IJMMS 31:5 (2002) 301–305
PII. S0161171202007925
http://ijmms.hindawi.com


# ON INCIDENCE ALGEBRAS AND DIRECTED GRAPHS

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Received 20 June 2001

The incidence algebra $I(X,\mathbb{R})$ of a locally finite poset $(X,\le)$ has been defined and studied by Spiegel and O'Donnell (1997). A poset $(V,\le)$ has a directed graph $(G_v,\le)$ representing it. Conversely, any directed graph $G$ without any cycle, multiple edges, and loops is represented by a partially ordered set $V_G$. So in this paper, we define an incidence algebra $I(G,\mathbb{Z})$ for $(G,\le)$ over $\mathbb{Z}$, the ring of integers, by $I(G,\mathbb{Z}) = \{f_i, f_i^* : V\times V \to \mathbb{Z}\}$ where $f_i(u,v)$ denotes the number of directed paths of length $i$ from $u$ to $v$ and $f_i^*(u,v) = -f_i(u,v)$. When $G$ is finite of order $n$, $I(G,\mathbb{Z})$ is isomorphic to a subring of $M_n(\mathbb{Z})$. Principal ideals $I_v$ of $(V,\le)$ induce the subdigraphs $\langle I_v\rangle$ which are the principal ideals $\mathcal{I}_v$ of $(G_v,\le)$. They generate the ideals $I(\mathcal{I}_v,\mathbb{Z})$ of $I(G,\mathbb{Z})$. These results are extended to the incidence algebra of the digraph representing a locally finite weak poset both bounded and unbounded.

2000 Mathematics Subject Classification: 05C20, 06A11.

**1. Introduction.** The incidence algebra $I(X,\mathbb{R})$ of a locally finite poset $X$ over a commutative ring $\mathbb{R}$ with identity is discussed in much detail by Spiegel and O'Donnell [5]. Every poset $(V,\le)$ has a directed graph $(G_v,\le)$, or $(G,\le)$ for short, representing it. The directed graph $(G,\le)$ is free of cycles and multiple arcs. It is natural for $(G,\le)$ to have an incidence algebra whose properties depend on those of the directed graph.

In this paper, we define and study an incidence algebra $I(G,\mathbb{Z})$ for $(G,\le)$ over the ring $\mathbb{Z}$ of all integers. Section 2 contains the basic results and definitions used in this paper. Section 3 deals with $I(G,\mathbb{Z})$, the incidence algebra of $(G,\le)$ representing a finite poset $(V,\le)$ of cardinality $n$. Ideals of $I(G,\mathbb{Z})$ are given through the principal ideals of $(G,\le)$ which are the subdigraphs induced by the principal ideals of $(V,\le)$. An extension of the results given in Section 3 to the digraph representing a locally finite weak poset $(V,\le)$ bounded or unbounded is the content of Section 4.

**2. Definitions and basic results.** We consider the directed graph $(G_v,\le)$ representing a locally finite partially ordered set $(V,\le)$. The terminology introduced here is used throughout the paper.

**Definition 2.1.** For any $u\le v$ in $(V,\le)$, $[u,v] = \{w : u\le w\le v\}$ is an interval of $(V,\le)$. The length of $[u,v]$ denoted by $l[u,v]$ is the length of the longest chain in $[u,v]$. A poset $(V,\le)$ is locally finite if $l[u,v]$ is finite for each $[u,v]$ in $(V,\le)$; and $(V,\le)$ is bounded if there is a $k>0$ such that $l[u,v]\le k$ for all $[u,v]$ in $(V,\le)$. Otherwise $(V,\le)$ is unbounded.

**Definition 2.2.** The directed graph $(G_v, \leq)$ associated with a locally finite poset $(V, \leq)$ is defined as $(G_v, \leq) = (V, E)$ where $V = (V, \leq)$ and $E = \{\text{arcs } \vec{uv}; u < v\}$; $(G_v, \leq)$ has no cycles nor multiple arcs.

**Lemma 2.3** (see [5]). *A finite poset* $(V, \leq)$ *can be labelled* $V = \{v_1, v_2, \dots, v_n\}$ *so that* $v_i \leq v_j$ *implies that* $i \leq j$.

**Proposition 2.4** (see [4]). *The vertices of a finite directed graph* $G = (V, E)$ *can be labelled* $v_1, v_2, \dots, v_n$ *such that* $\vec{v_i v_j} \in E$ *implies that* $i < j$, *if and only if* $G$ *has no cycles.*

**Note 2.5.** Lemma 2.3 and Proposition 2.4 have motivated the orientation of arcs in $(G_v, \leq)$.

**Definition 2.6** (see [5]). Let $(X, \leq)$ be a locally finite poset and $\mathbb{R}$ a commutative ring with identity. The incidence algebra $I(X, \mathbb{R})$ of $X$ over $\mathbb{R}$ is defined by

$$I(X, \mathbb{R}) = \{f : X \times X \longrightarrow \mathbb{R} : f(x, y) = 0 \text{ if } x \not\leq y\}, \tag{2.1}$$

with operations defined by

(i) $(f + g)(x, y) = f(x, y) + g(x, y)$;
(ii) $(f \cdot g)(x, y) = \sum_{x \leq u \leq y} f(x, u) g(u, y)$;
(iii) $(rf)(x, y) = rf(x, y)$, for all $r \in \mathbb{R}$; $f, g \in I(X, \mathbb{R})$.

**Definition 2.7.** For any $v \in (V, \leq)$, let $I_v = \{u \in V : u \leq v\}$; $I_v$ is called the principal ideal generated by $v$.

**Definition 2.8.** An ideal $\mathcal{I}$ of $(G_v, \leq)$ is an induced subdigraph of $G_v$ such that all directed paths with their terminal vertex in $\mathcal{I}$ are in $\mathcal{I}$.

If $I_v$ is a principal ideal of $(V, \leq)$, $\langle I_v \rangle$, the subdigraph induced by $I_v$ is the principal ideal of $G$ generated by $v$ in $(G_v, \leq)$. Denote $\langle I_v \rangle$ by $\mathcal{I}_v$.

**Notation 2.9.** For the remaining part of the paper, the digraphs $(G, \leq)$ and $(G_\infty, \leq)$ represent the finite poset $(V, \leq)$ and the locally finite poset $(V_\infty, \leq)$, respectively.

## 3. An incidence algebra for $(G, \leq)$.

In this section, we define an incidence algebra $I(G, \mathbb{Z})$ for the digraph $(G, \leq)$ representing the finite poset $(V, \leq)$. Subalgebras and ideals of $I(G, \mathbb{Z})$ are defined through principal ideals of $(G, \leq)$. Assume that $V = \{v_1, v_2, \dots, v_n\}$.

**Definition 3.1.** For $u, v \in V$, let $p_k(u, v)$ denote the number of directed paths of length $k$ from $u$ to $v$ and $p_k(v, u) = -p_k(u, v)$.

For $i = 0, 1, 2, \dots, n-1$, define $f_i, f_i^* : V \times V \to \mathbb{Z}$ by

$$f_i(u, v) = p_i(u, v), \qquad f_i^*(u, v) = -p_i(u, v). \tag{3.1}$$

The incidence algebra $I(G, \mathbb{Z})$ of $(G, \leq)$ over the commutative ring $\mathbb{Z}$ with identity is defined by $I(G, \mathbb{Z}) = \{fi, fi^* : V \times V \to \mathbb{Z}, i = 0, 1, 2, \dots, n-1\}$ with operations defined as follows:

(i) for $f \neq g$, $(f + g)(u, v) = f(u, v) + g(u, v)$;

(ii) $(f\cdot g)(u,v)=\sum_w f(u,w)g(w,v)$;
(iii) $(zf)(u,v)=zf(u,v)$, for all $z\in\mathbb{Z}$; $f,g\in I(G,\mathbb{Z})$.

**Remark 3.2.** The function $f_l$ is the digraph analogues of $\chi\in I(X,\mathbb{R})$ [5]. The matrix $[f_l,(v_i,v_j)]$ is the adjacency matrix of $(G,\le)$ and $f_l^k(v_i,v_j)=f_k(v_i,v_j)$. For any interval $[u,v]$ with $l[u,v]=k$, $f_l^k(u,v)=f_k(u,v)=0$. For every $f\in I(G,\mathbb{Z})$, there is a constant $m\in\mathbb{Z}$ such that $f^m(u,v)=0$, for all $(u,v)\in V\times V$.

**Definition 3.3.** With each ideal $\mathcal{I}_v=\langle I_v\rangle$ of $(G,\le)$ we associate an incidence algebra,

$$\begin{gathered} I(\mathcal{I}_v,\mathbb{Z})=\{f\in I(G,\mathbb{Z}):f:I_v\times I_v\longrightarrow\mathbb{Z}\},\\ \forall f\in I(\mathcal{I}_v,\mathbb{Z}),\ f(v_i,v_j)=0\quad\forall(v_i,v_j)\notin I_v\times I_v. \end{gathered}\tag{3.2}$$

**Remark 3.4.** If $(H,\le)$ is a subdigraph of $(G,\le)$, then $I(H,\mathbb{Z})$ is a subalgebra of $I(G,\mathbb{Z})$ [5]. In particular, $I(\mathcal{I}_v,\mathbb{Z})$ is a subalgebra of $I(G,\mathbb{Z})$; $I(\mathcal{I}_v,\mathbb{Z})$ is called the subalgebra generated by the vertex $v$.

**Remark 3.5** (see [5])**.** If $S$ is an ideal of $\mathbb{Z}$, $I(G,S)=\{f\in I(G,\mathbb{Z});f(u,v)\in S\}$ is a subalgebra of $I(G,\mathbb{Z})$.

**Proposition 3.6.** *For each principal ideal $\mathcal{I}_v$ of $(G,\le)$, $I(\mathcal{I}_v,\mathbb{Z})$ is an ideal of the ring $I(G,\mathbb{Z})$.*

**Proof.** Let $\mathcal{I}_v$ be a proper principal ideal of $(G,\le)$. Denote the elements of $I(G,\mathbb{Z})$ and $I(\mathcal{I}_v,\mathbb{Z})$ by $f$ and $f'$, respectively. For all $f'\in I(\mathcal{I}_v,\mathbb{Z})$, there is a unique $f\in I(G,\mathbb{Z})$ such that $f'(u,w)=f(u,w)$ for all $(u,w)\in I_v\times I_v$; and $f'(u,w)=0$ for all $(u,w)\notin I_v\times I_v$. □

Hence for any $(u,w)\in V\times V$, $f,g\in I(G,\mathbb{Z})$, $g'\in I(\mathcal{I}_v,\mathbb{Z})$

$$(fg')(u,w)=\begin{cases}(f,g)(u,w), & \text{if } (u,w)\in I_v\times I_v,\\ 0, & \text{otherwise,}\end{cases}\tag{3.3}$$

with similar values for $(g'f)(u,w)$ also.

Consequently, $fg'$ and $g'f\in I(\mathcal{I}_v,\mathbb{Z})$, and $I(\mathcal{I}_v,\mathbb{Z})$ is an ideal of $I(G,\mathbb{Z})$.

**Remark 3.7.** The incidence algebra $I(G,\mathbb{Z})$ is isomorphic to a subring of the ring of upper triangular matrices over $\mathbb{Z}$ [5]. In general, every ideal of $M_n(\mathbb{Z})$ has the form $M_n(S)$ for some ideal $S$ of $\mathbb{Z}$ [3].

**Proposition 3.8.** *Every ideal of $I(G,\mathbb{Z})$ has the form $I(\mathcal{I}_v,\mathbb{Z})$ for some principal ideal $\mathcal{I}_v$ of $(G,\le)$.*

**Proof.** Let $S$ be a proper ideal of the ring $I(G,\mathbb{Z})$. Then $S=I(H,\mathbb{Z})$ for some subdigraph $(H,\le)$ of $(G,\le)$. For all $f\in I(G,\mathbb{Z})$ and $g'\in S$, $fg'=g'f\in S$.

Consequently, there is an $h'\in S$ such that $fg'=h'$ satisfying $(fg')(u,v)=h'(u,v)=p_k(u,v)$ for some $k$ and for all $u,v$ in $V(H)$.

Then for all $(u,v) \in V(H)\times V(H)$, the number of paths of length $k$ in $H$ from $u$ to $v$ is the same as that in $G$.

Hence for any $v \in V(H)$, $H$ contains all the directed paths terminating in $v$. Then $H = \mathcal{I}_v$ for some $v$. □

**4. The incidence algebra $I(G_\infty,\mathbb{Z})$.** An extension of Proposition 2.4 is obtained by the author for locally finite directed graphs [2]. This provides an isomorphism between $I(G_\infty,\mathbb{Z})$ and a subring of the ring of upper triangular matrices over $\mathbb{Z}$. Also we have extended Propositions 3.6 and 3.8 to bounded as well as unbounded locally finite weak posets. It is assumed that the poset $(V,\le)$ is countable.

**Definition 4.1.** A locally finite poset $(V,\le)$ is weak if only finitely many chains intersect at every element $v$ in $V$.

**Definition 4.2.** A directed graph $G = (V,E)$ is locally finite if every vertex is of a finite degree.

**Definition 4.3.** A vertex $v$ of the directed graph $(G_\infty,\le)$, such that $d(v) \ne 0$, is called a source or atom if there is no other vertex $u$ such that $u \le v$.

**Proposition 4.4** (see [2]). *Let $G = (V,E)$ be a locally finite, acyclic digraph where $V$ is countable; $V$ can be labelled $\{v_1,v_2,v_3,\dots\}$ such that $\vec{v_iv_j} \in E$ implies that $i < j$ if and only if*

(i) *the set $S$ of its sources is nonempty and finite;*

(ii) *for each $v \in V$ and $s \in S$ such that $\vec{sv} \in E$, any maximal directed path terminating in $v$ is of finite length.*

**Definition 4.5.** Let $(V,\le)$ be a locally finite weak poset and $(G_\infty,\le)$ the digraph representing $(V,\le)$. The incidence algebra $I(G_\infty,\mathbb{Z})$ of $(G_\infty,\le)$ over the ring $\mathbb{Z}$ of integers is given by

$$I(G_\infty,\mathbb{Z}) = \{f_i, f_i^* : V\times V \longrightarrow \mathbb{Z}\} \tag{4.1}$$

satisfying the operations given in Definition 3.1.

**Definition 4.6.** An infinite matrix $A = [a_{ij}]$ is row (column) finite if $a_{ij} \ne 0$ for finitely many $j(i)$.

**Note 4.7.** (i) Each $f \in I(G_\infty,\mathbb{Z})$ is represented by a matrix $[f]$ where $[f]_{i,j} = f(v_i,v_j)$ and $[f]$ is both row and column finite, $M_\infty(\mathbb{Z})$ denotes the ring of row and column finite matrices over $\mathbb{Z}$. The incidence algebra $I(G_\infty,\mathbb{Z})$ is isomorphic to a subring of $M_\infty(\mathbb{Z})$.

(ii) A characterization of infinite directed graphs for which $[f]$ is nilpotent is obtained by the author in [1].

**Proposition 4.8.** *Let $(V,\le)$ be a weak, locally finite poset satisfying the following:*

(i) *the set $S$ of its atoms is nonempty and finite;*

(ii) *for each $v \in V$, $s \in S$ such that $s \le v$, any chain with $v$ as upperbound is of finite length. Then $I(G_\infty,\mathbb{Z})$ is isomorphic to a subring of the ring of upper triangular matrices over $\mathbb{Z}$.*

**Proof.** The digraph $(G_\infty, \leq)$ is locally finite. By Proposition 4.4, $f(v_r, v_s) \geq 0$ for all $r \leq s$ and for all $f \in I(G_\infty, \mathbb{Z})$. Hence every $f$ has a representation as an upper triangular matrix over $\mathbb{Z}$. □

**Definition 4.9.** In a bounded weak, locally finite poset $(V, \leq)$ a principal ideal $I_v = \{u : u \leq v\}$. The corresponding principal ideal of $(G_\infty, \leq)$ is given by $\mathcal{I}_v = \langle I_v \rangle$.

**Proposition 4.10.** *Let $(V, \leq)$ be any weak, locally finite bounded poset and $\mathcal{I}_v$ a principal ideal of $(G_\infty, \leq)$. Then*

(1) *$I(\mathcal{I}_v, \mathbb{Z})$ is a subalgebra of $I(G_\infty, \mathbb{Z})$;*
(2) *$I(\mathcal{I}_v, \mathbb{Z})$ is an ideal of $I(G_\infty, \mathbb{Z})$.*

**Note 4.11.** This follows from Propositions 3.6 and 4.4.

**Proposition 4.12** (see [5]). *Let $(V, \leq)$ be an unbounded weak poset. Then $V$ contains a subpartially ordered set isomorphic to $\mathbb{Z}^+$, $\mathbb{Z}^-$, or $UCn$ where $Cn$ denotes a chain of length $n$.*

**Remark 4.13.** When $(V, \leq)$ is unbounded, the principal ideals of $(V, \leq)$ and $(G_\infty, \leq)$ are not well defined. Hence Proposition 4.4 is not true for unbounded posets, in general.

But there are unbounded, locally finite posets satisfying Proposition 4.4. As an example we have $(V, \leq) = (\mathbb{Z}^+, \text{usual ordering})$.

Let $(V, \leq)$ be an unbounded, locally finite, weak poset such that $I(G_\infty, \mathbb{Z})$ is isomorphic to a subring of the ring of upper triangular matrices. Then

(i) principal ideals of $(G_\infty, \leq)$ are defined as $\mathcal{I}_v = \langle I_v \rangle$;
(ii) for each $\mathcal{I}_v$, $I(\mathcal{I}_v, \mathbb{Z})$ is a subalgebra of $I(G_\infty, \mathbb{Z})$;
(iii) for every $\mathcal{I}_v$, $I(\mathcal{I}_v, \mathbb{Z})$ is an ideal of $I(G_\infty, \mathbb{Z})$.

**Acknowledgment.** I would like to express my sincere gratitude to Prof. T. Thrivikraman, Head of the Department of Mathematics, Cochin University of Science and Technology, Kerala, India for his many valuable suggestions and continuing support throughout the preparation of this paper. Research under IX plan FIP of the UGC.

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