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ON A CLASS OF MEROMORPHIC p-VALENT STARLIKE FUNCTIONS INVOLVING CERTAIN LINEAR OPERATORS

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Let $\sum_p$ be the class of functions $f(z)$ which are analytic in the punctured disk $\mathbb{E}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Applying the linear operator D^{n+p} defined by using the convolutions, the subclass $\mathcal{T}_{n+p}(\alpha)$ of $\sum_p$ is considered. The object of the present paper is to prove that $\mathcal{T}_{n+p}(\alpha) \subset \mathcal{T}_{n+p-1}(\alpha)$. Since $\mathcal{T}_0(\alpha)$ is the class of meromorphic p-valent starlike functions of order α, all functions in $\mathcal{T}_{n+p-1}(\alpha)$ are meromorphic p-valent starlike in the open unit disk $\mathbb{E}$. Further properties preserving integrals and convolution conditions are also considered.

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1. Introduction. Let $\sum_p$ denote the class of functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{m=1-p}^{\infty} a_m z^m \quad (p \in N = \{1, 2, \dots\}), \tag{1.1}$$

which are analytic in the punctured disk $\mathbb{E}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$. The convolution of two power series $f(z)$, given by (1.1) and

$$g(z) = \frac{1}{z^p} + \sum_{m=1-p}^{\infty} b_m z^m, \tag{1.2}$$

is defined as the following power series:

$$(f * g)(z) = \frac{1}{z^p} + \sum_{m=1-p}^{\infty} a_m b_m z^m. \tag{1.3}$$

Let $\mathcal{S}_p^*(\alpha)$ denote the class of functions of the form (1.1), which satisfy the condition

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} < -p\alpha \quad (z \in \mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}) \tag{1.4}$$

for some α $(0 \leqq \alpha < 1)$. A function $f(z)$ in $\mathcal{S}_p^*(\alpha)$ is called a meromorphic p-valent starlike of order α in $\mathbb{E}$.

A function $f(z) \in \sum_p$ is said to be in the class $\mathcal{T}_{n+p-1}(\alpha)$ if it satisfies the inequality

$$\operatorname{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - \frac{n+2p}{n+p}\right\} < -\frac{p\alpha}{n+p} \quad (z \in \mathbb{E}), \tag{1.5}$$

where n is any integer greater than $-p$, $0 \leqq \alpha < 1$, and

$$D^{n+p-1}f(z) = \frac{1}{z^p(1-z)^{n+p}} * f(z) = \frac{\left(z^{n+2p-1}f(z)\right)^{(n+p-1)}}{(n+p-1)!z^p} = \frac{1}{z^p} + \sum_{m=1-p}^{\infty} \frac{(m+n+2p-1)!}{(n+p-1)!(m+p)!} a_m z^m. \tag{1.6}$$

The operator D^{n+p-1} when $p = 1$ was first introduced by Ganigi and Uralegaddi [1] and then generalized by Yang [9]. In recent years, many authors (e.g., [8, 10, 11]) have investigated certain subclasses of meromorphic functions defined by the operator D^{n+p-1}. In this paper, we show that a function $f(z) \in \mathcal{T}_{n+p-1}(\alpha)$ is meromorphic p-valent starlike of order α. More precisely, it is proved that

$$\mathcal{T}_{n+p}(\alpha) \subset \mathcal{T}_{n+p-1}(\alpha). \tag{1.7}$$

Since $\mathcal{T}_0(\alpha)$ is the class of functions that satisfy the condition

$$\operatorname{Re} \frac{zf'(z)}{f(z)} < -p\alpha \quad (z \in \mathbb{E}), \tag{1.8}$$

the starlikeness of members of $\mathcal{T}_{n+p-1}(\alpha)$ is a consequence of (1.7). Further, integral transforms of functions in $\mathcal{T}_{n+p-1}(\alpha)$ and convolution conditions are also considered.

2. Properties of the class $\mathcal{T}_{n+p-1}(\alpha)$.

In proving our main results, we will need the following lemma.

Lemma 2.1 (see [2, 3]). *Let $w(z)$ be nonconstant and analytic in $\mathbb{E}$, $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then $z_0 w'(z_0) = kw(z_0)$, where k is a real number and $k \geqq 1$.*

Theorem 2.2. *We have $\mathcal{T}_{n+p}(\alpha) \subset \mathcal{T}_{n+p-1}(\alpha)$.*

Proof. Let $f(z) \in \mathcal{T}_{n+p}(\alpha)$, then

$$\operatorname{Re}\left\{\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} - \frac{n+2p+1}{n+p+1}\right\} < -\frac{p\alpha}{n+p+1} \quad (z \in \mathbb{E}). \tag{2.1}$$

We have to show that (2.1) implies the inequality

$$\operatorname{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - \frac{n+2p}{n+p}\right\} < -\frac{p\alpha}{n+p} \quad (z \in \mathbb{E}). \tag{2.2}$$

Consider the analytic function $w(z)$ in $\mathbb{E}$ defined by

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - \frac{n+2p}{n+p} = -\frac{p}{n+p} \cdot \frac{1+(1-2\alpha)w(z)}{1-w(z)}. \tag{2.3}$$

It is clear that $w(0)=0$. Equation (2.3) may be written as

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}=\frac{1-(1+2p(1-\alpha)/(n+p))w(z)}{1-w(z)}. \tag{2.4}$$

Differentiating (2.4) logarithmically and using the identity (easy to verify)

$$z(D^{n+p-1}f(z))'=(n+p)D^{n+p}f(z)-(n+2p)D^{n+p-1}f(z), \tag{2.5}$$

we obtain

$$\begin{aligned}\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)}&-\frac{n+2p+1-p\alpha}{n+p+1}\\ &=\frac{1}{n+p+1}\bigg\{-\frac{p(1-\alpha)(1+w(z))}{1-w(z)}\\ &\qquad-\frac{2p(1-\alpha)}{n+p}\cdot\frac{zw'(z)}{(1-w(z))(1-(1+2p(1-\alpha)/(n+p))w(z))}\bigg\}.\end{aligned} \tag{2.6}$$

We claim that $|w(z)|<1$ in $\mathbb{E}$. For otherwise, by Lemma 2.1, there exists a point z_0 in $\mathbb{E}$ such that $z_0w'(z_0)=kw(z_0)$, where $|w(z_0)|=1$ and $k\geqq 1$. Equation (2.6) yields

$$\begin{aligned}\operatorname{Re}&\left\{\frac{D^{n+p+1}f(z_0)}{D^{n+p}f(z_0)}-\frac{n+2p+1-p\alpha}{n+p+1}\right\}\\ &=\frac{1}{n+p+1}\operatorname{Re}\bigg\{-\frac{p(1-\alpha)(1+w(z_0))}{1-w(z_0)}\\ &\qquad-\frac{2p(1-\alpha)}{n+p}\cdot\frac{kw(z_0)}{(1-w(z_0))(1-(1+2p(1-\alpha)/(n+p))w(z_0))}\bigg\}\\ &\geqq\frac{p(1-\alpha)}{2(n+p+1)(n+2p-p\alpha)}>0,\end{aligned} \tag{2.7}$$

which contradicts (2.1). Hence, $|w(z)|<1$ in $\mathbb{E}$ and it follows that $f(z)\in\mathcal{T}_{n+p-1}(\alpha)$. □

Theorem 2.3. *Let $c>0$ and let $f(z)\in\sum_p$ satisfy the condition*

$$\operatorname{Re}\left\{\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-\frac{n+2p}{n+p}\right\}<-\frac{p\alpha}{n+p}+\frac{p(1-\alpha)}{2(n+p)(c+p-p\alpha)}\quad(z\in\mathbb{E}). \tag{2.8}$$

Then,

$$F(z)=\frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)\,dt \tag{2.9}$$

belongs to $\mathcal{T}_{n+p-1}(\alpha)$.

Proof. From the definition of $F(z)$, we have

$$z\left(D^{n+p-1}F(z)\right)' = cD^{n+p-1}f(z)-(c+p)D^{n+p-1}F(z). \tag{2.10}$$

Using (2.5) and (2.10), condition (2.8) may be written as

$$\begin{aligned}\operatorname{Re}\left\{\frac{(n+p+1)\left(D^{n+p+1}F(z)/D^{n+p}F(z)\right)-(n+p+1-c)}{(n+p)-(n+p-c)\left(D^{n+p-1}F(z)/D^{n+p}F(z)\right)}-\frac{n+2p}{n+p}\right\}& \\ <-\frac{p\alpha}{n+p}+\frac{p(1-\alpha)}{2(n+p)(c+p-p\alpha)}.&\end{aligned} \tag{2.11}$$

We have to prove that (2.11) implies the inequality

$$\operatorname{Re}\left\{\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)}-\frac{n+2p}{n+p}\right\}<-\frac{p\alpha}{n+p}. \tag{2.12}$$

Define $w(z)$ in $\mathbb{E}$ by

$$\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)}-\frac{n+2p}{n+p}=-\frac{p}{n+p}\cdot\frac{1+(1-2\alpha)w(z)}{1-w(z)}. \tag{2.13}$$

Clearly, $w(z)$ is analytic and $w(0)=0$. Equation (2.13) may be written as

$$\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)}=\frac{1-\left(1+2p(1-\alpha)/(n+p)\right)w(z)}{1-w(z)}. \tag{2.14}$$

Differentiating (2.14) logarithmically and using (2.5), we obtain

$$\begin{aligned}&(n+p+1)\frac{D^{n+p+1}F(z)}{D^{n+p}F(z)}-(n+p)\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)}-1\\ &\quad=-\frac{2p(1-\alpha)}{n+p}\cdot\frac{zw'(z)}{(1-w(z))\left(1-\left(1+2p(1-\alpha)/(n+p)\right)w(z)\right)}.\end{aligned} \tag{2.15}$$

Using (2.14) and (2.15), we get

$$\begin{aligned}&\frac{(n+p+1)\left(D^{n+p+1}F(z)/D^{n+p}F(z)\right)-(n+p+1-c)}{(n+p)-(n+p-c)\left(D^{n+p-1}F(z)/D^{n+p}F(z)\right)}-\frac{n+2p-p\alpha}{n+p}\\ &\quad=\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)}-\frac{n+2p-p\alpha}{n+p}\\ &\qquad-\frac{2p(1-\alpha)}{n+p}\cdot\frac{zw'(z)}{(1-w(z))\left(1-(n+3p-2p\alpha)w(z)\right)}\\ &\qquad\qquad\cdot\frac{1}{(n+p)-(n+p-c)\left(D^{n+p-1}F(z)/D^{n+p}F(z)\right)}\\ &\quad=-\frac{p(1-\alpha)}{n+p}\cdot\frac{1+w(z)}{1-w(z)}-\frac{2p(1-\alpha)}{n+p}\cdot\frac{zw'(z)}{(1-w(z))\left(c-(c+2p-2p\alpha)w(z)\right)}.\end{aligned} \tag{2.16}$$

The remaining part of the proof is similar to that of Theorem 2.2. □

According to Theorem 2.2, we have the following corollary at once.

Corollary 2.4. *If $f(z) \in \mathcal{T}_{n+p-1}(\alpha)$, then the function $F(z)$ defined by (2.9) also belongs to $\mathcal{T}_{n+p-1}(\alpha)$.*

Theorem 2.5. *We have $f(z) \in \mathcal{T}_{n+p-1}(\alpha)$ if and only if*

$$F(z) = \frac{n+p}{z^{n+2p}} \int_0^z t^{n+2p-1} f(t)\, dt \in \mathcal{T}_{n+p}(\alpha). \tag{2.17}$$

Proof. From the definition of $F(z)$ we have

$$z\left(D^{n+p-1}F(z)\right)' = (n+p)D^{n+p-1}f(z) - (n+2p)D^{n+p-1}F(z). \tag{2.18}$$

Using identity (2.5), (2.18) reduces to $D^{n+p-1}f(z) = D^{n+p}F(z)$. Hence, $D^{n+p}f(z) = D^{n+p+1}F(z)$ and the result follows. □

Theorem 2.6. *Let $c > 0$ and $0 \leqq \alpha \leqq \beta < 1$. If $F(z) \in \mathcal{T}_{n+p-1}(\alpha)$, then the function $f(z)$, defined by (2.9), belongs to $\mathcal{T}_{n+p-1}(\beta)$ in $|z| < \rho$, where $\rho = \min(\rho_1, \rho_2)$, $\rho_1 = c/(c+2p-2p\alpha) \in (0,1)$, and $\rho_2 \in (0,1)$, is a minimum positive root of the equation*

$$\begin{aligned}(1+\beta-2\alpha)(c+2p-2p\alpha)r^3 - \left[(1-\alpha)(3c+4p-4p\alpha-2) + c(\beta-\alpha)\right]r^2 \\ + \left[2(1-\alpha)(c+1) + (1-\beta)(c+2p-2p\alpha)\right]r - c(1-\beta) = 0.\end{aligned} \tag{2.19}$$

Proof. If $F(z) \in \mathcal{T}_{n+p-1}(\alpha)$, then

$$\frac{D^{n+p}F(z)}{D^{n+p-1}F(z)} - \frac{n+2p}{n+p} = -\frac{p}{n+p}\left[\alpha + (1-\alpha)u(z)\right], \tag{2.20}$$

where $u(z)$ is analytic in $\mathbb{E}$ with $u(0) = 1$ and $\operatorname{Re}(u(z)) > 0$ for $z \in \mathbb{E}$. Using (2.5), (2.10), and (2.20), we have

$$\begin{aligned}(n+p)\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} - (n+2p) + p\beta \\ = p(\beta-\alpha) - p(1-\alpha)u(z) + \frac{p(1-\alpha)zu'(z)}{p(1-\alpha)u(z) - (p-p\alpha+c)}.\end{aligned} \tag{2.21}$$

It is well known that for $|z| = r < 1$,

$$\frac{1-r}{1+r} \leqq \operatorname{Re}(u(z)) \leqq \frac{1+r}{1-r}, \qquad |zu'(z)| \leqq \frac{2r}{1-r^2}\operatorname{Re}(u(z)). \tag{2.22}$$

Thus, from (2.21) we have for $|z| = r < \rho_1 = c/(c+2p-2p\alpha)$,

$$\begin{aligned}
&\operatorname{Re}\left\{(n+p)\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)}-(n+2p)+p\beta\right\} \\
&\quad \leqq p(\beta-\alpha)-p(1-\alpha)\operatorname{Re}u(z)+\left|\frac{p(1-\alpha)zu'(z)}{(p-p\alpha+c)-p(1-\alpha)u(z)}\right| \\
&\quad \leqq \frac{Q(r)}{(1-r^2)[c-(2p-2p\alpha+c)r]},
\end{aligned} \tag{2.23}$$

where

$$\begin{aligned}
Q(r) = {} & p(\beta-\alpha)(1-r^2)[c-(2p-2p\alpha+c)r] \\
& -p(1-\alpha)(1-r)^2[c-(2p-2p\alpha+c)r]+2p(1-\alpha)(r+r^2).
\end{aligned} \tag{2.24}$$

Since $Q(r)$ is continuous on $[0,1]$ with $Q(0) = -cp(1-\beta) < 0$ and $Q(1) = 4p(1-\alpha) > 0$, (2.19) has a minimum positive root $\rho_2 \in (0,1)$. This proves that $f(z)$ belongs to $\mathcal{T}_{n+p-1}(\beta)$ in $|z| < \rho$, where $\rho = \min(\rho_1, \rho_2)$. □

To prove Theorem 2.9, we need the following lemmas.

Lemma 2.7 (see [5]). *The function $(1-z)^\beta = e^{\beta\log(1-z)}$, $\beta \neq 0$, is univalent in $\mathbb{E}$ if and only if β is either in the closed disk $|\beta-1| \leqq 1$ or in the closed disk $|\beta+1| \leqq 1$.*

Lemma 2.8 (see [4]). *Let $q(z)$ be univalent in $\mathbb{E}$ and let $Q(w)$ and $\Phi(w)$ be analytic in a domain D containing $q(\mathbb{E})$, with $\Phi(w) \neq 0$ when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\Phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$ and suppose that*

(1) $Q(z)$ is starlike (univalent) in $\mathbb{E}$,

(2) $Q(z)$ and $h(z)$ satisfy

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left\{\frac{Q'(q(z))}{\Phi(q(z))}+\frac{zQ'(z)}{Q(z)}\right\} > 0, \quad z \in \mathbb{E}. \tag{2.25}$$

If $p(z)$ is analytic in $\mathbb{E}$, with $p(0) = q(0)$, $p(\mathbb{E}) \subset D$, and

$$\theta(p(z)) + zp'(z)\Phi(p(z)) \prec \theta(q(z)) + zq'(z)\Phi(q(z)) = h(z), \tag{2.26}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (2.26).

Theorem 2.9. *Let $f(z) \in \mathcal{T}_{n+p-1}(\alpha)$ and let β be a complex number with $\beta \neq 0$ and satisfy either $|2p\beta(1-\alpha)-1| \leqq 1$ or $|2p\beta(1-\alpha)+1| \leqq 1$. Then*

$$\left(z^p D^{n+p-1}f(z)\right)^\beta \prec (1-z)^{2p\beta(1-\alpha)} = q(z), \quad z \in \mathbb{E}, \tag{2.27}$$

and $q(z)$ is the best dominant.

Proof. Set

$$p(z) = \left(z^p D^{n+p-1} f(z)\right)^{\beta}, \quad z \in \mathbb{E}, \tag{2.28}$$

then $p(z)$ is analytic in $\mathbb{E}$ with $p(0) = 1$. Differentiating (2.28) logarithmically, we have

$$\frac{zp'(z)}{p(z)} = \beta\left[\frac{z\left(D^{n+p-1} f(z)\right)'}{D^{n+p-1} f(z)} + p\right]. \tag{2.29}$$

Since $f \in \mathcal{T}_{n+p-1}(\alpha)$, (2.29) is equivalent to

$$-p + \frac{zp'(z)}{\beta p(z)} \prec -p\frac{1+(1-2\alpha)z}{1-z}. \tag{2.30}$$

On the other hand, if we take $q(z) = (1-z)^{2p\beta(1-\alpha)}$, $\theta(w) = -p$, and $\Phi(w) = 1/\beta w$ in Lemma 2.8, then $q(z)$ is univalent by the condition of the theorem and Lemma 2.7. It is easy to see that $q(z)$, $\theta(w)$, and $\Phi(w)$ satisfy the conditions of Lemma 2.8. Since

$$Q(z) = zq'(z)\Phi(q(z)) = -\frac{2p(1-\alpha)z}{1-z} \tag{2.31}$$

is univalent starlike in $\mathbb{E}$ and

$$h(z) = \theta(q(z)) + Q(z) = -p\frac{1+(1-2\alpha)z}{1-z}, \tag{2.32}$$

it may be readily checked that conditions (1) and (2) of Lemma 2.8 are satisfied. Thus, the result follows from (2.30) and Lemma 2.8. □

Corollary 2.10. *Let $f \in \mathcal{T}_{n+p-1}(\alpha)$. Then*

$$\operatorname{Re}\left[z^p D^{n+p-1} f(z)\right]^{\beta} > 2^{2p\beta(1-\alpha)}, \quad z \in \mathbb{E}, \tag{2.33}$$

where β is a real number and $\beta \in [-1/2p(1-\alpha), 0)$. The result is sharp.

Proof. From Theorem 2.9, we have

$$\operatorname{Re}\left[z^p D^{n+p-1} f(z)\right]^{\beta} = \operatorname{Re}\left[(1-w(z))^{2p\beta(1-\alpha)}\right], \quad z \in \mathbb{E}, \tag{2.34}$$

where $w(z)$ is analytic in $\mathbb{E}$, $w(0) = 0$, and $|w(z)| < 1$ for $z \in \mathbb{E}$.

In view of $\operatorname{Re}(t^b) \geqq (\operatorname{Re} t)^b$ for $\operatorname{Re} t > 0$ and $0 < b \leqq 1$, (2.34) yields

$$\operatorname{Re}\left[z^p D^{n+p-1} f(z)\right]^{\beta} \geqq \left[\operatorname{Re}\frac{1}{1-w(z)}\right]^{-2p\beta(1-\alpha)} > 2^{2p\beta(1-\alpha)}, \quad z \in \mathbb{E}, \tag{2.35}$$

for $-1 \leqq 2p\beta(1-\alpha) < 0$.

To see that the bound $2^{2p\beta(1-\alpha)}$ cannot be increased, we consider the function $f(z)$ which satisfies $z^pD^{n+p-1}f(z) = (1-z)^{2p(1-\alpha)}$. We easily have $f(z) \in \mathcal{T}_{n+p-1}(\alpha)$ and

$$\operatorname{Re}\left[z^pD^{n+p-1}f(z)\right]^{\beta} \longrightarrow 2^{2p\beta(1-\alpha)} \tag{2.36}$$

as $z = \operatorname{Re}(z) \to -1$. The proof of the corollary is complete. □

3. Convolution conditions. In [7], Silverman, Silvia, and Telage considered some convolution conditions for starlikeness of analytic functions. Recently, Silverman and Silvia [6] showed many necessary and sufficient conditions in terms of convolution operators for an analytic function to be in classes of starlike and convex. In this section, we give some necessary and sufficient conditions in terms of convolution operators for meromorphic functions to be in $\mathcal{S}_p^*(\alpha)$ and $\mathcal{T}_{n+p-1}(\alpha)$.

Lemma 3.1. *Let $f(z) \in \sum_p$. Then $f \in \mathcal{S}_p^*(\alpha)$, $0 \leqq \alpha < 1$, and $p \geqq 1$, if and only if $f(z) * [(1-Az)/(z^p(1-z)^2)] \neq 0$ $(0 < |z| < 1)$, where*

$$A = \frac{1+x+2p(1-\alpha)}{2p(1-\alpha)}, \quad |x| = 1. \tag{3.1}$$

Proof. Let $f(z) \in \mathcal{S}_p^*(\alpha)$, then

$$\operatorname{Re}\left\{\frac{-zf'(z)/f(z)-p\alpha}{p-p\alpha}\right\} > 0, \tag{3.2}$$

which is equivalent to

$$\frac{zf'(z)/f(z)+p\alpha}{p-p\alpha} \neq \frac{1-x}{1+x}, \quad |x| = 1,\ x \neq -1. \tag{3.3}$$

This simplifies to

$$\left(zf'(z)+p\alpha f(z)\right)(1+x)-(p-p\alpha)(1-x)f(z) \neq 0 \tag{3.4}$$

in $0 < |z| < 1$. Using (2.5), we have

$$\begin{aligned} f(z) * \frac{1}{z^p(1-z)^2} &= zf'(z)+(1+p)f(z), \\ f(z) * \frac{1}{z^p(1-z)} &= f(z). \end{aligned} \tag{3.5}$$

Therefore, (3.4) is equivalent to

$$f(z) * \left\{\left\{\frac{1}{z^p(1-z)^2}-\frac{1+p}{z^p(1-z)}+\frac{p\alpha}{z^p(1-z)}\right\}(1+x)-(p-p\alpha)(1-x)\frac{1}{z^p(1-z)}\right\} \neq 0, \tag{3.6}$$

that is,

$$f(z) * \frac{1-((1+x+2p(1-\alpha))/2p(1-\alpha))z}{z^p(1-z)^2} \neq 0. \tag{3.7}$$

This proves Lemma 3.1. □

Theorem 3.2. *The function $f(z) \in \mathcal{T}_{n+p-1}(\alpha)$ if and only if*

$$f(z) * \frac{1+[(n+p)(1-A)-1]z}{z^p(1-z)^{n+p+1}} \neq 0 \tag{3.8}$$

for $0<|z|<1$, $|x|=1$, where A is given by (3.1).

Proof. Since $f(z) \in \mathcal{T}_{n+p-1}(\alpha)$ if and only if $D^{n+p-1}f \in \mathcal{S}_p^*(\alpha)$, an application of (1.6) to Lemma 3.1 yields

$$f(z) * \left(g(z) * \left(\frac{1}{z^p(1-z)^2} - \frac{Az}{z^p(1-z)^2}\right)\right) \neq 0, \tag{3.9}$$

where $g(z) = 1/z^p(1-z)^{n+p}$. In view of (3.5), we may write

$$\begin{aligned} g(z) * \left(\frac{1}{z^p(1-z)^2} - \frac{Az}{z^p(1-z)^2}\right) &= g(z) * \frac{1}{z^p(1-z)^2} - Ag(z) * \frac{z}{z^p(1-z)^2} \\ &= zg'(z) + (1+p)g(z) - A(zg'(z) + pg(z)) \\ &= (1-A)zg'(z) + (1+p-Ap)g(z) \\ &= (1-A)\cdot\frac{-p+(n+2p)z}{z^p(1-z)^{n+p+1}} \\ &\quad + (1+p-Ap)\cdot\frac{1}{z^p(1-z)^{n+p}} \\ &= \frac{1+[(n+p)(1-A)-1]z}{z^p(1-z)^{n+p+1}}. \end{aligned} \tag{3.10}$$

This completes the proof of the theorem. □

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