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A CLASS OF SIMPLE TRACIALLY AF C^*-ALGEBRAS

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The concept of a tracially AF (TAF) C^*-algebra was introduced recently to aid in the classification of nuclear C^*-algebras. Here, we construct and study a broad class of inductive-limit C^*-algebras. We give a numerical condition which, when satisfied, ensures that the corresponding algebra in our construction has the TAF property. We further give a necessary and sufficient condition under which certain of these C^*-algebras are TAF.

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1. Introduction. Much of recent C^*-algebra theory has been concentrated on the development of a noncommutative topology for C^*-algebras. The notions of real rank and topological stable rank, for example, have been successful in relating topological dimension to C^*-algebras. The AF algebras form a class of C^*-algebras, which may be viewed as an analogue to the dimension zero topological spaces. These C^*-algebras have a nice finite-dimensional approximation property but also have trivial K_1 groups. A broader class of C^*-algebras are the TAF algebras, which were introduced in [10] and provide a richer topological structure than that of the AF algebras (see Section 3 for the formal definition of TAF).

These TAF algebras have a large part approximable by finite-dimensional C^*-subalgebras, while the remaining part has arbitrarily small measure. They can have nontrivial K_1 groups and their K_0 groups may have torsion. Noncommutative analogues of both dimension and measure have influenced the development of the TAF C^*-algebra. The class of nuclear TAF C^*-algebras is broad but can in fact be classified. For example, all simple nuclear C^*-algebras classified in [5] are TAF, and every simple TAF C^*-algebra is quasidiagonal, has real rank zero, topological stable rank one, and weakly unperforated K_0 group. A classification theorem for TAF C^*-algebras can be found in [9].

In this paper, suggested by the work in [10, Section 4], we construct the inductive limit of matrix algebras over a separable, unital, residually finite-dimensional C^*-algebra (the source) using identity maps and finite-dimensional irreducible representations as the connecting monomorphisms. We investigate various properties of the inductive limit such as simplicity, real rank, topological stable rank, and TAF. We give a sufficient condition under which a large subclass of the constructed C^*-algebras are TAF, and show that this condition becomes necessary when the source has a bounded rank. This condition is numerically described by a single quantity.

In [7], Goodearl studied a similar construction using $C(X)$ for X separable, compact, Hausdorff, and connecting monomorphisms consisting of identity maps and point evaluations. There, it was shown that such an inductive limit has real rank zero exactly when $C(X)$ is an AF-algebra or the number of point evaluations eventually increases *much faster* than the number of identity maps. Here, we prove that the inductive limit of a separable, residually finite-dimensional, unital source with bounded rank is in fact TAF, exactly when, the source is AF or the dimensions of irreducible representations used in the connecting morphisms eventually increase more rapidly than the number of identity maps.

We begin by giving the details describing our construction and then proceed to investigate its properties. We then introduce a numerical condition to ensure the TAF property. Examples which help illustrating the breadth of these C^*-algebras are given. Finally, we prove the main characterization theorem of the paper: the numerical condition is a TAF-determinant when the source algebra has a bounded rank and is not AF.

2. The construction. We first establish notation and then proceed by induction to construct a direct sequence of C^*-algebras, which we use to form a C^*-inductive limit, the C^*-algebra of interest.

We begin with a unital, separable, residually finite-dimensional C^*-algebra B. A separable C^*-algebra is residually finite-dimensional if it has a countable separating family of finite-dimensional irreducible representations. Examples of such C^*-algebras include $C(X)$ and $M_n \otimes C(X)$. Of course, these have only finite-dimensional irreducible representations. More examples include $C^*(F_n)$ $(n > 1)$, the group C^*-algebra of the free group on n generators. In [2], this algebra was shown to be unital, separable, residually finite-dimensional, and primitive with an infinite-dimensional, faithful, irreducible representation.

Choose two sequences of positive integers, $\{m_i\}_{i=1}^{\infty}$ and $\{n_i\}_{i=1}^{\infty}$, with $\sup_i n_i = \infty$. Let $\mathcal{S}$ be a sequence of finite-dimensional irreducible representations of B, (π_n, H_n), such that for any nonzero $b \in B$, there is π_n for which $\pi_n(b) \neq 0$ and such that each representation appears infinitely many times in $\mathcal{S}$. For each n, let $d(n)$ be the dimension of H_n. Choose an identification of $B(H_n)$ with $M_{d(n)}$. Define $\psi_n : B \to B \otimes M_{d(n)}$ by $\psi_n(b) = 1_B \otimes \pi_n(b)$.

Let $\phi_1 : B \to M_{m_1}(B)$ be the inflation map defined by $\phi_1(b) = \operatorname{diag}(b, \dots, b)$. Let $J(1) = 1$ and set $J(2) = m_1 + \sum_{i=1}^{n_1} d(i)$. Define the monomorphism $h_1 : B \to M_{J(2)}(B)$ by

$$h_1(b) = \operatorname{diag}\left(\phi_1(b), \psi_1(b), \psi_2(b), \dots, \psi_{n_1}(b)\right). \tag{2.1}$$

Let $\phi_2 : M_{J(2)}(B) \to M_{J(2)\cdot m_2}(B)$ be the inflation map defined by $\phi_2(b) = \operatorname{diag}(b, \dots, b)$. Set $J(3) = J(2)(m_2 + \sum_{i=1}^{n_2} d(i))$ and define the monomorphism $h_2 : M_{J(2)}(B) \to M_{J(3)}(B)$ by

$$h_2(b) = \operatorname{diag}\left(\phi_2(b), \psi_1 \otimes I_{M_{J(2)}}(b), \dots, \psi_{n_2} \otimes I_{M_{J(2)}}(b)\right). \tag{2.2}$$

Continue inductively to form the direct sequence $(M_{J(s)}(B), h_s)$ of C^*-algebras. Notice that each member of the sequence of representations appears at some stage in the construction because $\sup_i n_i = \infty$. Since $m_i, n_i > 0$ for each i, at least one identity map and at least one representation appears at each stage of the construction.

For $s < t$, let $h_{s,t} : M_{J(s)}(B) \to M_{J(t)}(B)$ be the composition of monomorphisms $h_{t-1} \circ h_{t-2} \circ \cdots \circ h_s$. Let A be the C^*-inductive limit of the sequence $(M_{J(s)}(B), h_s)$ and let $h_{s,\infty} : M_{J(s)}(B) \to A$ be the monomorphism induced by the inductive limit construction. We then have the following theorem.

Theorem 2.1. *The algebra A is always unital and simple.*

Proof. Since $M_{J(s)}(B)$ and h_s are unital for each s, A is unital. Let $a \in A$ be nonzero. Let I be any (closed) ideal of A with the property that $\bigcup_s h_{s,\infty}(M_{J(s)}(B)) \cap I = 0$, and let $\pi : A \to A/I$ be the quotient map. Since $\bigcup_s h_{s,\infty}(M_{J(s)}(B))$ is dense in A, for every $\epsilon > 0$, there is b in $h_{s,\infty}(M_{J(s)}(B))$ for some s with $\|a-b\| < \epsilon$. Then $\|\pi(b)\| = \|b\|$ because $h_{s,\infty}(M_{J(s)}(B)) \cap I = 0$ and $h_{s,\infty}(M_{J(s)}(B))/(h_{s,\infty}(M_{J(s)}(B)) \cap I) \cong (h_{s,\infty}(M_{J(s)}(B)) + I)/I$. So,

$$\big|\, \|a\| - \|\pi(a)\| \,\big| \le \big|\, \|a\| - \|b\| \,\big| + \big|\, \|\pi(b)\| - \|\pi(a)\| \,\big| < 2\epsilon. \tag{2.3}$$

Thus, π is isometric and hence $I = 0$. Therefore, we may, without loss of generality, assume that there is $b \in M_{J(s)}(B)$ for some s such that $h_{s,\infty}(b) = a$.

By definition of $\mathcal{S}$, there is π_r belonging to $\mathcal{S}$ such that $(\pi_r \otimes I_{M_{J(s)}})(b) \neq 0$. Since π_r repeats infinitely many times, there is $t \ge s$ such that $h_{t-1} = \operatorname{diag}(\phi_{t-1}, \psi_1 \otimes I_{M_{J(t-1)}}, \dots, \psi_r \otimes I_{M_{J(t-1)}}, \dots)$. Since $\pi_r \otimes I_{M_{J(s)}}(b)$ is nonzero in $M_{d(r)\cdot J(s)}$, the ideal generated by $\psi_r \otimes I_{M_{J(t-1)}}(b)$ in $M_{J(t)} \otimes 1_B$ is $M_{J(t)} \otimes 1_B$. Since $h_{s,t-1}(b)$ has at least one diagonal block equal to b, then $h_{s,t}(b)$ has at least one nonzero scalar block matrix on its diagonal. So the ideal generated by $h_{s,t}(b)$ is $M_{J(t)}(B)$. Thus, the ideal generated by a is A. □

We will show in the next section that the construction always produces a C^*-algebra which has topological stable rank one.

Remark 2.2. This construction can be put in a much more general setting without affecting any of the results in this paper. For example, we could modify the construction in the following manner. Let $\mathfrak{D} = \{K_n\}_{n\in\mathbb{N}}$ be a dense sequence in $\operatorname{Prim}(B)$ consisting of kernels of finite-dimensional irreducible representations. We note that such a sequence exists. If $\{\pi_n\}_{n\in\mathbb{N}}$ is a separating family of finite-dimensional irreducible representations of B, then $\mathfrak{D} = \{\ker(\pi_n)\}_{n\in\mathbb{N}}$ is dense in $\operatorname{Prim}(B)$. Under the bijective correspondence, $J \to \operatorname{hull}(J)$, of the set of closed ideals of B onto the closed subsets of $\operatorname{Prim}(B)$, we have $\overline{\mathfrak{D}} = \operatorname{hull}(I)$ for some (closed) ideal I of B. Then $I \subset \ker(\pi_n)$ for every n, and so $I = 0$. Thus, $\mathfrak{D}$ is dense in $\operatorname{Prim}(B)$.

Let $\mathcal{S}$ be a sequence of finite-dimensional irreducible representations of B, $\{\pi_n\}_{n\in\mathbb{N}}$, so that $\ker(\pi_n) = K_n$ for each n. An arbitrary sequence of positive integers $\{n_i\}_{i=1}^{\infty}$ could then be chosen, from which a subsequence of $\mathcal{S}$ of size n_i, say $\{(\pi_{j_1}, H_{j_1}), \dots, (\pi_{j_{n_i}}, H_{j_{n_i}})\}$, should be chosen for each i. If the sequence $\{m_i\}$ and the maps ϕ_i are as before, then the matrix sizes $J(i)$ should be defined as $J(i) = J(i-1)(m_{i-1} + \sum_{k=1}^{n_{i-1}} d(j_k))$; and the maps $h_i : M_{J(i)}(B) \to M_{J(i+1)}(B)$ should be defined as

$$h_i(b) = \operatorname{diag}\big(\phi_i(b), \psi_{j_1} \otimes I_{M_{J(i)}}(b), \dots, \psi_{j_{n_i}} \otimes I_{M_{J(i)}}(b)\big). \tag{2.4}$$

All of the results in this paper still hold provided that $\bigcup_{i\geq n}(\bigcup_{k=1}^{n_i}\ker(\pi_{j_k}))$ is still dense in $\mathrm{Prim}(B)$ for every positive integer n. We have chosen to use the less general setting only for ease of notation.

3. The TAF property. We define the property TAF and give sufficient conditions for A (see Section 2 for the definition) to have this property. As a consequence, we show that A has always topological stable rank one. We also produce examples of C^*-algebras born from our construction, including many simple, unital, real rank zero, topological stable rank one, quasidiagonal (TAF) C^*-algebras.

Definition 3.1. A unital C^*-algebra C is TAF if for any $\epsilon > 0$, any positive integer n, any finite subset $\mathcal{F}$ of C containing $x_1 \neq 0$ and any full $a \in C_+$, there is a finite-dimensional C^*-subalgebra $F \subset C$ with $p = 1_F$ such that

(1) $\|px - xp\| < \epsilon$ for all $x \in \mathcal{F}$;

(2) (i) for each $x \in \mathcal{F}$, there exists $y \in F$ with $\|pxp - y\| < \epsilon$, and

(ii) $\|px_1p\| \geq \|x_1\| - \epsilon$;

(3) $n[1-p] \leq [p]$ in $D(C)$ and $1-p \preceq a$.

Here, $D(C)$ denotes the set of Murray-von Neumann equivalence classes of projections in C and $\mathrm{Her}(a)$ denotes the hereditary C^*-subalgebra of C generated by a. The condition that pCp contains n mutually orthogonal projections, each Murray-von Neumann equivalent to $1-p$ is denoted by $n[1-p] \leq [p]$, and the condition that $1-p$ is Murray-von Neumann equivalent to a projection in $\mathrm{Her}(a)$ is denoted by $1-p \preceq a$.

From this definition, it is easy to see that any unital AF C^*-algebra is TAF. When working with simple separable C^*-algebras, the definition can be somewhat simplified. In [10, Proposition 3.8], it is shown that a simple, unital, TAF C^*-algebra has cancellation of projections; and so a unital, simple C^*-algebra is TAF if and only if conditions (1) and (2)(i) in Definition 3.1 hold and condition (3) is replaced by

(3′) $1-p$ is unitarily equivalent to a projection in $\mathrm{Her}(a)$.

In the remainder of this paper, we use this latter definition of simple TAF, that is, we replace condition (3) in Definition 3.1 by condition (3′), because the C^*-algebra A is simple. More is found in [9, 10]. We have the following theorem.

Theorem 3.2. *If B is TAF, A is TAF.*

Proof. It is shown in [10, Theorem 3.10] that when B is TAF, $M_n(B)$ is TAF for all n, and a unital, simple direct limit of unital TAF C^*-algebras is TAF. □

Now, we give the numerical condition, which ensures that the TAF property holds for A. From the inductive construction, we have for each $s \in \mathbb{N}$,

$$J(s) = J(s-1)\left(m_{s-1} + \sum_{j=1}^{n_{s-1}} d(j)\right) = \prod_{i=1}^{s-1}\left(m_i + \sum_{j=1}^{n_i} d(j)\right). \tag{3.1}$$

For $s \in \mathbb{N}$, set

$$\lambda_s = \frac{m_1 \cdot m_2 \cdot \cdots \cdot m_s}{J(s+1)}. \tag{3.2}$$

A short computation shows that we also have

$$\lambda_s = \prod_{i=1}^{s}\left(1+\frac{\sum_{j=1}^{n_i} d(j)}{m_i}\right)^{-1}. \tag{3.3}$$

Since

$$\lambda_{s+1} = \frac{m_{s+1}\cdot J(s+1)}{J(s+2)}\cdot\lambda_s = \left(\frac{m_{s+1}}{m_{s+1}+\sum_{j=1}^{n_{s+1}} d(j)}\right)\cdot\lambda_s \le \lambda_s, \tag{3.4}$$

$\{\lambda_s\}_{s\in\mathbb{N}}$ is decreasing and bounded below by 0. So its limit exists. Let $\Lambda = \lim_{s\to\infty}\lambda_s$. From the theory of infinite products, we see $\Lambda \ne 0$ exactly when $\sum_{i=1}^{\infty}(m_i{}^{-1}\sum_{j=1}^{n_i} d(j))$ converges. Indeed both nonzero and zero values of Λ are possible. For example, if $m_i = i\sum_{j=1}^{n_i} d(j)$, then $\Lambda = 0$; and, if $m_i = 2^i\sum_{j=1}^{n_i} d(j)$, then $\Lambda > 1/3$.

Now, we give a sufficient condition for A to have the TAF property in the following theorem, which generalizes a result in [10, Section 4].

Theorem 3.3. *If $\Lambda = 0$, then A is TAF.*

Proof. Let $\epsilon > 0$. Let $\mathcal{F}$ be a nonempty finite subset of A containing a nonzero element x_1. Without loss of generality, we can assume $\mathcal{F} \subset h_{s,\infty}(M_{J(s)}(B))$. There is a finite subset $\mathcal{G} \subset M_{J(s)}(B)$ such that $h_{s,\infty}(\mathcal{G}) = \mathcal{F}$ with $y \in \mathcal{G}$ and $h_{s,\infty}(y) = x_1$. Since the representation $\oplus_{n\in\mathbb{N}}\pi_n$ is faithful on B, the representation $(\oplus_{n\in\mathbb{N}}\pi_n)\otimes I_{M_{J(s)}}$ is faithful on $M_{J(s)}(B)$. So, there is π_r belonging to the sequence $\mathcal{S}$ with $\|\pi_r\otimes I_{M_{J(s)}}(y)\| \ge \|y\|-\epsilon$. Choose $t > 1$ such that $h_{s+t-1} = \operatorname{diag}(\phi_{s+t-1}, \psi_1\otimes I_{M_{J(s)}}, \dots, \psi_r\otimes I_{M_{J(s)}}, \dots)$.

For any $z \in M_{J(s)}(B)$, there are at least $J(s+t) - m_s\cdot m_{s+1}\cdot\cdots\cdot m_{s+t-1}\cdot J(s)$ rows of the diagonal block matrix $h_{s,s+t}(z)$ with entries belonging solely to $\mathbb{C}\cdot 1_B$. If p is the diagonal projection in $M_{J(s+t)}$ corresponding to these rows so that the rank of p is $J(s+t) - m_s\cdot m_{s+1}\cdot\cdots\cdot m_{s+t-1}\cdot J(s)$, then $ph_{s,s+t}(M_{J(s)}(B))p$ may be identified with a C^*-subalgebra of $1_B\otimes M_{J(s+t)-m_s\cdots m_{s+t-1}J(s)}$. So $F = ph_{s,s+t}(M_{J(s)}(B))p$ is a finite-dimensional C^*-subalgebra of $M_{J(s+t)}(B)$ with $1_F = p$ and $\|h_{s,s+t}(y)\| \ge \|y\|-\epsilon$ since $h_{s,s+t}$ is injective. Then $h_{s+t,\infty}(F)$ is a finite-dimensional subalgebra of A with identity $h_{s+t,\infty}(p)$, and for all $x \in \mathcal{F}$, $\|h_{s+t,\infty}(p)x - xh_{s+t,\infty}(p)\| = 0$. Also, $\|h_{s+t,\infty}(p)x_1h_{s+t,\infty}(p)\| \ge \|x_1\|-\epsilon$ since $h_{s+t,\infty}$ is injective.

Note that $h_{s+t,\infty}(p)\mathcal{F}h_{s+t,\infty}(p) \subset h_{s+t,\infty}(F)$. So, A satisfies conditions (1) and (2) in Definition 3.1. By [10, Lemma 2.12], A has property (SP). It, therefore, suffices to show by heredity, given any nonzero projection q in A, that p can be chosen so that $h_{s+t,\infty}(1-p)$ is unitarily equivalent to a subprojection of q. Without loss of generality, we assume that q belongs to $h_{n,\infty}(M_{J(n)}(B))$ for some n. Then there is a projection $\hat{q} \in M_{J(n)}(B)$ with $h_{n,\infty}(\hat{q}) = q$. We may also assume, without loss of generality, that $s \ge n$ so that $h_{n,s}(\hat{q}) = \operatorname{diag}(q_1, q_2)$, where $q_2 \in M_n\otimes 1_B$ for some $1 \le n \le J(s)$ and q_2 is nonzero. We can then work in $M_{J(s)}$ by identifying $M_{J(s)}\otimes 1_B$ with $M_{J(s)}$ and in $M_{J(s+t)}$ similarly. Denote by tr the usual (nonnormalized) trace of a matrix in $M_{J(s)}$ or in $M_{J(s+t)}$.

Since $\Lambda = 0$, there is t such that

$$\lambda_{s+t-1} < \frac{\operatorname{tr}(q_2)}{J(s)^2}. \tag{3.5}$$

Then

$$\operatorname{tr}(1-p) = m_s \cdot m_{s+1} \cdot \cdots \cdot m_{s+t-1} \cdot J(s) < \frac{\operatorname{tr}(q_2) \cdot J(s+t)}{J(s)} = \operatorname{tr}\left(h_{s,s+t}(q_2)\right). \quad (3.6)$$

So, $1-p$ is unitarily equivalent to a subprojection of $h_{s,s+t}(q_2)$. Since we have $h_{s,s+t}(q_2) \leq h_{k,s+t}(\hat{q})$, $h_{s+t,\infty}(1-p)$ is unitarily equivalent to a subprojection of q. Therefore, A is TAF. □

Theorem 3.4. *If A is TAF, then A has real rank zero, topological stable rank one and is quasidiagonal.*

This is proved in [10, Theorem 3.4].

Now, we prove that A always has topological stable rank one using a proof which is fundamentally the same as in [7, Lemma 2, Theorem 3] with adjustments to accommodate the replacement of $C(X)$ by B.

Theorem 3.5. *A always has topological stable rank one.*

Proof. If $\Lambda = 0$, Theorem 3.3 and its corollary prove that A has topological stable rank one. So we can assume $\Lambda > 0$.

We show that $\operatorname{Inv}(A)$ is dense in A. Let $x \in M_{J(s)}(B)$. We assume that x is not invertible. Let $\epsilon > 0$. We first find $n \in \mathbb{N}$ and $y \in M_{J(s)}(B)$ such that $\|y-x\| < \epsilon$ and $\pi_n(y)$ is not invertible. Then we show that $y \in \overline{\operatorname{Inv}(A)}$. For convenience, denote $(\oplus_{n\in\mathbb{N}}\pi_n) \otimes \operatorname{Id}_{M_{J(s)}}$ by $\oplus_n \pi_n$. Then $\oplus_n\pi_n(x)$ is not invertible. So $\oplus_n\pi_n(x)$ is not bounded below or does not have dense range (see [8] for example).

Assume first that it is not bounded below. Then, there is a unit vector ξ with $\| \oplus_n \pi_n(x)(\xi)\| < \epsilon$. If $\xi_n \in H_{nJ(s)}$ are the components of ξ, then we have $\sum_n \|\pi_n(x)(\xi_n)\|^2 < \epsilon^2$. If for each n, $\|\pi_n(x)(\xi_n)\| \geq \epsilon \cdot \|\xi_n\|$, then we would have $\sum_n \|\pi_n(x)(\xi_n)\|^2 \geq \epsilon^2 \sum_n \|\xi_n\|^2 = \epsilon^2 \cdot \|\xi\|^2$. So, there is n for which $\|\pi_n(x)(\xi_n/\|\xi_n\|)\| < \epsilon$.

Set $\upsilon = \xi_n/\|\xi_n\|$ and extend to an ordered orthonormal basis Υ for $H_{d(n)J(s)}$. Let p be the projection of $H_{d(n)J(s)}$ on $\operatorname{span}(\upsilon)$, that is, $p = \langle \cdot, \upsilon\rangle \upsilon$. Set $Z = \pi_n(x)p$. Note that the matrix $(Z_{i,j})_\Upsilon$ has, as its first column, the first column of $(\pi_n(x)_{i,j})_\Upsilon$ and all other columns zero. So $\|Z\|^2 = \sum_{i=1}^{d(n)J(s)} \|(\pi_n(x))_{i,1}\|^2 = \|\pi_n(x)(\upsilon)\|^2 < \epsilon^2$ and $0 \in \sigma(\pi_n(x) - Z)$. Lift Z to $z \in M_{J(s)}(B)$ with $\|z\| = \|Z\|$. Set $y = x - z$. Then $\|y-x\| < \epsilon$ and $\pi_n(y) = \pi_n(x) - Z$ so that $0 \in \sigma(\pi_n(y))$.

Now, assume that the range of $\oplus_n\pi_n(x)$ is not dense. Then its range has a nonzero orthogonal complement. So $\oplus_n\pi_n(x^*)$ has a nontrivial kernel. Then there is a unit vector ξ with $\oplus_n\pi_n(x^*)(\xi) = 0$. So we can repeat the above to find $y^* \in M_{J(s)}(B)$ with $\|y^*-x^*\| < \epsilon$ and $0 \in \sigma(\pi_n(y^*))$. Then $\|y-x\| < \epsilon$ and $0 \in \sigma(\pi_n(y))$.

Choose $t > s$ such that

$$h_{t-1} = \operatorname{diag}\left(\phi_{t-1}, \psi_1 \otimes I_{M_{J(t-1)}}, \ldots, \psi_n \otimes I_{M_{J(t-1)}}, \ldots\right). \quad (3.7)$$

Since $0 \in \sigma(\pi_n(y))$, there are unitaries $u_1, u_2 \in M_{J(t)}$ so that $u_1 \cdot h_{s,t}(y) \cdot u_2$ is a block diagonal matrix, which differs from $h_{s,t}(y)$ in that the block $(\psi_n \otimes I_{M_{J(t-1)}})(h_{s,t-1}(y))$ of $h_{s,t}(y)$ has been replaced by the matrix $\operatorname{diag}(Y, 0)$, where Y is viewed as belonging to $(1_B \otimes M_{d(n)\cdot J(t-1)-1})$ and $0 \in \mathbb{C}$. Now, the matrix $u_1 \cdot h_{s,t}(y) \cdot u_2$ is a block diagonal with at least one (1×1) zero block. We show that we can use the inductive limit construction to find unitaries which will transform the image of this matrix in some later stage into one which is upper triangular. We next show how to choose this later stage.

Since $\Lambda > 0$, $\sum_{i=1}^{\infty} m_i{}^{-1} \sum_{j=1}^{n_i} d(j)$ converges. So $\{m_i{}^{-1}\sum_{j=1}^{n_i} d(j)\}_i \to 0$ and hence we also have $\{m_i{}^{-1} \cdot \max\{d(j) : j = 1,\dots,n_i\}\}_i \to 0$. Then $\{m_i(\max\{d(j) : j = 1,\dots, n_i\})^{-1}\}_i \to \infty$. Thus, we can choose $w > t$ so that

$$\prod_{i=s}^{w-1}\left(\frac{m_i}{\max\{d(j) : j = 1,\dots,n_i\}} + 1\right) > J(t). \tag{3.8}$$

Then $h_{t,w}(u_1 \cdot h_{s,t}(y) \cdot u_2)$ is a block diagonal matrix with at least $J(w)J(t)^{-1}$ zero rows, and all nonzero blocks have at most $J(s) \cdot \prod_{i=s}^{w-1} \max\{d(j) : j = 1,\dots,n_i\}$ rows (and columns).

There is a unitary $u_3 \in M_{J(w)}$ such that $u_3 \cdot h_{t,w}(u_1 \cdot h_{s,t}(y) \cdot u_2) \cdot u_3^* = \operatorname{diag}(0_{M_{J(w)J(t)^{-1}}(B)}, Y')$, where Y' is a block diagonal matrix with all blocks having row (and column) size at most $J(s) \cdot \prod_{i=s}^{w-1} \max\{d(j) : j = 1,\dots,n_i\}$. There is also a unitary $u_4 \in M_{J(w)}$ so that $u_4 \cdot u_3 \cdot h_{t,w}(u_1 \cdot h_{s,t}(y) \cdot u_2) \cdot u_3^*$ has as its first $J(w) - (J(w)/J(t))$ rows the last $J(w) - (J(w)/J(t))$ rows of $u_3 \cdot h_{t,w}(u_1 \cdot h_{s,t}(y) \cdot u_2) \cdot u_3^*$ and has as its last $J(w)/J(t)$ rows the first $J(w)/J(t)$ rows of $u_3 \cdot h_{t,w}(u_1 \cdot h_{s,t}(y) \cdot u_2) \cdot u_3^*$.

Each entry appearing in any of the first $J(w)J(t)^{-1}$ columns of the matrix $u_4 \cdot u_3 \cdot h_{t,w}(u_1 \cdot h_{s,t}(y) \cdot u_2) \cdot u_3^*$ is 0, and each (possibly) nonzero entry in column $J(w)J(t)^{-1} + k$, for $1 \le k \le J(w) - J(w)J(t)^{-1}$ appears in row l for $1 \le l \le k - 1 + J(s) \cdot \prod_{i=s}^{w-1} \max\{d(j) : j = 1,\dots,n_i\}$. Then, for each k with $1 \le k \le J(w) - J(w)J(t)^{-1}$, the entry in column $J(w)J(t)^{-1} + k$ and row $m + k$ is 0 for every $m \ge J(s) \cdot \prod_{i=s}^{w-1} \max\{d(j) : j = 1,\dots,n_i\}$.

Now $J(w)J(t)^{-1} > J(s) \cdot \prod_{i=s}^{w-1} \max\{d(j) : j = 1,\dots,n_i\}$ because

$$\begin{aligned}
J(w)&\left(\prod_{i=s}^{w-1} \max\{d(j) : j = 1,\dots,n_i\}\right)^{-1} \\
&= J(s) \cdot \prod_{i=s}^{w-1}\left(\left(m_i + \sum_{j=1}^{n_i} d(j)\right)\left(\max\{d(j) : j = 1,\dots,n_i\}\right)^{-1}\right) \\
&= J(s) \cdot \prod_{i=s}^{w-1}\left(\frac{m_i + \sum_{j=1}^{n_i} d(j)}{\max\{d(j) : j = 1,\dots,n_i\}}\right) \\
&\ge J(s) \cdot \prod_{i=s}^{w-1}\left(\frac{m_i}{\max\{d(j) : j = 1,\dots,n_i\}} + 1\right) \\
&> J(s) \cdot J(t).
\end{aligned} \tag{3.9}$$

So, the matrix $u_4 \cdot u_3 \cdot h_{t,w}(u_1 \cdot h_{s,t}(y) \cdot u_2) \cdot u_3^*$ is strictly upper triangular and hence nilpotent in $M_{J(w)}(B)$. Then $h_{w,\infty}(u_4 \cdot u_3 \cdot h_{t,w}(u_1 \cdot h_{s,t}(y) \cdot u_2) \cdot u_3^*)$ is nilpotent in A. Since A is unital, if a in A is nilpotent and κ is any complex scalar that is nonzero, then $\kappa - a$ is invertible. Consequently, $a \in \overline{\mathrm{Inv}(A)}$. So $h_{s,\infty}(y) \in \overline{\mathrm{Inv}(A)}$. Since $\|h_{s,\infty}(x) - h_{s,\infty}(y)\| < \epsilon$, $h_{s,\infty}(x) \in \overline{\mathrm{Inv}(A)}$. Thus, A has topological stable rank one. $\square$

We should note that the construction presented here may produce nonnuclear C^*-algebras. In particular, with $\Lambda = 0$ and $B = C^*(F_n)$ $(n > 1)$, a similar argument as in [10, Section 4] shows that this construction produces an example of a C^*-algebra which is unital, simple, has real rank zero and topological stable rank one, is quasidiagonal, nonnuclear, and which possesses a unique normalized trace. It is nonnuclear since no nonexact C^*-algebra can be embedded in an exact one. Also, this construction can produce non-TAF C^*-algebras (when $\Lambda > 0$). Easy examples will be apparent after the work in the next section, which contains a necessary and sufficient condition for A to possess the TAF property when the source has a bounded rank.

4. A TAF determinant. The notation in this section follows that of [11, Chapters 4–6]. Recall that if B is a C^*-algebra, an element $x \in B_+$ has continuous trace if $\check{x} \in C_b(\mathrm{Irr}(B))$, where $\check{x} : \mathrm{Irr}(B) \to [0,\infty]$ is defined by $\check{x}(t) = \mathrm{tr}(\pi(x))$ whenever $(\pi, H_\pi) \in t$. A C^*-algebra B has continuous trace if the set of elements with continuous trace is dense in B_+.

Let ${}_n\check{B}$ be the subset of $\mathrm{Prim}(B)$ corresponding to irreducible representations of B with dimension less than or equal to n. Then $\ker({}_n\check{B}) = \cap \ker(\pi)$, where the intersection is taken over all irreducible representations of B with dimension at most n, so that it is closed as in [11, Theorem 4.4.10]. Let $\check{B}_n = ({}_n\check{B} \backslash {}_{n-1}\check{B})$. Then $\check{B}_n$ is the set of n-dimensional irreducible representations of B. Furthermore, $B_n = \ker({}_{n-1}\check{B})/\ker({}_n\check{B})$ has primitive spectrum homeomorphic to $\check{B}_n$ by [11, Theorem 4.1.11] and has all its irreducible representations of dimension n. It, therefore, has continuous trace by [6, Theorem 4.3] and so $\check{B}_n$ is locally compact Hausdorff by [11, Theorems 6.1.11 and 6.1.5].

Theorem 4.1. *When A has real rank zero and Λ is nonzero, $\check{B}_n$ is totally disconnected for every $n \in \mathbb{N}$.*

Proof. Let $\Lambda \geq \sigma > 0$. Note that $\sigma \leq 1$ by definition of Λ. Assume that there is an integer n for which $\check{B}_n$ is not totally disconnected. Then it contains a connected component containing more than one point, $\mathcal{N}$. Since $\check{B}_n$ is locally compact Hausdorff, for any compact subset $\mathcal{S}$ of $\mathcal{N}$, $\mathcal{S}$ is closed. So, under the homeomorphism $\Theta : \mathrm{Irr}(B_n) \to \check{B}_n$, where $\Theta([H_\xi, \xi]) = \ker(\xi)$, there is a closed ideal I of B_n with $\mathcal{S} = \mathrm{hull}(I)$.

The C^*-algebra B_n/I has primitive spectrum homeomorphic to $\mathrm{hull}(I)$ by [11, Theorem 4.1.11], which is compact. Since B_n/I has each irreducible representation of rank n, it is isomorphic to $C_0(\mathcal{B})$, the set of all continuous cross-sections of $\mathcal{B}$ which vanish at infinity, where $\mathcal{B}$ is a fiber bundle with base space $\mathrm{Irr}(B_n/I) = \Theta^{-1}(\mathrm{hull}(I))$, fiber space M_n and group $\mathrm{Aut}(M_n)$ by [6, Theorem 3.2]. Therefore, B_n/I is locally trivial. So, each point of $\mathcal{N}$ possesses a compact neighborhood $\mathcal{U}$ on which the maximal full algebra of operator fields defined by $\mathrm{hull}(I)$, which is B_n/I, is isomorphic to the constant field, $C(\mathcal{U}) \otimes M_n$.

It follows that there is a closed ideal J of B_n with $\mathcal{U} = \mathrm{hull}(J)$. Since $\mathrm{hull}(J) \subset \mathrm{hull}(I)$, $I \subset J$ and B_n/J is isomorphic to $C(\mathrm{Irr}(B_n/J)) \otimes M_n$. For any two points π and $\rho \in \mathrm{Irr}(B_n/J)$, there is $\check{x} \in (C(\mathrm{Irr}(B_n/J)) \otimes M_n)$ with $\check{x}$ positive and $\check{x} \le 1$, $\check{x}(\pi) = 1_{M_n}$, and $\check{x}(\rho) = 0$. Note that π and ρ can not be separated by clopen sets in $\check{B}_n$. Viewing $\check{x}$ as an element of B_n/J, we can lift $\check{x}$ to $\hat{x}$ in $(B_n)_+$. Then $\pi(\hat{x}) = 1_{M_n}$ and $\rho(\hat{x}) = 0$. Since B_n has continuous trace, for every $\epsilon > 0$, there is $\hat{b}_\epsilon \in (B_n)_+$ with continuous trace such that $\|\hat{b}_\epsilon - \hat{x}\| < \epsilon$. Next, $\hat{b}_\epsilon$ is lifted to b_ϵ in B_+ and $\hat{x}$ is lifted to x in B with $\|b_\epsilon - x\| < \epsilon$ so that $\pi(x) = 1_{M_n}$ and $\rho(x) = 0$.

Choose $\epsilon = \sigma/12$ and set $a = b_\epsilon$. Since A has real rank zero, every hereditary subalgebra has also real rank zero [1, Corollary 2.8]. In particular, if K is the hereditary C^*-subalgebra of A generated by $h_{1,\infty}(a^{1/2})$, then K has real rank zero. Then there is $y \in K$ such that $y = \sum_{i=1}^k \mu_i q_i$ is a linear combination of projections $q_i \in K$ with $\mu_i \in \mathbb{C}$ and

$$||y - h_{1,\infty}(a)|| < \frac{\sigma}{12}. \tag{4.1}$$

So

$$||h_{1,\infty}(x) - y|| < \frac{\sigma}{6}. \tag{4.2}$$

Since $q_i \in K$, for each $i = 1, 2, \ldots, k$, there is $y_i \in A_{sa}$ with

$$||q_i - h_{1,\infty}(a^{1/2}) y_i h_{1,\infty}(a^{1/2})|| < \frac{\sigma}{216k\mu}, \tag{4.3}$$

where $\mu = \max\{\{|\mu_i|\}_{i=1}^k, 1\}$.

Then there are $z = \sum_{i=1}^k \mu_i p_i$ and $\{z_i\}_{i=1}^k \in (M_{J(s)}(B))_{sa}$ for large s such that z is a linear combination of projections p_i,

$$\begin{gathered} ||h_{s,\infty}(p_i) - q_i|| < \frac{\sigma}{108k\mu}, \\ ||h_{s,\infty}(z) - y|| < \frac{\sigma}{108}, \\ ||h_{1,\infty}(a^{1/2}) h_{s,\infty}(z_i) h_{1,\infty}(a^{1/2}) - h_{1,\infty}(a^{1/2}) y_i h_{1,\infty}(a^{1/2})|| < \frac{\sigma}{108k\mu}. \end{gathered} \tag{4.4}$$

Thus

$$\begin{gathered} ||z - h_{1,s}(x)|| < \frac{\sigma}{3}, \\ ||p_i - h_{1,s}(a^{1/2}) z_i h_{1,s}(a^{1/2})|| < \frac{\sigma}{36k\mu}. \end{gathered} \tag{4.5}$$

Since $\lim_{s\to\infty} \lambda_s \ge \sigma$, we have $m_1 \cdot m_2 \cdot \cdots \cdot m_{s-1} \ge \sigma \cdot J(s)$. So the matrix $h_{1,s}(x)$ contains at least $\sigma \cdot J(s)$ diagonal entries equal to x. Then $\pi(h_{1,s}(x)) - \rho(h_{1,s}(x))$ contains at least $\sigma \cdot J(s)n$ diagonal entries equal to 1 while all others are zero. If tr denotes the standard trace, then using the computation above,

$$\mathrm{tr}\left(\pi(h_{1,s}(x)) - \rho(h_{1,s}(x))\right) = n \cdot \prod_{i=1}^{s-1} m_i \ge \sigma \cdot J(s)n. \tag{4.6}$$

There are projections $\chi_i = \chi_{[1-\sigma/36k\mu, 1+\sigma/36k\mu]}(h_{1,s}(a^{1/2})z_i h_{1,s}(a^{1/2}))$ in the C^*-algebra generated by $h_{1,s}(a^{1/2})z_i h_{1,s}(a^{1/2})$ for each $i = 1,2,\dots,k$, by spectral theory, with

$$||p_i - \chi_i|| < \frac{\sigma}{12k\mu}. \tag{4.7}$$

Furthermore, since $\sigma/36k\mu \leq 1$, $0 \leq \chi_i \leq 2 \cdot h_{1,s}(a^{1/2})z_i h_{1,s}(a^{1/2})$ for each $i = 1,2,\dots,k$. Note that $h_{1,s}(a)$ is a diagonal block matrix with blocks of the form a or blocks belonging to $1_B \otimes M_{d(i)\cdot J(s-1)}$ (for $1 \leq i \leq s-1$). So, if π is an n-dimensional representation of B, $\operatorname{tr}(\pi(h_{1,s}(a))) = \sum_{m_1 \cdots m_{s-1}} \operatorname{tr}(\pi(a)) + \operatorname{tr}(D)$, where $D \in M_{n(J(s)-m_1\cdots m_{s-1})}$. Since the trace function of a evaluated on the set of n-dimensional representations of B is continuous (see the beginning of this section for an explanation), the trace function of $h_{1,s}(a)$ evaluated on this set is also continuous.

As $0 \leq h_{1,s}(a^{1/2})z_i h_{1,s}(a^{1/2}) \leq \|z_i\| h_{1,s}(a)$, the trace function of the elements $h_{1,s}(a^{1/2})z_i h_{1,s}(a^{1/2})$ and $\|z_i\| h_{1,s}(a) - h_{1,s}(a^{1/2})z_i h_{1,s}(a^{1/2})$ evaluated on the set of n-dimensional representations of B is lower semicontinuous and is continuous on their sum. Thus, for each $i = 1,2,\dots,k$, the trace function of $h_{1,s}(a^{1/2})z_i h_{1,s}(a^{1/2})$ evaluated on that set is continuous. Since for each $i = 1,2,\dots,k$, $0 \leq \chi_i \leq 2 \cdot h_{1,s}(a^{1/2})z_i h_{1,s}(a^{1/2})$, a similar argument shows that the same holds for the trace of each χ_i.

Since the trace of these projections evaluated on the set of n-dimensional representations of B is $\mathbb{Z}$-valued, it is constant on connected components, and so, for each $i = 1,2,\dots,k$, $\operatorname{tr}(\pi(\chi_i)) = \operatorname{tr}(\rho(\chi_i))$. Then

$$\begin{aligned} |\operatorname{tr}(\pi(z) - \rho(z))| &\leq \left|\operatorname{tr}\left(\pi(z) - \sum_{i=1}^{k} \mu_i \pi(\chi_i)\right)\right| + \left|\operatorname{tr}\left(\sum_{i=1}^{k} \mu_i \rho(\chi_i) - \rho(z)\right)\right| \\ &\leq 2 \cdot \left\|z - \sum_{i=1}^{k} \mu_i \chi_i\right\| \cdot J(s)n \\ &\leq \frac{J(s)n \cdot \sigma}{6}. \end{aligned} \tag{4.8}$$

So

$$\begin{aligned} |\operatorname{tr}(\pi(h_{1,s}(x)) - \rho(h_{1,s}(x)))| &\leq |\operatorname{tr}(\pi(h_{1,s}(x)) - \pi(z))| + |\operatorname{tr}(\pi(z) - \rho(z))| \\ &\quad + |\operatorname{tr}(\rho(z) - \rho(h_{1,s}(x)))| \\ &\leq 2||h_{1,s}(x) - z|| \cdot J(s)n + |\operatorname{tr}(\pi(z) - \rho(z))| \\ &< \frac{2\sigma \cdot J(s)n}{3} + \frac{\sigma \cdot J(s)n}{6} \\ &\leq \frac{5\sigma \cdot J(s)n}{6}, \end{aligned} \tag{4.9}$$

which contradicts the previous minimal estimate of $\sigma \cdot J(s)n$. So $\check{B}_n$ is totally disconnected for each n. □

Corollary 4.2. *If A is TAF and $\check{B}_n$ is not totally disconnected for some n, then $\Lambda = 0$.*

Proof. By Theorem 3.4 A has real rank zero. $\square$

Now, we give two preliminary results for certain C^*-algebras with bounded rank.

Theorem 4.3. *A separable C^*-algebra C, which has each irreducible representation of the same finite rank and with totally disconnected spectrum, is an AF-algebra.*

Proof. Note that $\check{C}$ is second countable, locally compact, and has finite topological dimension. Let $K = K(H)$, where H is an infinite-dimensional, separable Hilbert space. Then $C \otimes K$ is homogeneous of degree $\aleph_0$ with totally disconnected spectrum. Consequently, $C \otimes K \cong C_0(\check{C}, K)$ by [3, Corollary 10.9.6].

Since $C_0(\check{C}, K) \cong C_0(\check{C}) \otimes K$, and the tensor product of AF-algebras is AF, we have $C \otimes K$ is AF. Note that $C \otimes M_n$ is a hereditary subalgebra for each n. By a well-known theorem in [4], $C \otimes M_n$ is AF as it is a hereditary subalgebra of an AF-algebra. Thus, C is AF. $\square$

Theorem 4.4. *Assume that the dimensions of the irreducible representations of B are bounded by $m < \infty$ and that $\check{B}_n$ is totally disconnected for each n. Then B is an AF-algebra.*

Proof. Note that $\ker({}_m\check{B})$ is AF. By Theorem 4.3, $\ker({}_{m-1}\check{B})/\ker({}_m\check{B})$ is also AF. Then we have a short exact sequence of C^*-algebras with endpoints AF,

$$0 \longrightarrow \ker({}_m\check{B}) \longrightarrow \ker({}_{m-1}\check{B}) \longrightarrow \ker({}_{m-1}\check{B})/\ker({}_m\check{B}) \longrightarrow 0. \tag{4.10}$$

Since the extension of an AF C^*-algebra by an AF C^*-algebra is AF, we have, by a well-known theorem of Brown [1], $\ker({}_{m-1}\check{B})$ is AF.

For each $1 \leq j \leq m-1$, we have $\ker({}_{m-j-1}\check{B})/\ker({}_{m-j}\check{B})$ is AF due to Theorem 4.3 and the short exact sequence

$$0 \longrightarrow \ker({}_{m-j}\check{B}) \longrightarrow \ker({}_{m-j-1}\check{B}) \longrightarrow \ker({}_{m-j-1}\check{B})/\ker({}_{m-j}\check{B}) \longrightarrow 0. \tag{4.11}$$

A simple induction then shows that B is indeed AF. $\square$

We conclude with our main characterization theorem.

Theorem 4.5. *Assume that B has each finite-dimensional irreducible representation of dimension at most $m < \infty$. Then A is TAF if and only if B is AF or $\Lambda = 0$.*

Proof. If A is TAF, it has real rank zero. If Λ is positive, Theorems 4.1 and 4.4 prove that B is AF. If B is AF, it is TAF and Theorem 3.2 applies. If $\Lambda = 0$, Theorem 3.3 shows that A is TAF. $\square$

5. A variation of the construction.

Finally, it is useful to generalize our construction using direct sums of matrix algebras over B. In keeping with our notation and assumptions in Section 2, we would have $A = \lim_{n\to\infty}(\oplus_{i=1}^{s_n} M_{J_i(n)}(B), \oplus_{i=1}^{s_n} h_n^{(i,j)})$, where s_n is the number of direct summands at stage n of the construction, and the homomorphism $h_n^{(i,j)} : M_{J_i(n)}(B) \to M_{J_j(n+1)}(B)$ uses $m(n,j)$ identity maps in its definition.

Let

$$\lambda(t) = \max\left\{ \frac{\left(\prod_{w=1}^{t-1} \sum_{q=1}^{s_w} m(w,q)\right) \cdot m(t,j)}{J_j(t+1)} : j = 1,\dots,s_t \right\}. \tag{5.1}$$

With $\Lambda = \lim_{t\to\infty} \lambda(t)$, it is easy to see that this variation has the same properties as that in Section 2. In particular, if $\Lambda = 0$, then A is TAF and hence has real rank zero.

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