IJMMS 31:3 (2002) 157–166
PII. S0161171202111318
http://ijmms.hindawi.com

CONTROLLABILITY OF SEMILINEAR STOCHASTIC DELAY EVOLUTION EQUATIONS IN HILBERT SPACES

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Received 11 November 2001

The controllability of semilinear stochastic delay evolution equations is studied by using a stochastic version of the well-known Banach fixed point theorem and semigroup theory. An application to stochastic partial differential equations is given.

2000 Mathematics Subject Classification: 93C40, 93E35.

1. Introduction. The fixed point technique is widely used as a tool to study the controllability of nonlinear systems in finite- and infinite-dimensional Banach spaces, see the early survey paper by Balachandran and Dauer [5]. Also, Anichini [2] and Yamamoto [14] studied the controllability of the classical nonlinear system by means of Schaefer's theorem and Schauder's theorem, respectively. Several authors have extended the finite-dimensional controllability results to infinite-dimensional controllability results represented by evolution equations with bounded and unbounded operators in Banach spaces (e.g., see Balachandran et al. [4] and Dauer and Balasubramaniam [7]).

The semigroup theory gives a unified treatment of a wide class of stochastic parabolic, hyperbolic, and functional differential equations. Much effort has been devoted to the study of the controllability of such evolution equations (Rabah and Karrakchou [11]). Controllability of nonlinear stochastic systems has been a well-known problem and frequently discussed in the literature (e.g., Aström [3], Wonham [13], and Zabczyk [15]). The stochastic control theory is a stochastic generalization of the classical control theory. The purpose of this paper is to consider the controllability of semilinear stochastic delay systems represented by evolution equations with unbounded linear operators in Hilbert spaces. The Banach fixed point theorem (see [1]) is employed to obtain the suitable controllability conditions.

The system considered in this paper is an abstract formulation of the stochastic partial differential equation discussed by Liu [8]. For an example, a stochastic model for drug distribution was described in [12]. This model is a closed biological system with a simplified heart, a one organ or capillary bed, and recirculation of the blood with a constant rate of flow, where the heart is considered as a mixing chamber of constant volume. The drug concentration in the plasma in given areas of the system are assumed to be a random function of time. It is further assumed that for $t \geq 0$, $x_1(s,t;\omega)$ is the concentration in moles per unit volume at points (represented by s) in the capillary at time t with $\omega \in \Omega$, the supporting set of a complete probability measure space $(\Omega, A, \mathbf{P})$ with A being the σ-algebra and $\mathbf{P}$ the probability measure.

The heart is considered as a mixing chamber of constant volume given by

$$V = \frac{V_e}{\ln(1+V_e/V_r)}, \tag{1.1}$$

where V_r is the residual volume of the heart and V_e is the injection volume. It is assumed that an initial injection is given at the entrance of the heart resulting in a concentration $x(t)$, $0 \le t \le T$, of drug in plasma entering the heart, where T is the duration of injection. Let the time required for the blood to flow from the heart exit to the entrance of the organ be $\tau > 0$, and also let τ be the time required for blood to flow from the exit of the organ to the heart entrance. Then, the drug concentration in the plasma leaving the heart $x(\cdot;\omega)$ satisfies the integral equation (see [6])

$$x(t;\omega) = G(t) + \int_0^T K(s,x(s;\omega);\omega)\,ds, \quad 0 \le t \le T, \tag{1.2}$$

where

$$\begin{gathered} G(t) = \int_0^{T(t)} \frac{C}{V} x(s)\,ds, \quad T(t) = \{t, \text{ for } 0 \le t \le T, \text{ and } T, \text{ for } t \ge T\}, \\ K(s,x(s;\omega);\omega) = -\frac{C}{V}[x(s;\omega) - x_1(l,s-\tau;\omega)], \end{gathered} \tag{1.3}$$

and $x_1(l,s;\omega) = 0$ if $s < 0$. Here, C is the constant volume flow rate of plasma in the capillary bed and $x_1(l,s;\omega)$ is the concentration of drug in plasma leaving the organ at time s. The mild solutions of such integral equations are of the form in stochastic integral equations.

Stochastic delay equations serve as an abstract formulation of many partial differential equations that arise in problems of heat flow in material with memory, viscoelasticity, and many other physical phenomena (for details, see [8, 12] and the references therein). The main objective of this paper is to derive controllability conditions for semilinear stochastic delay evolution equations in Hilbert spaces.

2. Preliminaries.

Consider the semilinear stochastic delay evolution equation

$$\begin{aligned} \frac{dx(t)}{dt} + Ax(t) &= (Bu)(t) + f(t,x(t),x(t-\tau(t))) \\ &\quad + g(t,x(t),x(t-\tau(t)))\frac{dw(t)}{dt}, \quad t \in J = [0,T], \\ x(t) &= \psi(t), \quad t \in [-r,0], \end{aligned} \tag{2.1}$$

where $T > 0$ and A is a linear operator (in general unbounded), defined on a given Hilbert space X with an infinitesimal generator of an analytic semigroup $S(t)$, $t \ge 0$. The state $x(\cdot)$ takes its values in the Hilbert space X, and the control function $u(\cdot)$ is in $L^2(J,U)$, the Hilbert space of admissible control functions with U a Hilbert space. B is a bounded linear operator from U into X.

Let K be a separable Hilbert space, and let $(\Omega,\mathfrak{I},\mathfrak{I}_t,\mathbf{P})$ be a complete probability space furnished with a complete family of right continuous increasing sigma algebras $\{\mathfrak{I}_t\}$ satisfying $\mathfrak{I}_t \subset \mathfrak{I}$ for $t \ge 0$. The process $\{w(t),\ t \ge 0\}$ is a K-valued, $\mathfrak{I}_t$-adapted

Brownian motion with $\mathbf{P}\{w(0) = 0\} = 1$, and $\psi(\cdot)$ is an X-valued $\Im_0$-measurable random variable independent of the Brownian motion $w(\cdot)$.

For any Banach space F, let $L_2(\Omega, F)$ denote the space of strongly measurable, F-valued, square integrable random variables equipped with the norm topology

$$\|x\|_{L_2(\Omega,F)} = \{E\|x\|_F^2\}^{1/2}, \tag{2.2}$$

where E is defined as integration with respect to the probability measure $\mathbf{P}$. Then $L_2(\Omega, F)$ is also a Hilbert space since F is a Hilbert space. Let $\tau(\cdot)$ be a continuous nonnegative function on $\mathbb{R}^+$ and define $r = \sup\{\tau(t) - t : t \geq 0\} < \infty$. Let $\psi \in L_2^0([-r,0], X_\alpha)$, the family of all continuous square integrable stochastic processes $\psi(\cdot)$ such that $\sup\{E\|\psi\|_\alpha^2\} < \infty$, for $-r \leq t \leq 0$. Let $I = [-r, T]$ and $M(I,F)$ denote the space of $\Im_t$-adapted stochastic processes defined on I, taking values in F, having square integrable norms, that are continuous in t on I in the mean square sense. This is a Banach space with respect to the norm topology

$$\|\xi\|_{M(I,F)} = \left\{\sup_{t\in I} E\|\xi(t)\|_F^2\right\}^{1/2}, \quad \xi \in M(I,F). \tag{2.3}$$

Assume the following conditions:

(i) for $0 \leq \alpha < 1/2$, $X_\alpha = [D(A^\alpha)]$ is a Banach space with respect to the graph topology induced by the graph norm

$$\|x\|_\alpha = \|A^\alpha x\| + \|x\|, \quad \text{for } x \in D(A^\alpha); \tag{2.4}$$

(ii) the function f maps X_α to X and there exists a constant $C > 0$ such that

$$\begin{aligned} \|f(t,x,y) - f(t,\bar{x},\bar{y})\|_X &\leq C(\|x-\bar{x}\|_\alpha + \|y-\bar{y}\|_\alpha), \\ \|f(t,x,y)\|_X &\leq C\{1 + \|x\|_\alpha + \|y\|_\alpha\} \quad \forall x, y \in X_\alpha; \end{aligned} \tag{2.5}$$

(iii) the function g maps X_α to $L(K,X)$ and there exists a constant $C > 0$ such that

$$\begin{aligned} \|g(t,x,y) - g(t,\bar{x},\bar{y})\|_{L(K,X)} &\leq C(\|x-\bar{x}\|_\alpha + \|y-\bar{y}\|_\alpha), \\ \|g(t,x,y)\|_{L(K,X)} &\leq C\{1 + \|x\|_\alpha + \|y\|_\alpha\}; \end{aligned} \tag{2.6}$$

(iv) the linear operator W from $L^2(J,U)$ into X defined by

$$Wu = \int_0^T S(T-s)Bu(s)\,ds \tag{2.7}$$

has an invertible operator W^{-1} defined on $X \setminus \ker W$ (see [9]) and there exist the positive constants N_1, N_2 such that

$$\|B\|^2 \leq N_1, \qquad \|W^{-1}\|^2 \leq N_2. \tag{2.8}$$

Here, $L(K,X)$ is the family of all bounded linear operators from K into X, equipped with the usual operator norm topology, and w is a $\Im_t$-adapted Brownian motion having a nuclear covariance operator $Q \in L_n^+(F)$.

By the assumptions (i), (ii), and (iii), there exists a unique stochastic process $x(\cdot) \in M(I, X_\alpha)$, that is, a solution of (2.1) (see [1, 8]) such that $x(\cdot)$ is $\Im_t$-adapted, measurable, and almost surely that $\int_{-r}^{T} \|x(s)\|_\alpha^2 ds < \infty$, with

$$\begin{aligned}
x(t) &= S(t)\psi(0) + \int_0^t S(t-s)\left[(Bu)(s) + f(s, x(s), x(s-\tau(s)))\right] ds \\
&\quad + \int_0^t S(t-s) g(s, x(s), x(s-\tau(s))) dw(s), \quad t \geq 0, \\
x(t) &= \psi(t), \quad t \in [-r, 0].
\end{aligned} \tag{2.9}$$

Definition 2.1. The stochastic system (2.1) is said to be *controllable* on J, if for every continuous initial random process $\psi(\cdot)$ defined on $[-r, 0]$, there exists a control $u \in L^2(J, U)$ such that the solution of (2.1) satisfies $x(T) = x_1$, where x_1 and T are preassigned terminal state and time, respectively. If the system is controllable for all x_1 at $t = T$, it is called completely controllable on J.

3. Main results

Theorem 3.1. *Suppose that conditions (i), (ii), (iii), and (iv) are satisfied, then system (2.1) is completely controllable on J.*

Proof. Using assumption (iv), define the control

$$\begin{aligned}
u(t) = W^{-1}\Big[x_1 - S(T)\psi(0) &- \int_0^T S(T-s) f(s, x(s), x(s-\tau(s))) ds \\
&- \int_0^T S(T-s) g(s, x(s), x(s-\tau(s))) dw(s)\Big](t).
\end{aligned} \tag{3.1}$$

Now, it is shown that when using this control the operator defined by

$$\begin{aligned}
(\Phi x)(t) &= S(t)\psi(0) \\
&\quad + \int_0^t S(t-\mu) B W^{-1}\Big[x_1 - S(T)\psi(0) - \int_0^T S(T-s) f(s, x(s), x(s-\tau(s))) ds \\
&\qquad\qquad - \int_0^T S(T-s) g(s, x(s), x(s-\tau(s))) dw(s)\Big](\mu) d\mu \\
&\quad + \int_0^t S(t-s) f(s, x(s), x(s-\tau(s))) ds \\
&\quad + \int_0^t S(t-s) g(s, x(s), x(s-\tau(s))) dw(s), \quad t \in J, \\
(\Phi x)(t) &= \psi(t), \quad -r \leq t \leq 0,
\end{aligned} \tag{3.2}$$

has a fixed point. This fixed point is a solution of (2.1). Clearly $(\Phi x)(0) = \psi(0)$, which means that the control $u(\cdot)$ steers the semilinear stochastic delay differential system from the initial state $\psi(\cdot)$ to x_1 in time T provided the nonlinear operator Φ has a fixed point.

First, it must be shown that Φ maps $M(I, X_\alpha)$ into $M(I, X_\alpha)$. Without loss of generality, assume that $0 \in \rho(A)$. Otherwise, if $0 \notin \rho(A)$, for the identity operator I add the term νI to A giving $A_\nu = A + \nu I$, then $0 \in \rho(A_\nu)$. This simplifies the graph norm to $\|\zeta\|_\alpha = \|A^\alpha \zeta\|$, for $\zeta \in D(A^\alpha)$. Since $S(t)$, $t \geq 0$, is an analytic semigroup and A^α is a closed operator, there exist numbers $C_1 \geq 1$ and C_α such that

$$\sup_{t\in J} \|S(t)\|_{L(X)}^2 \leq C_1, \quad \|A^\alpha S(t)\|_{L(X)} \leq C_\alpha t^{-\alpha}, \quad \text{for } t > 0. \tag{3.3}$$

Further, $|a+b+c|^2 \leq 9(|a|^2 + |b|^2 + |c|^2)$ for any real numbers a, b, c. Hence, for $x \in M(I, X_\alpha)$,

$$E\left(\sup_{t\in[-r,0]} \|(\Phi x)(t)\|_X^2\right) \leq E\left(\sup_{t\in[-r,0]} \|\psi(t)\|_X^2\right) < \infty, \quad \text{for } -r \leq t \leq 0, \tag{3.4}$$

and for $t \in J$,

$$\begin{aligned}
&E\left(\sup_{t\in J} \|(\Phi x)(t)\|_X^2\right) \\
&\quad \leq 9 \sup_{t\in J} E\left(\|S(t)\psi(0)\|_\alpha^2\right) \\
&\qquad + 9E\bigg\| \int_0^t S(t-\mu) B W^{-1} \bigg[x_1 - S(T)\psi(0) \\
&\qquad\qquad\qquad\qquad - \int_0^T S(T-s) f(s, x(s), x(s-\tau(s)))\, ds \\
&\qquad\qquad\qquad\qquad - \int_0^T S(T-s) g(s, x(s), x(s-\tau(s)))\, dw(s) \bigg](\mu) \bigg\|_\alpha^2 d\mu \\
&\qquad + 9E\left\| \int_0^t S(t-s) f(s, x(s), x(s-\tau(s)))\, ds \right\|_\alpha^2 \\
&\qquad + 9 T_r Q \int_0^t E\left(\|A^\alpha S(t-s) g(s, x(s), x(s-\tau(s)))\|_{L(K,X)}^2\right) ds \\
&\quad \leq 9 \sup_{t\in J} E\left(\|A^\alpha S(t)\psi(0)\|_X^2\right) \\
&\qquad + 9 N_1 N_2 \int_0^t \|A^\alpha S(T-\mu)\|_{L(X)}^2 d\mu \\
&\qquad \times \bigg[E\|x_1\|_\alpha^2 + E\|A^\alpha S(T)\psi(0)\|_X^2 \\
&\qquad\quad + \left(\int_0^T \|A^\alpha S(T-s)\|_{L(X)}^2 ds \right) E \int_0^T \|f(s, x(s), x(s-\tau(s)))\|_X^2 ds \\
&\qquad\quad + E \int_0^T \|A^\alpha S(T-s) g(s, x(s), x(s-\tau(s)))\, dw(s)\|_X^2 \bigg] \\
&\qquad + 9\left(\int_0^t \|A^\alpha S(t-s)\|_{L(X)}^2 ds \right) E \int_0^t \|f(s, x(s), x(s-\tau(s)))\|_X^2 ds \\
&\qquad + 9 T_r Q \int_0^t E\left(\|A^\alpha S(t-s) g(s, x(s), x(s-\tau(s)))\|_{L(K,X)}^2\right) ds
\end{aligned}$$

$$\begin{aligned}
&\leq 9C_1E\Big(\|\psi(0)\|_\alpha^2\Big)+(9N_1N_2C_\alpha^2)\frac{T^{(1-2\alpha)}}{(1-2\alpha)}\\
&\quad\times\Big[E\|x_1\|_\alpha^2+C_1E\Big(\|\psi(0)\|_\alpha^2\Big)\\
&\qquad+[C_\alpha C]^2\frac{T^{2(1-\alpha)}}{(1-2\alpha)}\Big\{1+\sup_{0\leq s\leq t}E\|x(s)\|_\alpha^2+\sup_{0\leq s\leq t}E\|x(s-\tau(s))\|_\alpha^2\Big\}\\
&\qquad+T_rQ\Big[2(C_\alpha C)^2\Big]\frac{T^{(1-2\alpha)}}{(1-2\alpha)}\Big\{1+\sup_{0\leq s\leq t}E\|x(s)\|_\alpha^2+\sup_{0\leq s\leq t}E\|x(s-\tau(s))\|_\alpha^2\Big\}\Big]\\
&\quad+9\Big\{[C_\alpha C]^2\frac{t^{2(1-\alpha)}}{(1-2\alpha)}\Big\}\Big\{1+\sup_{0\leq s\leq t}E\|x(s)\|_\alpha^2+\sup_{0\leq s\leq t}E\|x(s-\tau(s))\|_\alpha^2\Big\}\\
&\quad+9T_rQ\Big[2(C_\alpha C)^2\Big]\frac{T^{(1-2\alpha)}}{(1-2\alpha)}\Big\{1+\sup_{0\leq s\leq t}E\|x(s)\|_\alpha^2+\sup_{0\leq s\leq t}E\|x(s-\tau(s))\|_\alpha^2\Big\}\\
&\leq 9C_1E\Big(\|\psi(0)\|_\alpha^2\Big)+(9N_1N_2C_\alpha^2)\frac{T^{(1-2\alpha)}}{(1-2\alpha)}\\
&\quad\times\Big[E\|x_1\|_\alpha^2+C_1E\Big(\|\psi(0)\|_\alpha^2\Big)+[C_\alpha C]^2\frac{T^{2(1-\alpha)}}{(1-2\alpha)}\Big\{1+2\|x\|_{M(I,X_\alpha)}^2\Big\}\\
&\qquad+2T_rQ(C_\alpha C)^2\frac{T^{(1-2\alpha)}}{(1-2\alpha)}\Big\{1+2\|x\|_{M(I,X_\alpha)}^2\Big\}\Big]\\
&\quad+9[C_\alpha C]^2\frac{T^{2(1-\alpha)}}{(1-2\alpha)}\Big\{1+2\|x\|_{M(I,X_\alpha)}^2\Big\}\\
&\quad+18T_rQ\Big\{(C_\alpha C)^2\frac{T^{(1-2\alpha)}}{(1-2\alpha)}\Big\}\Big\{1+2\|x\|_{M(I,X_\alpha)}^2\Big\}\\
&\leq 9C_1E\Big(\|\psi(0)\|_\alpha^2\Big)+(9N_1N_2C_\alpha)\frac{T^{(1-2\alpha)}}{(1-2\alpha)}\\
&\quad\times\Big[E\|x_1\|_\alpha^2+C_1E\Big(\|\psi(0)\|_\alpha^2\Big)+\eta(T_rQ)\Big]+9\eta(T_rQ),
\end{aligned}\tag{3.5}$$

where T_rQ represents the trace of the operator Q and

$$\eta(T_rQ)=[C_\alpha C]^2\{T+2T_rQ\}\frac{T^{(1-2\alpha)}}{(1-2\alpha)}\Big\{1+2\|x\|_{M(I,X_\alpha)}^2\Big\}.\tag{3.6}$$

Hence $\sup_{t\in I}\|(\Phi x)(t)\|_\alpha^2<\infty$ for $x\in M(I,X_\alpha)$.

Since $\psi(\cdot)$ is continuous in $[-r,0]$, to complete the proof it remains to show that $\Phi\in C((0,T),L_2(\Omega,X_\alpha))$. To accomplish that, let $t\in(0,T)$, $h>0$ and $t+h\in J$. For analytic semigroups, there exists a constant $\nu_\beta>0$ such that

$$\|(S(h)-I)\xi\|_X\leq\nu_\beta h^\beta\|A^\beta\xi\|_X\quad\forall\xi\in D(A^\beta)\tag{3.7}$$

and for all $\beta\geq 0$ and all $\zeta\in X$ with $S(t)\zeta\in D(A^\beta)$ for $t>0$ (see Pazy [10, Theorem 6.13]). Thus, for $t>0$, the closedness of A^α and the fact that $S(t)$ commutes with A^α

on $D(A^\alpha)$ yields that by choosing $\beta > 0$ such that $0 \le \alpha+\beta \le 1/2$, we have

$$
\begin{aligned}
&E\Big\{||(\Phi x)(t+h)-(\Phi x)(t)||_\alpha^2\Big\}\\
&\quad\le 9E\Big(||(S(h)-I)S(t)A^\alpha\psi(0)||_\alpha^2\Big)\\
&\qquad+9E\bigg|\bigg|\int_0^t (S(h)-I)A^\alpha S(t-\mu)BW^{-1}\\
&\qquad\qquad\times\bigg[x_1-S(T)\psi(0)-\int_0^T S(T-s)f(s,x(s),x(s-\tau(s)))ds\\
&\qquad\qquad\quad-\int_0^T S(T-s)g(s,x(s),x(s-\tau(s)))dw(s)\bigg](\mu)\bigg|\bigg|_\alpha^2 d\mu\\
&\qquad+9E\bigg|\bigg|\int_t^{t+h} A^\alpha S(t+h-\mu)BW^{-1}\\
&\qquad\qquad\times\bigg[x_1-S(T)\psi(0)-\int_0^T S(T-s)f(s,x(s),x(s-\tau(s)))ds\\
&\qquad\qquad\quad-\int_0^T S(T-s)g(s,x(s),x(s-\tau(s)))dw(s)\bigg](\mu)\bigg|\bigg|_\alpha^2 d\mu\\
&\qquad+9E\bigg|\bigg|\int_0^t (S(h)-I)A^\alpha S(t-s)f(s,x(s),x(s-\tau(s)))ds\bigg|\bigg|_\alpha^2\\
&\qquad+9E\bigg|\bigg|\int_t^{t+h} A^\alpha S(t+h-s)f(s,x(s),x(s-\tau(s)))ds\bigg|\bigg|_\alpha^2\\
&\qquad+9E\bigg|\bigg|\int_0^t (S(h)-I)A^\alpha S(t-s)g(s,x(s),x(s-\tau(s)))dw(s)\bigg|\bigg|_{L(K,X)}^2\\
&\qquad+9E\bigg|\bigg|\int_t^{t+h} A^\alpha S(t+h-s)g(s,x(s),x(s-\tau(s)))dw(s)\bigg|\bigg|_{L(K,X)}^2\\
&\quad\le 9\nu_\beta^2 h^{2\beta}||A^\beta S(t)||^2 E||A^\alpha\psi(0)||_\alpha^2\\
&\qquad+9N_1N_2\nu_\beta^2 C_{\alpha+\beta}^2 h^{2\beta}\int_0^t\bigg[\frac{1}{(t-\mu)}\bigg]^{2(\alpha+\beta)}d\mu\\
&\qquad\times\bigg[E\Big(||x_1||_\alpha^2\Big)+C_1E\Big(||\psi(0)||_\alpha^2\Big)+[C_\alpha C]^2\{T+2T_rQ\}\frac{T^{(1-2\alpha)}}{(1-2\alpha)}\\
&\qquad\quad\times\Big\{1+\sup_{0\le s\le t}E||x(s)||_\alpha^2+\sup_{0\le s\le t}E||x(s-\tau(s))||_\alpha^2\Big\}\bigg]\\
&\qquad+9N_1N_2C_\alpha^2\int_t^{t+h}\bigg[\frac{1}{(t+h-\mu)}\bigg]^{2\alpha}d\mu\\
&\qquad\times\bigg[E\Big(||x_1||_\alpha^2\Big)+C_1E\Big(||\psi(0)||_\alpha^2\Big)+\Big\{[C_\alpha C]^2\Big\}\{T+2T_rQ\}\frac{T^{(1-2\alpha)}}{(1-2\alpha)}\\
&\qquad\quad\times\Big\{1+\sup_{0\le s\le t}E||x(s)||_\alpha^2+\sup_{0\le s\le t}E||x(s-\tau(s))||_\alpha^2\Big\}\bigg]\\
&\qquad+9\nu_\beta^2C_{\alpha+\beta}^2h^{2\beta}E\bigg(\int_0^t\bigg[\frac{1}{(t-s)}\bigg]^{2(\alpha+\beta)}||f(s,x(s),x(s-\tau(s)))||_\alpha^2 ds\bigg)\\
&\qquad+9C_\alpha^2E\bigg(\int_t^{t+h}\bigg[\frac{1}{(t+h-s)}\bigg]^{2\alpha}||f(s,x(s),x(s-\tau(s)))||_\alpha^2 ds\bigg)\\
&\qquad+9T_rQ\nu_\beta^2C_{\alpha+\beta}^2h^{2\beta}E\bigg(\int_0^t\bigg[\frac{1}{(t-s)}\bigg]^{2(\alpha+\beta)}||g(s,x(s),x(s-\tau(s)))||_{L(K,X)}^2 ds\bigg)\\
&\qquad+9T_rQC_\alpha^2E\bigg(\int_t^{t+h}\bigg[\frac{1}{(t+h-s)}\bigg]^{2\alpha}||g(s,x(s),x(s-\tau(s)))||_{L(K,X)}^2 ds\bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\le 9(\nu_\beta C_\beta)^2\left(\frac{h}{t}\right)^{2\beta} E\|\psi(0)\|_\alpha^2 + 9N_1N_2 \\
&\quad\times\left\{(\nu_\beta C_{\alpha+\beta})^2\left(\frac{h}{t}\right)^{2\beta}\frac{t^{(1-2\alpha-2\beta)}}{(1-2\alpha-2\beta)} + C_\alpha^2\frac{h^{2(1-\alpha)}}{(1-2\alpha)}\right\} \\
&\quad\times\left\{E\left(\|x_1\|_\alpha^2 + C_1E\|\psi(0)\|_\alpha^2 + \eta(T_rQ)\right)\right\} \\
&\quad+9(\nu_\beta CC_{\alpha+\beta})^2\left(\frac{h}{t}\right)^{2\beta}\frac{t^{2(1-\alpha-\beta)}}{(1-2\alpha-2\beta)}\left\{1+2\sup_{s\in I}E\|x(s)\|_\alpha^2\right\} \\
&\quad+9(CC_\alpha)^2\frac{h^{2(1-\alpha)}}{(1-2\alpha)}\left\{1+2\sup_{s\in I}E\|x(s)\|_\alpha^2\right\} \\
&\quad+9T_rQ(\nu_\beta CC_{\alpha+\beta})^2\left(\frac{h}{t}\right)^{2\beta}\frac{t^{2(1-\alpha-\beta)}}{(1-2\alpha-2\beta)}\left\{1+2\sup_{s\in I}E\|x(s)\|_\alpha^2\right\} \\
&\quad+9T_rQ(CC_\alpha)^2\frac{h^{(1-2\alpha)}}{(1-2\alpha)}\left\{1+2\sup_{s\in I}E\|x(s)\|_\alpha^2\right\}
\end{aligned}
\tag{3.8}
$$

for $t\in(0,T)$. Thus, letting $h\to 0$, the desired continuity follows. Hence Φ maps $M(I,X_\alpha)$ into itself.

Now, it is shown that for sufficiently small T, defining the interval I leads to a contraction in $M(I,X_\alpha)$. Indeed, for $x,y\in M(I,X_\alpha)$ satisfying $x(t)=y(t)=\psi(t)$ for $-r\le t\le 0$ it can be easily seen that

$$
\sup_{t\in J}E\|(\Phi x)(t)-(\Phi y)(t)\|^2 \le K_\alpha \sup_{t\in J}E\|x(t)-y(t)\|_\alpha^2, \tag{3.9}
$$

where

$$
K_\alpha = 9N_1N_2C_\alpha^2[C_\alpha C]^2\{T+2T_rQ\}\frac{T^{2(1-2\alpha)}}{(1-2\alpha)^2} + 9[C_\alpha C]^2\{T+2T_rQ\}\frac{T^{(1-2\alpha)}}{(1-2\alpha)}. \tag{3.10}
$$

Thus, for sufficiently small T, $K_\alpha<1$ and Φ is a contraction in $M(I,X_\alpha)$ and so, by the Banach fixed point theorem (see [1]), Φ has a unique fixed point $x\in M(I,X_\alpha)$. Any fixed point of Φ is a solution of (2.1) on J satisfying $(\Phi x)(t)=x(t)\in X$, for all $\psi(\cdot)$ and $T>0$. Thus, system (2.1) is completely controllable on J. □

4. Example. Consider a stochastic Burgers-type equation with constant time delay (i.e., $\tau(t)=2h>0$). Assume $\nu>0$, $\psi(t,\xi):[-2h,0]\times\Omega\to X=L^2[0,1]$ is a suitable $\Im_0$-measurable process and for $t\ge 0$, $\xi\in[0,1]$,

$$
\begin{aligned}
\frac{dY_t(\xi)}{dt} &= \nu\frac{\partial^2Y_t(\xi)}{\partial\xi^2}+\frac{1}{2}\frac{\partial Y_t^2(\xi)}{\partial\xi} \\
&\qquad + Y_{t-2h}(\xi)+(Bu)(t)+2t^3e^{-\eta\lambda_0 t}\frac{dw_t(\xi)}{dt},
\end{aligned}
\tag{4.1}
$$
$$
\begin{aligned}
&Y_t(0)=Y_t(1)=0, \quad t>0, \\
&Y_t(\xi)=\psi(t,\xi), \quad \xi\in[0,1],\ t\in[-2h,0],
\end{aligned}
$$

with the following assumptions:

(1) let $\operatorname{dom} A = H^2(0,1) \cap H_0^1(0,1)$ and $(A\phi)\xi = \nu(\partial^2 Y_t(\xi)/\partial\xi^2)$, $\phi \in \operatorname{dom} A$, and let B be a bounded linear operator from the control space $U = L^2(0,1)$ into H satisfying the hypothesis (iv);

(2) define the functions

$$\begin{aligned} f(t, Y_t(\xi), Y_{t-2h}(\xi)) &= \frac{1}{2}\frac{\partial Y_t^2(\xi)}{\partial\xi} + Y_{t-2h}(\xi), \\ g(t, Y_t(\xi), Y_{t-2h}(\xi)) &= 2t^3 e^{-\eta\lambda_0 t}, \end{aligned} \tag{4.2}$$

with

$$\lambda_0 = \inf_{y \in D(A)} \frac{|\nabla y(\xi)|^2}{|y(\xi)|^2}; \tag{4.3}$$

(3) let $w_t(\xi)$ be a Wiener process with a bounded, continuous covariance $q(\xi,\zeta)$; namely, there exists a constant $c > 0$ such that $|q(\xi,\zeta)| \le c$.

Then, system (4.1) has an abstract formulation given by the following semilinear stochastic equation in Hilbert space

$$\begin{aligned} \frac{dx(t)}{dt} &= Ax(t) + (Bu)(t) + f(t, x(t), x(t-\tau(t))) \\ &\quad + g(t, x(t), x(t-\tau(t)))\frac{dw(t)}{dt}, \quad t \in J = [0,T], \\ x(t) &= \psi(t), \quad -2h \le t \le 0, \end{aligned} \tag{4.4}$$

where the linear operator A is the infinitesimal generator of a strongly continuous semigroup e^{At}, $t \ge 0$ in H. Thus (4.4) has a unique solution (see [8]).

All the conditions stated in the Theorem 3.1 are satisfied, and so system (4.1) is completely controllable on J.

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