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SUPER AND SUBSOLUTIONS FOR ELLIPTIC EQUATIONS ON ALL OF $\mathbb{R}^n$

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By construction sub and supersolutions for the following semilinear elliptic equation $-\triangle u(x) = \lambda g(x)f(u(x))$, $x \in \mathbb{R}^n$, which arises in population genetics, we derive some results about the theory of existence of solutions as well as asymptotic properties of the solutions for every n and for the function $g : \mathbb{R}^n \to \mathbb{R}$ such that g is smooth and is negative at infinity.

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1. Introduction. In this paper, we discuss the existence and nonexistence of solutions as well as asymptotic properties of the solutions of the equation

$$-\triangle u(x) = \lambda g(x)f(u(x)), \quad x \in \mathbb{R}^n,\ 0 \le u(x) \le 1 \tag{1.1}$$

which arises in population genetics (see [7, 11]). The unknown function u corresponds to the relative frequency of an allele and is hence constrained to have values between 0 and 1. The real parameter $\lambda > 0$ corresponds to the reciprocal of a diffusion coefficient.

We assume throughout that $g : \mathbb{R}^n \to \mathbb{R}$ is smooth which changes sign on $\mathbb{R}^n$. Also we will assume throughout that f satisfies the condition $f : [0,1] \to \mathbb{R}$ is a smooth function such that $f(0) = f(1) = 0$, $f'(0) > 0$, $f'(1) < 0$, and $f(u) > 0$ for all $0 < u < 1$.

By the definition of f, it is clear that (1.1) has the trivial solutions $u \equiv 0$ and $u \equiv 1$.

The existence of solutions for (1.1) in the bounded region case with Dirichlet or Neumann boundary conditions is discussed in [7, 11], but in this case all of $\mathbb{R}^n$ is much more complicated (see [3, 6, 7, 8, 9, 12, 13]). The results obtained in [7] with the assumption that g is negative at infinity show that the existence theory for solutions of (1.1) is very different for the two cases $n = 1,2$ and $n \ge 3$.

Some of the nontrivial solutions were bifurcating off the trivial solution $u \equiv 0$. In order to investigate these bifurcation phenomena, it was necessary to understand the eigenvalues and eigenfunctions of the corresponding linearized problem

$$-\triangle u(x) = \lambda g(x)f'(0)u(x), \quad x \in \mathbb{R}^n. \tag{1.2}$$

The existence of positive principal eigenfunctions of (1.2) with the following conditions on g was considered in [6]:

(i) g is negative and bounded away from zero at infinity; or

(ii) $|g(x)| \le k/(1+|x|^2)^\alpha$, $n \ge 3$,

for some constants $k > 0$ and $\alpha > 1$, and these results for the case $g^+ \in L^{n/2}(\mathbb{R}^n)$, $n \ge 3$ where $g^+(x) = \max\{g(x),0\}$ are extended in [3].

In this paper, we investigate the existence of solutions of (1.1) with the assumption that g or g^+ are small at infinity.

Our analysis is based on the construction of sub and supersolutions.

It is proved in [2] that the positive principal eigenvalue of the Dirichlet boundary value problem

$$\begin{aligned} -\triangle u(x) &= \lambda g(x)u(x), \quad x \in D, \\ u(x) &= 0, \quad x \in \partial D, \end{aligned} \tag{1.3}$$

where D is a bounded domain with smooth boundary has the variational characterisation

$$\lambda_1^+(D) = \inf\left\{\int_D |\nabla u(x)|^2\,dx : u \in H_0^1(D),\ \int_D g u^2\,dx = 1\right\}. \tag{1.4}$$

Also, it is well known that the above infimum is attained and a minimizer $\phi_1 > 0$ is smooth, that is, $c^2(\overline{D})$. Hence ϕ_1 satisfies the Dirichlet boundary value problem (1.3), so ϕ_1 is a principal eigenfunction corresponding to principal eigenvalue $\lambda_1^+(D)$.

Suppose, however, that $g = g^+ - g^-$ where $g^+(x) = \max\{g(x),0\}$ and $g^-(x) = \min\{g(x),0\}$.

If $n \geq 3$ and $g^+ \in L^{n/2}(\mathbb{R}^n)$, then for all $u \in H_0^1(D)$ such that $\int_D gu^2 dx = 1$ we have

$$\begin{aligned} 1 = \int_D g u^2\,dx &\leq \int_D g^+ u^2\,dx \\ &\leq \|g^+\|_{L^{n/2}(D)} \|u\|^2_{L^{2n/(n-2)}(D)} \\ &\leq c(n)\|g^+\|_{L^{n/2}(D)} \|\nabla u\|^2_{L^2(D)}, \end{aligned} \tag{1.5}$$

where $c(n)$ is the embedding constant of $H_0^1(D)$ into $L^{2n/(n-2)}(D)$ and is independent of D (see Brézis and Nirenberg [5, page 443]). Thus

$$\lambda_1^+(D) \geq \|\nabla u\|^2_{L^2(D)} \geq \left\{c(n)\|g^+\|_{L^{n/2}(D)}\right\}^{-1} > 0. \tag{1.6}$$

Also, it is well known (see [1]) that if $g^+ \in L^{n/2}(\mathbb{R}^n)$, then $\lambda^* = \lim_{\mathbb{R}\to\infty} \lambda_1^+(B_R(0))$ exists and λ^* is the principal eigenvalue of the equation

$$-\triangle u(x) = \lambda g(x)u(x), \quad x \in \mathbb{R}^n \tag{1.7}$$

and there exists a corresponding principal eigenfunction ϕ such that $\phi(x) \to 0$ as $|x| \to \infty$. In addition, λ^* can be characterized as follows (see [1, Lemma 2.7])

$$\lambda^* = \inf\left\{\int_{\mathbb{R}^n} |\nabla u(x)|^2\,dx : u \in c_0^\infty(\mathbb{R}^n),\ \int_{\mathbb{R}^n} g u^2\,dx = 1\right\}. \tag{1.8}$$

Theorem 1.1 (see [10]). *If $\lambda > \lambda^*$, then there exists $\underline{u} \geq 0$ $(\underline{u} \neq 0)$ with compact support such that $\underline{u}$ is a subsolution of*

$$\begin{aligned} -\triangle u(x) &= \lambda g(x) f(u(x)), \quad x \in B_R(0), \\ u(x) &= 0, \quad x \in \partial B_R(0) \end{aligned} \tag{1.9}$$

for all $\mathbb{R}$ sufficiently large, also we can choose $\underline{u}$ sufficiently small.

2. Sub and supersolutions for $n \geq 3$. We assume $D \subset \mathbb{R}^n$ is a bounded region with smooth boundary. We consider the following boundary value problem:

$$\begin{aligned} -\triangle u(x) &= \lambda g(x) f(u(x)), \quad x \in D, \\ u(x) &= 0, \quad x \in \partial D. \end{aligned} \tag{2.1}$$

If $\lambda > 0$ be fixed, we can choose $c > 0$ such that for u, $0 \leq u \leq 1$, the function $u \to \lambda g(x) f(u) + cu$, for every $x \in D$, is an increasing function.

Let $h(x,u) = \lambda g(x) f(u) + cu$, then we have $h(x,0) \equiv 0$ and $h(x,1) \equiv c$. We can write (2.1) as

$$\begin{aligned} -\triangle u(x) + cu(x) &= h(x,u(x)), \quad x \in D, \\ u(x) &= 0, \quad x \in \partial D. \end{aligned} \tag{2.2}$$

It is well known that (2.2) has a unique solution $u = Kf$ (see Amann [4]), where K is given by an integral operator whose kernel is the Green's function for the problem, that is,

$$(Kf)(x) = \int_D G(x,y) h(y,u(y))\, dy. \tag{2.3}$$

In (2.3), $G(x,y)$ is the Green's function of the operator $-\triangle + c$ with Dirichlet boundary condition, also we can write (2.3) as $u = KN(u)$ in where $K : c(\overline{D}) \to c^\alpha(\overline{D})$ is a compact linear integral operator with kernel G (see [4]) and $N : c(\overline{D}) \to c(\overline{D})$ is the Nemytskii operator corresponding to h. Since $h(x,\cdot)$ is increasing, it is easy to see that N is an increasing operator, that is, if $u_1 \geq u_2$, then $Nu_1 \geq Nu_2$.

We call $u \in c^2(D)$ is a subsolution of (2.2) or equivalently (2.1) if we have

$$\begin{aligned} -\triangle u(x) + cu(x) &\leq h(x,u(x)), \quad x \in D, \\ u(x) &\leq 0, \quad x \in \partial D, \end{aligned} \tag{2.4}$$

and $u \in c(\overline{D})$ is a subsolution of (2.3) if

$$u(x) \leq \int_D G(x,y) h(y,u(y))\, dy, \quad x \in D, \tag{2.5}$$

that is, $u \leq KN(u)$. The definition of supersolution is quite similar.

It is well known that if v, w are sub and supersolutions of (2.2) (or for (2.3)), respectively, and $v \leq w$, then there exists a solution u of (2.2) (of (2.3)) such that $v \leq u \leq w$.

3. The case when $n = 1,2$. In this section, we consider the problem

$$-\triangle u(x) = \lambda g(x) f(u(x)), \quad x \in \mathbb{R}^n, \\ 0 \le u(x) \le 1, \quad x \in \mathbb{R}^n, \tag{3.1}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a continuous function which changes sign on $\mathbb{R}^n$ and it has the following condition: (G) there exists $R_0 > 0$ such that $g(x) < 0$ for all of $x \in \mathbb{R}^n$, whenever $|x| > R_0$.

Also $f \in c^1([0,1])$ with the conditions

$$f(0) = 0 = f(1), \qquad f'(0) > 0, \qquad f'(1) < 0, \qquad f(u) > 0, \quad 0 < u < 1. \tag{3.2}$$

Theorem 3.1 (see [7]). *Let u be a nontrivial solution of (4.1). Then there exists a real constant k such that $0 < u(x) < k < 1$ for all of x in $\mathbb{R}^n$.*

Now by using Theorem 3.1 and condition (G) on g, we conclude that

$$\triangle u(x) > 0 \tag{3.3}$$

for all of $x \in \mathbb{R}^n$ with $|x| > R_0$.

Theorem 3.2. *Let u be a nontrivial solution of (4.1). Then u is nonconstant in out of the ball $B_{R_0}(0)$.*

Proof. Using assumption on g, we have $\triangle u(x) > 0$ for all of $x \in \mathbb{R}^n$ with $|x| > R_0$, so $|\nabla u(x)| > 0$ whenever $|x| > R_0$. Hence u is a nonconstant function in out of the ball $B_{R_0}(0)$. □

Theorem 3.3. *Let $n = 1$ and u be a nontrivial solution of (4.1). Then u is a strictly decreasing function on (R_0, ∞) and increasing function on $(-\infty, -R_0)$.*

Proof. By using assumption on g, we have $u''(x) > 0$ for all of $x \in \mathbb{R}^n$ with $|x| > R_0$. So, u can have only one of the possibilities (a) and (b) in Figure 3.1.

Figure 3.1(a) is impossible because we must have $0 \le u(x) \le 1$ for all $x \in \mathbb{R}^n$. So u satisfy in Figure 3.1(b), thus u is strictly decreasing in out of ball $B_{R_0}(0)$. □

Theorem 3.4. *Let $n = 2$ and u be a solution of (4.1) which is radially symmetric, then u is a strictly monotone function in out of the ball $B_{R_1}(0)$, where $R_1 > R_0$.*

Proof. It is obvious by using maximum principle. □

4. The case when $n \ge 3$. Let g satisfy condition (G). It is easy to see that

$$\overline{u}(x) = \begin{cases} 1, & |x| \le R_0, \\ \left(\dfrac{R_0}{|x|}\right)^{(n-2)}, & |x| > R_0, \end{cases} \tag{4.1}$$

is a supersolution of (4.1), so we are ready to prove the following theorem.

Theorem 4.1. *If $\lambda > \lambda^*$, then there exists a nonconstant solution u of (4.1) such that*

$$\lim_{|x| \to \infty} u(x) = 0. \tag{4.2}$$

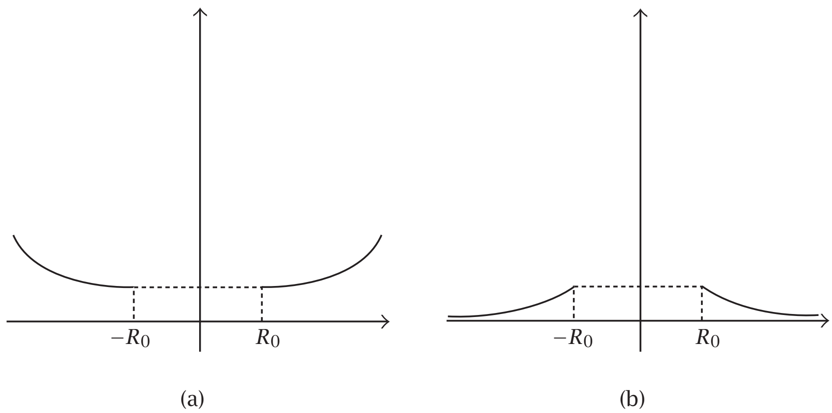

FIGURE 3.1

PROOF. We consider $\overline{u}$ as a supersolution of (4.1). Also there exists a subsolution $\underline{u}$ of (4.1) with compact support and sufficiently small (see [10]). So we can choose $\underline{u}$ such that $\underline{u} \leq \overline{u}$, so there exists a solution u of (4.1) such that $\underline{u} \leq u \leq \overline{u}$. Also by using the definition of $\overline{u}$, we have $\lim_{|x|\to\infty} u(x) = 0$. □

THEOREM 4.2. *Let $\alpha > 1$ and $\lambda > 0$ be arbitrary. Then there exists a supersolution $\overline{u}$ of (4.1) such that $|\overline{u}(x)| \leq c|x|^{-\beta}$ for a constant $c > 0$, and*

$$\beta = \begin{cases} n-2, & n < 2\alpha, \\ 2\alpha - 2, & n > 2\alpha. \end{cases} \tag{4.3}$$

PROOF. Using condition (G) of the function g, we have

$$|g^+(x)| \leq \frac{k}{(1+|x|^2)^{\alpha}}, \tag{4.4}$$

where $k \geq M(1+R_0^2)^{\alpha}$, $M = \max g^+(x)$. So using [10, Lemma 4.3], the proof is complete. □

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