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# ON SOME NEW GENERALIZATIONS OF OSTROWSKI INTEGRAL INEQUALITY

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In this paper, we give some new generalizations of Ostrowski integral inequality on multivariate.

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**1. Introduction.** In 1938, Ostrowski proved the following integral inequality [2, page 468].

**Theorem 1.1.** *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on $(a,b)$ whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on $(a,b)$, that is, $\|f'\|:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\frac{(x-(a+b)/2)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

*for all $x\in[a,b]$. Here the constant $1/4$ is the best possible.*

Recently, Dragomir [1] gave a generalization of Ostrowski integral inequality for mappings whose derivatives belong to $L_p[a,b]$.

**Theorem 1.2.** *Let $I_k: a=x_0<x_1<\cdots<x_{k-1}<x_k$ be a division of the interval $[a,b]$ and $\alpha_i$ (where $i=0,\dots,k+1$) be $(k+2)$ points so that $\alpha_0=a$, $\alpha_i\in[x_{i-1},x_i]$ (where $i=1,\dots,k$), and $\alpha_{k+1}=b$. If $f:[a,b]\to\mathbb{R}$ is absolutely continuous on $[a,b]$. Then*

$$\begin{aligned}&\left|\int_a^b f(x)\,dx-\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)f(x_i)\right|\\&\quad\le\frac{1}{(q+1)^{1/q}}\left[\sum_{i=0}^{k-1}\left[(\alpha_{i+1}-x_i)^{q+1}+(x_{i+1}-\alpha_{i+1})^{q+1}\right]\right]^{1/q}\|f'\|_p\\&\quad\le\frac{1}{(q+1)^{1/q}}\left[\sum_{i=0}^{k-1}h_i^{q+1}\right]^{1/q}\|f'\|_p\le\frac{\nu(h)(b-a)^{1/q}}{(q+1)^{1/q}}\|f'\|_p,\end{aligned} \tag{1.2}$$

*where $h_i:=x_{i+1}-x_i$ (where $i=0,1,\dots,k-1$), $\nu(h):=\max\{h_i\mid i=0,\dots,k\}$, $p>1$, $1/p+1/q=1$, and $\|\cdot\|_p$ is the usual $L_p[a,b]$-norm.*

In this paper, we show other generalizations of the Ostrowski inequality on multivariate.

**2. Main results.** First we introduce the lemma in [1, page 609].

**Lemma 2.1.** *Let the function* $g(t) = (t-a)^{p+1} + (b-t)^{p+1}$, $t \in [a,b]$, $p > 1$. *Then*

$$\frac{(b-a)^{p+1}}{2^p} \le g(t) \le (b-a)^{p+1}. \tag{2.1}$$

Now we show the main results.

**Theorem 2.2.** *Let* $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$, $c = y_0 < y_1 < \cdots < y_{k-1} < x_k = d$ *be a division of* $[a,b]\times[c,d]$ *and* $\alpha_i, \beta_j$ *(where* $i,j = 0,\dots,k+1$*) be* $(2k+2)$ *points, so that* $\alpha_0 = a$, $\alpha_i \in [x_{i-1}, x_i]$ *(where* $i = 1,\dots,k$*),* $\alpha_{k+1} = d$. *If* $f : [a,b]\times[c,d] \to \mathbb{R}$ *is continuous on* $[a,b]\times[c,d]$, $f''_{x,y} = \partial^2 f/\partial x\partial y$ *exist on* $(a,b)\times(c,d)$ *and* $\|f''_{x,y}\|_\infty := \sup_{(x,y)\in(a,b)\times(c,d)} |\partial^2 f(x,y)/\partial x\partial y| < \infty$. *Then*

$$\begin{aligned}
&\Bigg| \int_a^b \int_c^d f(s,t)\,ds\,dt + \sum_{i=0}^{k}\sum_{j=0}^{k} (\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j) f(x_i,y_j) \\
&\quad - \sum_{i=0}^{k} \int_c^d (\alpha_{i+1}-\alpha_i) f(x_i,t)\,dt - \sum_{j=0}^{k} \int_a^b (\beta_{j+1}-\beta_j) f(s,y_j)\,ds \Bigg| \\
&\qquad \le \frac{((b-a)(d-c))^{1/q}}{(p+1)^{2/p}} \|f''_{s,t}\|_\infty \left[ \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} (x_{i+1}-x_i)^{p+1} (y_{j+1}-y_j)^{p+1} \right]^{1/p} \\
&\qquad \le \frac{u(x)v(y)(b-a)(d-c)}{(p+1)^{2/p}} \|f''_{s,t}\|_\infty,
\end{aligned} \tag{2.2}$$

*where* $u(x) := \max\{x_{i+1} - x_i \mid i = 0,\dots,k-1\}$, $v(y) = \max\{y_{j+1} - y_j \mid j = 0,\dots,k-1\}$, $p > 1$, $1/p + 1/q = 1$.

**Proof.** Define the mapping $K : [a,b]\times[c,d] \to \mathbb{R}$ given by

$$K(s,t) = \begin{cases} (s-\alpha_i)(t-\beta_j), & (s,t) \in [x_{i-1},x_i)\times[y_{j-1},y_j)\ (i,j = 1,\dots,k-1), \\ (s-\alpha_i)(t-\beta_k), & (s,t) \in [x_{i-1},x_i)\times[y_{k-1},y_k]\ (i = 1,\dots,k-1,\ j = k), \\ (s-\alpha_k)(t-\beta_j), & (s,t) \in [x_{k-1},x_k]\times[y_{j-1},y_j)\ (i = k,\ j = 1,\dots,k-1), \\ (s-\alpha_k)(t-\beta_k), & (s,t) \in [x_{k-1},x_k]\times[y_{k-1},\ y_k]\ (i = k,\ j = k). \end{cases} \tag{2.3}$$

Integrating by parts, we get

$$
\begin{aligned}
&\int_a^b\int_c^d K(s,t)f''_{s,t}(s,t)\,ds\,dt\\
&\quad=\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}(s-\alpha_{i+1})(t-\beta_{j+1})f''_{s,t}(s,t)\,ds\,dt=\int_a^b\int_c^d f(s,t)\,ds\,dt\\
&\qquad+\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\Bigg\{(y_{j+1}-\beta_{j+1})\Bigg[(x_{i+1}-\alpha_{i+1})f(x_{i+1},y_{j+1})+(\alpha_{i+1}-x_i)f(x_i,y_{j+1})\\
&\qquad\qquad-\int_{x_i}^{x_{i+1}}f(s,y_{j+1})\,ds\Bigg]\\
&\qquad\quad+(\beta_{j+1}-y_j)\Bigg[(x_{i+1}-\alpha_{i+1})f(x_{i+1},y_j)+(\alpha_{i+1}-x_i)f(x_i,y_j)\\
&\qquad\qquad-\int_{x_i}^{x_{i+1}}f(s,y_j)\,ds\Bigg]\\
&\qquad\quad-\int_{y_j}^{y_{j+1}}[(x_{i+1}-\alpha_{i+1})f(x_{i+1},t)+(\alpha_{i+1}-x_i)f(x_i,t)]\,dt\Bigg\}\\
&\quad=\int_a^b\int_c^d f(s,t)\,ds\,dt\\
&\qquad+\sum_{i=0}^{k-1}(y_k-\beta_k)\Bigg[(x_{i+1}-\alpha_{i+1})f(x_{i+1},y_k)+(\alpha_{i+1}-x_i)f(x_i,y_k)\\
&\qquad\qquad-\int_{x_i}^{x_{i+1}}f(s,y_k)\,ds\Bigg]\\
&\qquad+\sum_{i=0}^{k-1}\sum_{j=1}^{k-1}(\beta_{j+1}-\beta_j)\Bigg[(x_{i+1}-\alpha_{i+1})f(x_{i+1},y_j)+(\alpha_{i+1}-x_i)f(x_i,y_j)\\
&\qquad\qquad-\int_{x_i}^{x_{i+1}}f(s,y_j)\,ds\Bigg]\\
&\qquad+\sum_{i=0}^{k-1}(\beta_1-y_0)\Bigg[(x_{i+1}-\alpha_{i+1})f(x_{i+1},y_0)+(\alpha_{i+1}-x_i)f(x_i,y_0)-\int_{x_i}^{x_{i+1}}f(s,y_0)\,ds\Bigg]\\
&\qquad-\sum_{j=0}^{k-1}\int_{y_j}^{y_{j+1}}[(x_{i+1}-\alpha_{i+1})f(x_{i+1},t)+(\alpha_{i+1}-x_i)f(x_i,t)]\,dt\\
&\quad=\int_a^b\int_c^d f(s,t)\,ds\,dt\\
&\qquad+\sum_{i=0}^{k-1}\sum_{j=0}^{k}(\beta_{j+1}-\beta_j)\Bigg[(x_{i+1}-\alpha_{i+1})f(x_{i+1},y_j)+(\alpha_{i+1}-x_i)f(x_i,y_j)\\
&\qquad\qquad-\int_{x_i}^{x_{i+1}}f(s,y_j)\,ds\Bigg]\\
&\qquad+\sum_{j=0}^{k-1}[(x_{i+1}-\alpha_{i+1})f(x_{i+1},t)+(\alpha_{i+1}-x_i)f(x_i,t)]\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= \int_a^b \int_c^d f(s,t)\,ds\,dt + \sum_{i=0}^{k} \sum_{j=0}^{k} (\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j) f(x_i,y_j) \\
&\quad - \sum_{i=0}^{k} \int_c^d (\alpha_{i+1}-\alpha_i) f(x_i,t)\,dt - \sum_{j=0}^{k} \int_a^b (\beta_{j+1}-\beta_j) f(s,y_j)\,ds,
\end{aligned}
\tag{2.4}
$$

and then we get the integral equality

$$
\begin{aligned}
\int_a^b \int_c^d f(s,t)\,ds\,dt &= \int_a^b \int_c^d K(s,t) f''_{s,t}(s,t)\,ds\,dt - \sum_{i=0}^{k} \sum_{j=0}^{k} (\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j) f(x_i,y_j) \\
&\quad + \sum_{i=0}^{k} \int_c^d (\alpha_{i+1}-\alpha_i) f(x_i,t)\,dt + \sum_{j=0}^{k} \int_a^b (\beta_{j+1}-\beta_j) f(s,y_j)\,ds.
\end{aligned}
\tag{2.5}
$$

On the other hand, we have

$$
\begin{aligned}
\left| \int_a^b \int_c^d K(s,t) f''_{s,t}(s,t)\,ds\,dt \right| &= \left| \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} (s-\alpha_{i+1})(t-\beta_{j+1}) f''_{s,t}(s,t)\,ds\,dt \right| \\
&\leq \sum_{i=0}^{k-1} \sum_{j=0}^{k-1} \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \left| (s-\alpha_{i+1})(t-\beta_{j+1}) \right| \left| f''_{s,t}(s,t) \right| ds\,dt.
\end{aligned}
\tag{2.6}
$$

Using Hölder's integral inequality, we get

$$
\begin{aligned}
&\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \left| (s-\alpha_{i+1})(t-\beta_{j+1}) \right| \left| f''_{s,t}(s,t) \right| ds\,dt \\
&\quad \leq \left( \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \left| (s-\alpha_{i+1})(t-\beta_{j+1}) \right|^p ds\,dt \right)^{1/p} \left( \int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \left| f''_{s,t}(s,t) \right|^q ds\,dt \right)^{1/q}.
\end{aligned}
\tag{2.7}
$$

But

$$
\begin{aligned}
&\int_{x_i}^{x_{i+1}} \int_{y_j}^{y_{j+1}} \left| (s-\alpha_{i+1})(t-\beta_{j+1}) \right|^p ds\,dt \\
&\qquad = \left[ \int_{x_i}^{\alpha_{i+1}} (\alpha_{i+1}-s)^p\,ds + \int_{\alpha_{i+1}}^{x_{i+1}} (s-\alpha_{i+1})^p\,ds \right] \\
&\qquad\quad \times \left[ \int_{y_j}^{\beta_{j+1}} (\beta_{j+1}-t)^p\,dt + \int_{\beta_{j+1}}^{y_{j+1}} (t-\beta_{j+1})^p\,dt \right] \\
&\qquad = \frac{1}{(p+1)^2} \left[ (\alpha_{i+1}-x_i)^{p+1} + (x_{i+1}-\alpha_{i+1})^{p+1} \right] \\
&\qquad\quad \times \left[ (\beta_{j+1}-y_j)^{p+1} + (y_{j+1}-\beta_{j+1})^{p+1} \right]
\end{aligned}
\tag{2.8}
$$

and using relations (2.6), (2.7), (2.8), and Hölder's discrete inequality, we get

$$\begin{aligned}
&\left|\int_a^b\int_c^d K(s,t)f''_{s,t}(s,t)\,ds\,dt\right| \\
&\quad\le \frac{1}{(p+1)^{2/p}}\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\Big\{\Big[(\alpha_{i+1}-x_i)^{p+1}+(x_{i+1}-\alpha_{i+1})^{p+1}\Big] \\
&\qquad\qquad \times\Big[(\beta_{j+1}-y_j)^{p+1}+(y_{j+1}-\beta_{j+1})^{p+1}\Big]\Big\}^{1/p} \\
&\qquad\qquad \times\left(\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}|f''_{s,t}(s,t)|^q\,ds\,dt\right)^{1/q} \\
&\quad\le \frac{1}{(p+1)^{2/p}}\Bigg[\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\Big(\Big\{\Big[(\alpha_{i+1}-x_i)^{p+1}+(x_{i+1}-\alpha_{i+1})^{p+1}\Big] \\
&\qquad\qquad \times\Big[(\beta_{j+1}-y_j)^{p+1}+(y_{j+1}-\beta_{j+1})^{p+1}\Big]\Big\}^{1/p}\Big)^p\Bigg]^{1/p} \qquad (2.9)\\
&\qquad \times\left[\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\left(\left(\int_{x_i}^{x_{i+1}}\int_{y_j}^{y_{j+1}}|f''_{s,t}(s,t)|^q\,ds\,dt\right)^{1/q}\right)^q\right]^{1/q} \\
&\quad\le \frac{1}{(p+1)^{2/p}}\Bigg(\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\Big[(\alpha_{i+1}-x_i)^{p+1}+(x_{i+1}-\alpha_{i+1})^{p+1}\Big] \\
&\qquad\qquad \times\Big[(\beta_{j+1}-y_j)^{p+1}+(y_{j+1}-\beta_{j+1})^{p+1}\Big]\Bigg)^{1/p} \\
&\qquad \times\big[(b-a)(d-c)\big]^{1/q}\|f''_{s,t}\|_\infty.
\end{aligned}$$

By Lemma 2.1, the first inequality in (2.2) is obtained.

For the second inequality in (2.2) we only remark that

$$\begin{aligned}
\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}(x_{i+1}-x_i)^{p+1}(y_{j+1}-y_j)^{p+1}&\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}(x_{i+1}-x_i)(y_{j+1}-y_j) \\
&=(b-a)(d-c)\big(u(x)v(y)\big)^p.
\end{aligned} \qquad (2.10)$$

The theorem is completely proved. □

Under the assumption that the points of the division $I_k$ are fixed, the best inequality which is obtained from Theorem 2.2 is embodied in the following corollaries.

**Corollary 2.3.** *Let* $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$, $c = y_0 < y_1 < \cdots < y_{k-1} < x_k = d$ *be a division of* $[a,b]\times[c,d]$. *If* $f$ *is as above, then*

$$\begin{aligned}
&\Bigg|\int_a^b\int_c^d f(s,t)\,ds\,dt+\frac{1}{4}\Bigg\{(x_1-a)\sum_{j=1}^{k-1}(y_{j+1}-y_{j-1})f(a,y_j)+(x_1-a)(y_1-c)f(a,c)\\
&\quad+(y_1-c)\sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i,c)+(b-x_{k-1})(y_1-c)f(b,c)\\
&\quad+\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(x_{i+1}-x_{i-1})(y_{j+1}-y_{j-1})f(x_i,y_j)\\
&\qquad+(b-x_{k-1})(d-y_{k-1})f(b,d)\\
&\quad+(d-y_{k-1})\sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i,d)\\
&\quad+(b-x_{k-1})\sum_{j=1}^{k-1}(y_{j+1}-y_{j-1})f(b,y_j)\\
&\quad-2(x_1-a)\int_c^d f(a,t)\,dt-2\sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})\int_c^d f(x_i,t)\,dt\\
&\quad-2(b-x_{k-1})\int_c^d f(b,t)\,dt-2(y_1-c)\int_a^b f(s,c)\,ds\\
&\quad-2\sum_{j=1}^{k-1}(y_{j+1}-y_{j-1})\int_a^b f(s,y_j)\,ds-2(d-y_{k-1})\int_a^b f(s,d)\,ds\Bigg\}\Bigg|\\
&\le\frac{(b-a)(d-c)^{1/q}}{4(p+1)^{2/p}}\|f''_{s,t}\|_\infty\left[\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\big((x_{i+1}-x_i)(y_{j+1}-y_j)\big)^{p+1}\right]^{1/p}\\
&\le\frac{u(x)v(y)(b-a)(d-c)}{4(p+1)^{2/p}}\|f''_{s,t}\|_\infty.
\end{aligned}\tag{2.11}$$

**Proof.** In Theorem 2.2, let

$$\begin{gathered}
\alpha_0=a,\qquad \alpha_1=\frac{a+x_1}{2},\qquad \alpha_2=\frac{x_1+x_2}{2},\ldots,\\
\alpha_{k-1}=\frac{x_{k-2}+x_{k-1}}{2},\qquad \alpha_k=\frac{x_{k-1}+x_k}{2},\qquad \alpha_{k+1}=b,\\
\beta_0=c,\qquad \beta_1=\frac{c+y_1}{2},\qquad \beta_2=\frac{y_1+y_2}{2},\ldots,\\
\beta_{k-1}=\frac{y_{k-2}+y_{k-1}}{2},\qquad \beta_k=\frac{y_{k-1}+y_k}{2},\qquad \beta_{k+1}=d.
\end{gathered}\tag{2.12}$$

Then we get

$$\begin{aligned}
&\sum_{i=0}^{k}\sum_{j=0}^{k}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)f(x_i,y_j)-\sum_{i=0}^{k}\int_c^d(\alpha_{i+1}-\alpha_i)f(x_i,t)\,dt\\
&\quad-\sum_{j=0}^{k}\int_a^b(\beta_{j+1}-\beta_j)f(s,y_j)\,ds\\
&=\frac{1}{4}\Bigg\{(x_1-a)\sum_{j=1}^{k-1}(y_{j+1}-y_{j-1})f(a,y_j)+(x_1-a)(y_1-c)f(a,c)
\end{aligned}$$

$$
\begin{aligned}
&+(y_1-c)\sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i,c)+(b-x_{k-1})(y_1-c)f(b,c)\\
&+\sum_{i=1}^{k-1}\sum_{j=1}^{k-1}(x_{i+1}-x_{i-1})(y_{j+1}-y_{j-1})f(x_i,y_j)+(b-x_{k-1})(d-y_{k-1})f(b,d)\\
&+(d-y_{k-1})\sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})f(x_i,d)+(b-x_{k-1})\sum_{j=1}^{k-1}(y_{j+1}-y_{j-1})f(b,y_j)\\
&-2(x_1-a)\int_c^d f(a,t)\,dt-2\sum_{i=1}^{k-1}(x_{i+1}-x_{i-1})\int_c^d f(x_i,t)\,dt\\
&-2(b-x_{k-1})\int_c^d f(b,t)\,dt-2(y_1-c)\int_a^b f(s,c)\,ds\\
&\left.-2\sum_{j=1}^{k-1}(y_{j+1}-y_{j-1})\int_a^b f(s,y_j)\,ds-2(d-y_{k-1})\int_a^b f(s,d)\,ds\right\}.
\end{aligned}
\tag{2.13}
$$

Applying inequality (2.2), we obtain the desired result. □

**Corollary 2.4.** *Let $I_k : x_i = a+i((b-a)/k)$, $y_j = c+j((d-c)/k)$ (where $i,j = 0,\dots,k$) be a division of $[a,b]\times[c,d]$. If $f$ is as above, then*

$$
\begin{aligned}
&\Bigg|\int_a^b\int_c^d f(s,t)\,ds\,dt+\frac{(b-a)(d-c)}{4k^2}\Bigg[f(a,c)+f(a,d)+f(b,c)+f(b,d)\\
&\qquad+2\sum_{i=1}^{k-1}\sum_{j=0}^{k} f\left(\frac{(k-i)a+ib}{k},\frac{(k-j)c+jd}{k}\right)\\
&\qquad+2\sum_{i=0}^{k}\sum_{j=1}^{k-1} f\left(\frac{(k-i)a+ib}{k},\frac{(k-j)c+jd}{k}\right)\Bigg]\\
&-\frac{b-a}{2k}\left[\int_c^d f(a,t)\,dt+2\sum_{i=1}^{k-1}\int_c^d f\left(\frac{(k-i)a+ib}{k},t\right)dt+\int_c^d f(b,t)\,dt\right]\\
&-\frac{d-c}{2k}\left[\int_a^b f(s,c)\,ds+2\sum_{j=1}^{k-1}\int_a^b f\left(s,\frac{(k-j)c+jd}{k}\right)ds+\int_a^b f(s,d)\,ds\right]\Bigg|\\
&\qquad\le\frac{((b-a)(d-c))^2}{4k^2(p+1)^{2/p}}\|f''_{s,t}\|_\infty.
\end{aligned}
\tag{2.14}
$$

Similarly, we can obtain the following theorem.

**Theorem 2.5.** *Let $I_k : a = x_0 < x_1 < \cdots < x_{k-1} < x_k = b$, $c = y_0 < y_1 < \cdots < y_{k-1} < y_k = d$, $l = z_0 < z_1 < \cdots < z_{k-1} < z_k = m$ be a division of $[a,b]\times[c,d]\times[l,m]$ and $\alpha_i,\beta_j,\gamma_h$ $(i,j,h = 0,\dots,k+1)$ be $(3k+3)$ points so that $\alpha_0 = a$, $\alpha_i \in [x_{i-1},x_i]$*

*$(i=1,\dots,k)$, $\alpha_{k+1}=b$, $\beta_0=c$, $\beta_j\in[y_{j-1},y_j]$ $(j=1,\dots,k)$, $\beta_{k+1}=d$, $\gamma_0=l$, $\gamma_h\in[z_{h-1},z_h]$ $(h=1,\dots,k)$, $\gamma_{k+1}=m$. If $f:[a,b]\times[c,d]\times[l,m]\to\mathbb{R}$ is continuous on $[a,b]\times[c,d]\times[l,m]$, $f'''_{x,y,z}=\partial^3 f/\partial x\partial y\partial z$ exist on $(a,b)\times(c,d)\times(l,m)$ and $\|f'''_{x,y,z}\|_\infty:=\sup_{(x,y,z)\in(a,b)\times(c,d)\times(l,m)}|\partial^3 f(x,y,z)/\partial x\partial y\partial z|<\infty$. Then*

$$\begin{aligned}
&\Bigg|\int_a^b\int_c^d\int_l^m f(r,s,t)\,dr\,ds\,dt-\sum_{i=0}^{k}\sum_{j=0}^{k}\sum_{h=0}^{k}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)(\gamma_{h+1}-\gamma_h)f(x_i,y_j,z_h)\\
&+\sum_{i=0}^{k}\sum_{h=0}^{k}(\alpha_{i+1}-\alpha_i)(\gamma_{h+1}-\gamma_h)\int_c^d f(x_i,s,z_h)\,ds\\
&+\sum_{j=0}^{k}\sum_{h=0}^{k}(\beta_{j+1}-\beta_j)(\gamma_{h+1}-\gamma_h)\int_a^b f(r,y_j,z_h)\,dr\\
&+\sum_{i=0}^{k}\sum_{j=0}^{k}(\alpha_{i+1}-\alpha_i)(\beta_{j+1}-\beta_j)\int_l^m f(x_i,y_j,t)\,dt\\
&-\sum_{i=0}^{k}(\alpha_{i+1}-\alpha_i)\int_c^d\int_l^m f(x_i,s,t)\,ds\,dt\\
&-\sum_{l=0}^{k}(\gamma_{h+1}-\gamma_h)\int_a^b\int_c^d f(r,s,z_h)\,dr\,ds\\
&-\sum_{j=0}^{k}(\beta_{j+1}-\beta_j)\int_a^b\int_l^m f(r,y_j,t)\,dr\,dt\Bigg|\\
&\qquad\le\frac{((b-a)(d-c)(m-l))^{1/q}}{(p+1)^{3/p}}\\
&\qquad\quad\times\left[\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\sum_{h=0}^{k-1}((x_{i+1}-x_i)(y_{j+1}-y_j)(z_{h+1}-z_h))^{p+1}\right]^{1/p}\|f'''_{r,s,t}\|_\infty\\
&\qquad\le\frac{u(x)v(y)w(z)(b-a)(d-c)(m-l)}{(p+1)^{3/p}}\|f'''_{r,s,t}\|_\infty,
\end{aligned}\tag{2.15}$$

*where $u(x):=\max\{x_{i+1}-x_i\mid i=0,\dots,k-1\}$, $v(y)=\max\{y_{j+1}-y_j\mid j=0,\dots,k-1\}$, $w(z):=\max\{z_{h+1}-z_h\mid h=0,\dots,k-1\}$, $p>1$, $1/p+1/q=1$.*

**Theorem 2.6.** *Let $I_k: a_{i1}=x_{i,0}<x_{i,1}<\cdots<x_{i,k-1}<x_{i,k}=a_{i2}$, $(i=1,\dots,k)$ be a division of $[a_{11},a_{12}]\times\cdots\times[a_{n1},a_{n2}]$ and $\alpha_{ij}$ $(i=1,\dots,k;\ j=0,\dots,k+1)$ be "$nk+n$" points so that $\alpha_{i0}=a_{i1}$, $\alpha_{ij}\in[x_{i,j-1},x_{i,j}]$ $(j=1,\dots,k)$, $\alpha_{i,k+1}=a_{i2}$. If $f:[a_{11},a_{12}]\times\cdots\times[a_{n1},a_{n2}]\to\mathbb{R}$ is continuous on $[a_{11},a_{12}]\times\cdots\times[a_{n1},a_{n2}]$, $f^{(n)}_{s_1,\dots,s_n}=\partial^n f/\partial s_1\cdots\partial s_n$ exist on $(a_{11},a_{12})\times\cdots\times(a_{n1},a_{n2})$ and*

$$\|f^{(n)}_{s_1,\dots,s_n}\|_\infty:=\sup_{(s_1,\dots,s_n)\in(a_{11},a_{12})\times\cdots\times(a_{n1},a_{n2})}\left|\frac{\partial^n f(s_1,\dots,s_n)}{\partial s_1\cdots\partial s_n}\right|<\infty.\tag{2.16}$$

*Then,*

$$\Bigg| \int_{a_{11}}^{a_{12}} \int_{a_{21}}^{a_{22}} \cdots \int_{a_{n1}}^{a_{n2}} f(s_1, s_2, \ldots, s_n)\, ds_1\, ds_2 \cdots ds_n$$

$$+ (-1)^n \sum_{i_1=0}^{k} \sum_{i_2=0}^{k} \cdots \sum_{i_n=0}^{k} (\alpha_{1,i_1+1} - \alpha_{1,i_1})(\alpha_{2,i_2+1} - \alpha_{2,i_2}) \cdots (\alpha_{n,i_n+1} - \alpha_{n,i_n}) f(x_{1,i_1}, x_{2,i_2}, \ldots, x_{n,i_n})$$

$$+ (-1)^{n-1} \sum_{C(i_{j_1} i_{j_2} \ldots i_{j_{n-1}})} \sum_{i_{j_1}, i_{j_2}, \ldots, i_{j_{n-1}}=0}^{k} (\alpha_{j_1, i_{j_1}+1} - \alpha_{j_1, i_{j_1}}) \cdots (\alpha_{j_{n-1}, i_{j_{n-1}}+1} - \alpha_{j_{n-1}, i_{j_{n-1}}})$$

$$\times \int_{a_{jn1}}^{a_{jn2}} f(x_{j_1, i_{j_1}}, \ldots, s, \ldots, x_{j_{n-1}, i_{j_{n-1}}})\, ds + \cdots$$

$$- \sum_{h=1}^{n} \sum_{i_h=0}^{k} \int_{a_{11}}^{a_{12}} \cdots \int_{a_{h-1,1}}^{a_{h-1,2}} \int_{a_{h+1,1}}^{a_{h+1,2}} \cdots \times \int_{a_{n1}}^{a_{n2}} f(s_1, \ldots, s_{h-1}, x_{i_h}, s_{h+1}, \ldots, s_n)\, ds_1 \cdots ds_{h-1}\, ds_{h+1} \cdots ds_n \Bigg|$$

$$\leq \frac{((a_{12} - a_{11}) \cdots (a_{n2} - a_{n1}))^{1/q}}{(p+1)^{n/p}} \|f^{(n)}_{s_1, \ldots, s_n}\|_\infty \times \left[ \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{k-1} \cdots \sum_{i_n=0}^{k-1} ((x_{1,i_1+1} - x_{1,i_1}) \cdots (x_{n,i_n+1} - x_{n,i_n}))^{p+1} \right]^{1/p}$$

$$\leq \frac{(u_1(x_1) \cdots u_n(x_n))(a_{12} - a_{11}) \cdots (a_{n2} - a_{n1})}{(p+1)^{n/p}} \|f^{(n)}_{s_1, \ldots, s_n}\|_\infty, \tag{2.17}$$

*where $C(i_{j_1} i_{j_2} \cdots i_{j_{n-1}})$ is combination dependent on $i_{j_1} i_{j_2} \cdots i_{j_{n-1}}$, $\sum_{C(i_{j_1} i_{j_2} \cdots i_{j_{n-1}})}$ is combinatorial sum dependent on $C(i_{j_1} i_{j_2} \cdots i_{j_{n-1}})$, and $u_i(x_j) := \max\{x_{j,i_j+1} - x_{j,i_j} \mid j = 0, \ldots, k-1\}$.*

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