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SUBMODULES OF SECONDARY MODULES

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Let R be a commutative ring with nonzero identity. Our objective is to investigate representable modules and to examine in particular when submodules of such modules are representable. Moreover, we establish a connection between the secondary modules and the pure-injective, the Σ-pure-injective, and the prime modules.

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1. Introduction. In this paper, all rings are commutative rings with identity and all modules are unital. The notion of associated prime ideals and the related one of primary decomposition are classical. In a dual way, we define the attached prime ideals and the secondary representation. This theory is developed in the appendix to Section 6 in Matsumura [6] and in Macdonald [5]. Now we define the concepts that we will need.

Let R be a ring and let $0 \neq M$ be an R-module. Then M is called a secondary module (second module) provided that for every element r of R the homothety $M \xrightarrow{r\cdot} M$ is either surjective or nilpotent (either surjective or zero). This implies that $\operatorname{nilrad}(M) = P$ ($\operatorname{Ann}(M) = P'$) is a prime ideal of R, and M is said to be P-secondary (P'-second), so every second module is secondary (the concept of second module is introduced by Yassemi [14]). A secondary representation for an R-module M is an expression for M as a finite sum of secondary modules (see [5]). If such a representation exists, we will say that M is representable.

If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ will be denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M, $\operatorname{Ann}(M)$. A proper submodule N of a module M over a ring R is said to be prime submodule (primary submodule) if for each $r \in R$ the homothety $M/N \xrightarrow{r\cdot} M/N$ is either injective or zero (either injective or nilpotent), so $(0 : M/N) = P$ ($\operatorname{nilrad}(M/N) = P'$) is a prime ideal of R, and N is said to be P-prime submodule (P'-primary submodule). So N is prime in M if and only if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $rM \subseteq N$. We say that M is a prime module (primary module) if zero submodule of M is prime (primary) submodule of M, so N is a prime submodule of M if and only if M/N is a prime module. Moreover, every prime module is primary.

Let R be a ring, and let N be an R-submodule of M. Then N is pure in M if for any finite system of equations over N which is solvable in M, the system is also solvable in N. A module is said to be absolutely pure if every embedding of it into any other modules is pure embedding. A submodule N of an R-module M is called relatively divisible (or an RD-submodule) if $rN = N \cap rM$ for all $r \in R$. Every RD-submodule of a P-secondary module over a commutative ring R is P-secondary (see [2, Lemma 2.1]).

A module M is pure-injective if and only if any system of equations in M which is finitely solvable in M, has a global solution in M [7, Theorem 2.8]. The module N is a pure-essential extension of M if M is pure in N and for all nonzero submodules L of N, if $M \cap L = 0$, then $(M \oplus L)/L$ is not pure in N/L. A pure-injective hull $H(M)$ of a module M is a pure essential extension of M which is pure-injective. Every module M has a pure-injective hull which is unique to isomorphism over M [12].

Given an R-module M and index set I, the direct sum of the family $\{M_i : i \in I\}$ where $M_i = M$ for each $i \in I$ will be denoted by $M^{(I)}$. Given a module property $\mathcal{P}$, we will say that a module M is $\sum$-$\mathcal{P}$ if $M^{(I)}$ satisfies $\mathcal{P}$ for every index set I.

Let R be a commutative ring. An element $a \in R$ is said to be regular if there exists $b \in R$ such that $a = a^2 b$, and R is said to be regular if each of its elements is regular. An important property of regular rings is that every module is absolutely pure (see [13, Theorem 37.6]).

Let R be a ring and M an R-module. A prime ideal P of R is called an associated prime ideal of M if P is the annihilator $\mathrm{Ann}(x)$ of some $x \in M$. The set of associated primes of M is written $\mathrm{Ass}(M)$. For undefined terms, we refer to [6, 7].

2. Secondary submodules.

In general, a nonzero submodule of a representable (even secondary) R-module is not representable (secondary), but we have the following results.

Lemma 2.1. *Let R be a commutative ring and let $0 \neq N$ be an RD-submodule of R-module M. Then M is P-secondary if and only if N and M/N are P-secondary.*

Proof. If M is P-secondary, then N and M/N are P-secondary by [2, Lemma 2.1] and [5, Theorem 2.4], respectively. Conversely, suppose that $r \in R$. If $r \in P$, then $r^n(M/N) = 0$ and $r^n N = 0$ for some n, hence $r^n M \subseteq N$ and $0 = r^n N = r^n M \cap N = r^n M$. If $r \notin P$, then $rM + N = M$, $rN = N$, and $N = rN = rM \cap N$, so we have $rM = M$, as required. □

Corollary 2.2. *Let R be a commutative regular ring, and let $0 \neq N$ be a submodule of R-module M. Then M is P-secondary if and only if N and M/N are P-secondary.*

Proof. This follows from Lemma 2.1. □

Theorem 2.3. *Let R be a commutative regular ring. Then every nonzero submodule of a representable R-module is representable.*

Proof. Let M be a representable R-module and let $M = \sum_{i=1}^{n} M_i$ be a minimal secondary representation with $\mathrm{nilrad}(M_i) = P_i$. There is an element $r_1 \in P_1$ such that $r_1 \notin \cup_{i=2}^{n} P_i$. Otherwise $P_1 \subseteq \cup_{i=2}^{n} P_i$, so by [10, Theorem 3.61], $P_1 \subseteq P_j$ for some j, and hence $P_1 = P_j$, a contradiction. Thus there exists a positive integer m_1 such that $r_1^{m_1} \in \mathrm{Ann}(M_1)$ and the module $r_1^{m_1} M = \sum_{i=2}^{n} r_1^{m_1} M_i$ is representable. By using this process for the ideals $P_2, \dots, P_{n-1}$, there are integers $m_2, \dots, m_{n-1}$ and elements $r_2 \in P_2, \dots, r_{n-1} \in P_{n-1}$ such that $s_n M = M_n$, where $0 \neq s_n = r_1^{m_1} r_2^{m_2} \cdots r_{n-1}^{m_{n-1}}$, $s_n \in \cap_{i=1}^{n-1} P_i$ and $s_n \notin P_n$. Therefore by a similar argument, there are elements $s_1, \dots, s_{n-1}$

such that $M = \sum_{i=1}^{n} s_i M$, where for each i, where $i = 1,\dots,n$, $s_i \notin P_i$, $s_i M = M_i$, and $s_i \in \cap_{i=1_{i\neq j}}^{n} \mathrm{Ann}(M_j)$.

Let N be a nonzero submodule of M and $0 \neq a \in N$. Then $a = s_1 b_1 + \cdots + s_n b_n$ for some $b_i \in M$, $i = 1,\dots,n$. By assumption, there exists $t_1,\dots,t_n \in R$ such that for each i, $s_i = s_i^2 t_i$. As $0 \neq a$, $s_i b_i \neq 0$ for some i and $s_i t_i a = s_i^2 t_i b_i = s_i b_i$, so $s_i N \neq 0$. We can assume that $s_{i_1} N \neq 0,\dots,s_{i_k} N \neq 0$, where $\{i_1,\dots,i_k\} \subseteq \{1,\dots,n\}$. By a similar argument as above, if $a \in N$, then $a = \sum_{j=1}^{k} s_{i_j} t_{i_j} a \in \sum_{j=1}^{k} s_{i_j} N$, and hence $N = \sum_{j=1}^{k} s_{i_j} N$. Since for each j, where $j = 1,\dots,k$, $s_{i_j} N$ is pure in the P_{i_j}-secondary module M_{i_j}, it is P_{i_j}-secondary by [2, Lemma 2.1], as required. □

Theorem 2.4. *Let R be a commutative ring and let N be a prime submodule of secondary R-module of M. Then N is $(N:M)$-secondary.*

Proof. Suppose that M is a P-secondary module over R. Let $r \in R$. If $r \in P$, then $r^n N \subseteq r^n M = 0$ for some n. If $r \notin P$, then $rM = M$. Suppose that $n \in N$, so there is an element $m \in M$ such that $n = rm$. As N is a prime submodule of M and $N \neq rM = M$, $m \in N$, so $rN = N$, hence N is P-secondary.

By [4, Lemma 1], the ideal $P' = (N:M) = \{r \in R : rM \subseteq N\}$ is prime. Clearly, $P' \subseteq P$. Let $s \in P$. Then $s^n N = s^n M = 0$ for some n. There is an element $m \in M$ such that $m \notin N$ and $s^n m = 0 \in N$, so $s^n \in P'$, hence $s \in P'$. Thus $P = P'$, as required. □

Proposition 2.5. *Let R be a commutative ring and let N be a prime submodule of P-second R-module of M. Then N is an* RD*-submodule of M.*

Proof. Let $r \in R$. If $r \in P$, then $rN \subseteq rM = 0$, so $rN = N \cap rM = 0$. If $r \notin P$, then $rM = M$, so the homothety $M/N \xrightarrow{r\cdot} M/N$ is not zero since N is prime. It follows that the above homothety is injective. If $a \in N \cap rM$, then there is $b \in M$ such that $a = rb$. Since $r(b+N) = 0$, so $b \in N$, hence $rN = N \cap rM$, as required. □

Theorem 2.6. *Let M be a P-second module over a commutative ring R, and let N be a prime submodule of M. Then every submodule of M properly containing N is an* RD*-submodule. In particular, it is P-second.*

Proof. Let K be a submodule of M properly containing N. Then K/N is a prime submodule of prime and P-second module M/N, so by Proposition 2.5, K/N is an RD-submodule of M/N. Now the assertion follows from [3, Consequences 18-2.2(c)] and Proposition 2.5. □

Lemma 2.7. *Let M be a nonzero module over a commutative domain R. Then M is (0)-second if and only if M is (0)-secondary.*

Proof. The proof is completely straightforward. □

By [3, Proposition 11-3.11] and [11, Proposition 12, page 506] (see also [14]), and the definitions of secondary and primary modules, we obtain the following corollary.

Corollary 2.8. *Let R be a commutative ring.*

(i) *Every Artinian primary module over R is secondary.*

(ii) *Every Noetherian secondary module over R is primary.*

(iii) *Every finitely generated secondary module is primary.*

Lemma 2.9. *Let R be a commutative ring. Let K and N be submodules of an R-module M such that N is prime and K is P-secondary. Then $N\cap K$ is P-secondary.*

Proof. Let $r\in R$. If $r\in P$, then $r^n(N\cap K)\subseteq r^nK=0$ for some n. Suppose $r\notin P$ and $t\in N\cap K$. Then $t=rs$ for some $s\in K$ since K P-secondary. As N is prime, we have $s\in N$, and hence $t\in r(N\cap K)$. This gives, $N\cap K=r(N\cap K)$. □

Theorem 2.10. *Let M be a representable module over a commutative ring R, and let N be a prime submodule of M with $(N:M)=P$. Then the following hold:*

(i) *N is representable;*

(ii) *M/N is P-secondary.*

Proof. (i) Let M be a representable R-module and let $M=\sum_{i=1}^m M_i$ be a minimal secondary representation with $\text{nilrad}(M_i)=P_i$. For each i, $i=1,2,\dots,m$, let $m_i\in M_i$ and $r_i\in P_i$. Then $r_i^{n_i}m_i=0$ for some n_i, and we have $(r_i^{n_i}+P)(m_i+M_i)=0$ and hence either $P_i\subseteq P$ or $M_i\subseteq N$ $(i=1,2,\dots,m)$. It follows that $M_i\not\subseteq N$ for some i (otherwise $M=N$). If $M_i\not\subseteq N$ and $M_j\not\subseteq N$ for $i\neq j$, then $P=P_i=P_j$, a contradiction (for if $t\in P-P_i$ then $M_i=tM_i\subseteq tM\subseteq N$). Therefore, without loss of generality, we can assume that $M_1\not\subseteq N$ and $M_i\subseteq N$, so $P_1=P$ and $P_i\not\subseteq P$ $(i=2,3,\dots,m)$. Then $M_2+M_3+\cdots+M_m\subseteq N$ and

$$N=N\cap M=N\cap(M_1+\cdots+M_m)=M_2+\cdots+M_m+(N\cap M_1). \tag{2.1}$$

Now the assertion follows from Lemma 2.9.

(ii) Since $M=M_1+N$, we have $M/N=(M_1+N)/N\cong M_1/(M_1\cap N)$, as required. □

Proposition 2.11. *Let R be a Dedekind domain, and let M be a $0\neq P$-secondary R-module. Then M is a P-primary module.*

Proof. Let $r\in R$. If $r\in P$, then the homothety $M\xrightarrow{r\cdot}M$ is nilpotent since M is secondary. Suppose that $r\notin P$. If $ra=0$ for some $0\neq a\in M$, then by [6, Theorem 6.1], there exists $0\neq b\in M$ and $Q\in\text{Ass}(M)$ such that $r\in Q$ and $Q=(0:_R b)$. As $(0:M)\subseteq(0:b)=Q$, we have $P=Q$, a contradiction. So the homothety $M\xrightarrow{r\cdot}M$ is injective, as required. □

Remarks. (i) Let R be a domain which is not a field. Then R is a prime R-module (since R is torsion-free) but it is not secondary (even it is not pure-injective).

(ii) Let R be a local Dedekind domain with maximal ideal $P=Rp$. We show that the module $E(R/P)$ is not prime (but it is (0)-secondary). Set $E=E(R/P)$ and $A_n=(0:_E P^n)$ $(n\geq 1)$. Then by [2, Lemma 2.6], $PA_{n+1}=A_n$, $A_n\subseteq E$ is a cyclic R-module with $A_n=Ra_n$ such that $pa_{n+1}=a_n$, every nonzero proper submodule L of E is of the form $L=A_m$ for some m and E is Artinian module with a strictly increasing sequence of submodules

$$A_1\subset A_2\subset\cdots\subset A_n\subset A_{n+1}\subset\cdots. \tag{2.2}$$

We claim that $(A_n:_R E)=0$ for every n. Suppose that $r\in(A_n:_R E)$ with $r\neq 0$. Then $rE\subseteq A_n$ and for all $a\in M$, we have $a=rb$ for some $b\in M$ since E is injective (= divisible). Thus $a=rb\in A_n$, so $E=A_n$, a contradiction. Therefore $(A_n:_R E)=0$ for every integer $n \geq 1$. However no A_n is a prime submodule of E, for if m is any positive integer, then $p^m \notin (A_n :_R E) = 0$ and $a_{n+m} \notin A_n$, but $p^m a_{m+n} = a_n \in A_n$.

Theorem 2.12. *Let R be a Dedekind domain, and let M be an R-module. Then M is $0 \neq P$-second if and only if M is P-prime.*

Proof. By Proposition 2.11, it is enough to show that if M is P-prime, then M is P-second. Since $(0 : M) = P$ is a maximal ideal in R, so M is a vector space over R/P, hence M is P-second. □

Proposition 2.13. *Let R be a Dedekind domain. Then any $0 \neq P$-prime R-module is a direct sum of copies of $R_P/PR_P \cong R/P$.*

Proof. By the proof of Proposition 2.11, every element of $R - P$ acts invertibly on M, so the R-module structure of M extends naturally to a structure of M as a module over the localisation R_P of R at P. Therefore, we can assume that R is a commutative local Dedekind domain with maximal ideal $P = Rp$. Let M_j denote the indecomposable summand of M, so M_j is P-prime. Let m_j be a nonzero element of M_j, hence $(0 : m_j) = (0 : M) = P$. Then $Rm_j \cong R/P$ is pure in M_j since m_j is not divisible by p in M_j, but by [1, Proposition 1.3], the module R/P is itself pure-injective, so Rm_j is a direct summand of M_j, and hence $M_j \cong Rm_j$, as required. □

3. Pure-injective modules

Proposition 3.1. *Let M be a P-secondary module over a commutative ring R. Then $H = H(M)$, the pure-injective hull, is P-secondary.*

Proof. Let $r \in R$. If $r \notin P$, then $rM = M$, so M satisfies the sentence for all x there exists y $(x = ry)$, and hence so does H (because any module and its pure-injective hull satisfy the same sentences [7, Chapter 4]). If $r \in R$, then $r^n M = 0$, so M satisfies the sentence for all x $(r^n x = 0)$, hence so does in H, as required. □

Theorem 3.2. *The following conditions are equivalent for a Prufer domain R:*

(i) *the ring R is a Dedekind domain;*

(ii) *every secondary R-module is pure-injective.*

Proof. Let R be a Dedekind domain and M a secondary R-module. If $\mathrm{Ann}(M) = 0$, then M is divisible, hence injective. If $\mathrm{Ann}(M) \neq 0$, then M is a torsion R-module of bounded order, so that M is Σ-pure-injective (see [15]). In both cases, M is Σ-pure-injective (so pure-injective).

Conversely, let R be a Prufer domain with the property that every secondary module is pure-injective. In order to prove that R is Dedekind domain, it suffices to show that every divisible R-module is injective. Let M be a divisible R-module. Then M is secondary, Hence pure-injective. Since R is Prufer, pure-injective modules are RD-injective (see [7]). The embedding of M in its injective envelope $E(M)$ is an RD-pure monomorphism, because for every nonzero $r \in R$ we have that $M = rM$, so that $rE(M) \cap M \subseteq M \subseteq rM$. Since M is the RD-injective, M is a direct summand of $E(M)$. Thus M is injective. This shows that R is a Dedekind domain. □

Remarks. (i) There is a module over a commutative regular ring which is injective but not secondary (see [9, Theorem 2.3]). The commutative regular ring $R = F \times F$, F a field, is an Artinian Gorenstein, that is, R is injective (so pure-injective) as an R-module. But R is not secondary, because multiplication by $(1,0)$ is neither nilpotent nor surjective.

(ii) The above consideration thus leads us to the following question: are secondary modules pure-injective? The answer is yes because of the following reason. Every non-Noetherian Prufer domain has secondary modules that are not pure-injective. For instance, every non-Noetherian valuation domain has secondary modules that are not pure-injective.

Proposition 3.3. *Let M be an R-module.*

(i) *M is $\sum$-secondary if and only if M is secondary.*

(ii) *Let M be a direct sum of modules M_i ($i \in I$) where for each i, M_i is secondary and $\operatorname{Ann}(M_i) = \operatorname{Ann}(M_j)$ for all $i, j \in I$. Then M is secondary.*

Proof. (i) The necessity is immediate by the definition. Conversely, suppose that M is P-secondary. Given an index set J, and let $r \in R$. If $r \in P$, then $r^n M = 0$ for some n, so $r^n M^{(J)} = 0$. If $r \notin P$ then $rM = M$, so $rM^{(J)} = M^{(J)}$, as required.

(ii) Since the annihilators of all direct summands coincide, we can assume that M_i is P-secondary (say) for all $i \in I$. Now the proof of (ii) is similar to that (i) and we omit it. □

Corollary 3.4. *Let M be an indecomposable Σ-pure-injective module over a commutative Prufer ring R. Then M is secondary.*

Proof. Set $P = \{r \in R : \operatorname{Ann}_M r \neq 0\}$ and $P' = \cap_n P^n$. Then P and P' are prime ideals in R by [8, Fact 3.1 and Lemma 2.1]. By [8, Fact 3.2], M is either P-secondary or P'-secondary, as required. □

Corollary 3.5. *Every Σ-pure-injective module over a Prufer ring is representable.*

Proof. Suppose M is a Σ-pure-injective module over a commutative Prufer ring R. By [8, page 967], we can write $M = M_1 \oplus \cdots \oplus M_m$ where M_i is secondary for all i by Proposition 3.3 and Corollary 3.4, as required. □

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