximant p^* to $f_{H,\beta}$ from π_m^*.*

If q^* is another best uniform approximant to $f_{H,\beta}$ from π_m^*, then there exist N points $\alpha_i \in \operatorname{crit}(e_{f_{H,\beta},q^*})$, $i = 1,2,\dots,N$, satisfying Theorem 3.1(a), (b), and (c). Using the properties of best approximant to $f_{H,\beta}$, we observe that $|f_H(\alpha_i) - p^*(\alpha_i)| \le \|e_{f_{H,\beta},p^*}\|_\infty = \|e_{f_{H,\beta},q^*}\|_\infty = |f_{H,\beta}(\alpha_i) - q^*(\alpha_i)|$ for each $i = 1,2,\dots,N$, and consequently,

$$|E_{f_{H,\beta},p^*}(\alpha_i)| \le |E_{f_{H,\beta},q^*}(\alpha_i)|. \tag{5.1}$$

We set $D(x) := R_{p^*}(x) - R_{q^*}(x)$. Then $D \in \pi_{m-s}$ and

$$D(\alpha_i) = E_{f_{H,\beta},q^*}(\alpha_i) - E_{f_{H,\beta},p^*}(\alpha_i), \quad i = 1,2,\dots,N. \tag{5.2}$$

Note that if $D(\alpha_i) \ne 0$ for any $i = 1,2,\dots,N$, then by (5.2), $\operatorname{sgn}(D(\alpha_i)) = \operatorname{sgn}(E_{f_{H,\beta},q^*}(\alpha_i))$. Thus, the polynomial D has either a double zero at α_i, or it has a zero in (α_i,α_{i+1}) implying that $D \equiv 0$. Hence $R_{p^*}(x) = R_{q^*}(x)$, and consequently, $p^* = q^*$.

6. Convergence. In this section, we discuss the convergence of the sequence of best uniform approximants $\{q_k^*\}_{k=s-1}^{\infty}$ to $f_{H,\beta}$ with the conditions that f is *sufficiently differentiable* and the set S_β (see Problem 1.1) is replaced by

$$S_f = \{f^{(j)}(u_i) : j \in N_i,\ i \in I_k\}. \tag{6.1}$$

In this case, we write $f_{H,f}$ and $\Phi_{m,f}$ instead of $f_{H,\beta}$ and $\Phi_{m,\beta}$ (see (2.3)).

Theorem 6.1. *Assume that $f \in C^{n^*}[a,b]$ with $n^* = (\max_{i\in I_k} n_i) - 1$ and that the set S_β (see (1.3)) is replaced by S_f. If $q_m^* \in \pi_m^*$ is the best approximant to $f_{H,f}$ in the sense of Theorem 3.1, then*

$$\lim_{m\to\infty} ||q_m^* - f_{H,f}||_\infty = 0. \tag{6.2}$$

Consequently, the sequence $\{q_m^* + H_{s-1}(\cdot, S_f)\}_{m=s-1}^{\infty}$ will converge uniformly to f.

The crux of the proof of this theorem is in an extension of a result based on the Weierstrass approximation theorem [10, page 160]. We state it in the next lemma without proof as it is a routine exercise.

Lemma 6.2. *For any $f \in C^r[a,b]$, and for a given $\varepsilon > 0$, there exists a polynomial p such that*

$$||f^{(j)} - p^{(j)}||_\infty < \varepsilon \tag{6.3}$$

for all $j = 0,1,2,\dots,r$.

Proof of Theorem 6.1. In the notations of (2.3) and (6.1), we can write

$$f_{H,f}(x) := f(x) - H_{s-1}(x, S_f). \tag{6.4}$$

The polynomial $H_{s-1}(x,S_f)$ due to its interpolation properties may be expressed as

$$H_{s-1}(x,S_f) = \sum_{l\in I_k}\sum_{n\in N_l} f^{(n)}(u_l) L_{n,l}(x), \tag{6.5}$$

where $L_{n,l}(x)$ are the fundamental polynomials of degree $s-1$ satisfying the conditions

$$L_{n,l}^{(j)}(u_i) = \begin{cases} 1 & \text{for } l = i,\ n = j, \\ 0 & \text{otherwise.} \end{cases} \tag{6.6}$$

For a given $\varepsilon > 0$, we can fix a polynomial p of degree $r > s$ such that (see Lemma 6.2)

$$||f^{(j)} - p^{(j)}||_\infty < \frac{\varepsilon}{2\tau}, \quad j = 0,1,\dots,n^*, \tag{6.7}$$

where $\tau = \max\{1,\lambda\}$ with $\lambda = \max_{x\in[a,b]} \sum_{l\in I_k}\sum_{n\in N_l} |L_{n,l}(x)|$. From (6.1), (6.5), and (6.7) it follows that

$$\max_{x\in[a,b]} |H_{s-1}(x,S_f) - H_{s-1}(x,S_p)| < \frac{\varepsilon}{2}. \tag{6.8}$$

We set $q(x) := p(x) - H_{s-1}(x,S_p)$. Then $q \in \pi_r$ and $q^{(j)}(u_i) = 0$ for $j \in N_i$ and $i \in I_k$. Hence, W as defined in (2.1) is a factor of the polynomial q. This shows that $q \in \pi_r^*$. Using (6.4), (6.5), and (6.8) it can be seen that

$$||f_{H,f} - q||_\infty < \varepsilon. \tag{6.9}$$

Now consider the best uniform approximant q_r^* to f_H from π_r^* (see Theorem 3.1) and note that

$$\varepsilon > ||f_{H,f} - q||_\infty \geq ||f_{H,f} - q_r^*||_\infty \geq ||f_{H,f} - q_m^*||_\infty \tag{6.10}$$

for all $m \geq r$. The last inequality follows from the relation $\pi_r^* \subseteq \pi_m^*$. This proves the desired result. □

Acknowledgment. The author acknowledges with gratitude the research facilities available at King Fahd University of Petroleum and Minerals, Saudi Arabia, and the University of Alberta, Canada, during the preparation of this paper.

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