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MORPHISMS OF MISLIN GENERA INDUCED BY FINITE NORMAL SUBGROUPS

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We correct an erroneous statement about induced morphisms of Mislin genera and give the correct statement, even under more general hypotheses.

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As in [9], we denote the class of all finitely generated groups with finite commutator subgroups by $\mathscr{X}_0$, and for an $\mathscr{X}_0$-group H, we let $\chi(H)$ be the set of isomorphism classes of groups K for which $K \times \mathbb{Z} \cong H \times \mathbb{Z}$. If H is a *nilpotent* $\mathscr{X}_0$-group, the Mislin genus (i.e., the genus as defined in [4]) of H is denoted by $\mathscr{G}(H)$. By a result of Warfield [6], we know that if H is a nilpotent $\mathscr{X}_0$-group, then $\chi(H) = \mathscr{G}(H)$. Furthermore, for an $\mathscr{X}_0$-group H, in [9] it is shown that there is an abelian group structure on $\chi(H)$ which coincides with the Hilton-Mislin group structure [3] on $\mathscr{G}(H)$ if H is nilpotent.

In [8, Section 3], it was shown how to define a function $\eta : \chi(H) \to \chi(H/F)$ if H is an infinite $\mathscr{X}_0$-group and F is a finite normal subgroup of H. It was also shown that the function is not always a homomorphism [8, Example 5.4]. This is in conflict with [2, Theorem 1.3]. In fact there is an error in [2, Theorem 1.1] in that the function $\alpha_* : \mathscr{G}(N) \to \mathscr{G}(N/F)$ is not always well defined. The counterexample of [9] suggests a way to show explicitly how things may go wrong. (To merely show that α_* is not always well defined there are simpler examples, but for a simpler example one may find that there is nevertheless some epimorphisms $\mathscr{G}(N) \to \mathscr{G}(N/F)$.) We will show that the results of [2, Section 1] remain valid.

In order to ensure that the relation α_* of [2, Section 1] is a well-defined function, we could follow the option of replacing the domain $\mathscr{G}(N)$ with a different set, which we briefly describe as follows.

Let $\mathscr{N}_0$ be the subclass of $\mathscr{X}_0$ consisting of all infinite nilpotent groups. For an $\mathscr{N}_0$-group H and a suitable finite group F, we fix a monomorphism $h : F \to H$ with $h(F) \lhd H$. Now let K be a group in the Mislin genus of H, and let $k : F \to K$ be any monomorphism with $k(F) \lhd K$ which admits, for every prime p, an isomorphism $f : K_p \to H_p$ for which $f \circ k_p = h_p$. We denote the class of all such pairs (K,k) by $\mathscr{H}_0$. If $l : F \to L$ is another such homomorphism, then we say that $l \sim k$ if there is an isomorphism $\phi : L \to K$ for which $\phi \circ l = k$. Then $\sim$ is an equivalence relation. Let $\mathscr{G}(H,h)$ be the *set* $\mathscr{G}(H,h) = \mathscr{H}_0/\sim$ of all equivalence classes of such endomorphisms. Since $\mathscr{G}(H)$ is finite and since there are only finitely many embeddings of F into H, it is easy to prove that $\mathscr{G}(H,h)$ is a finite set. At least then we can follow [2, Theorem 1.1]. The association $(K,k) \mapsto K/k(F)$ determines a function $\alpha_* : \mathscr{G}(H,h) \mapsto \mathscr{G}(H/h(F))$. There is of course the difficulty that the set $\mathcal{G}(H,h)$ is not well understood, for example, we do not know whether $\mathcal{G}(H,h)$ has a suitable group structure. Anyway, we are interested in $\mathcal{G}(H)$, and we will follow a different option.

We know (see, e.g., [7]) that if F is a characteristic subgroup of the torsion subgroup T_H of H, then we do have a homomorphism $\mathcal{G}(H) \to \mathcal{G}(H/F)$, in fact, an epimorphism. In the calculation that leads up to [2, Theorem 3.1], the subgroup $\ker\alpha$ of N that is being factored out is, indeed, a characteristic subgroup of T (see Proposition 7). Further we note that $\tilde{N}$ is of the form $H \times (\mathbb{Z}_2)$ for some group H, and then by [7, Corollary 4.2] we have an isomorphism $\mathcal{G}(H) \to \mathcal{G}(\tilde{N})$. For such a group H we have (see [1]) that $\mathcal{G}(H) = (\mathbb{Z}_{\tilde{t}})^*/\{1,-1\}$. Thus it follows that [2, Theorem 3.1] is valid. In this paper, we will find a more general condition on the pair $F \triangleleft H$ in order to have a homomorphism $\mathcal{G}(H) \to \mathcal{G}(H/F)$, in fact, an epimorphism. Our result in this regard is more general in that we do not require the group H to be nilpotent.

We recall the following invariant of an $\mathcal{X}_0$-group.

Definition 1 (see [9]). For an $\mathcal{X}_0$-group H, let n_1 be the exponent of the torsion subgroup T_H, let n_2 be the exponent of the group $\mathrm{Aut}(T_H)$, and let n_3 be the exponent of the torsion subgroup of the center of H. We define the natural number $n(H) = n_1 n_2 n_3$.

Note that if H is an $\mathcal{X}_0$-group and K is a group for which $K \times \mathbb{Z} \cong H \times \mathbb{Z}$, then K is also an $\mathcal{X}_0$-group and $T_K \cong T_H$, so that $n(K) = n(H)$. Also note that for such groups H and K, if $\epsilon : H \to K$ is an embedding then the index $[K : \epsilon(H)]$ is finite.

Theorem 2. *Let H be an infinite $\mathcal{X}_0$-group, and let $n = n(H)$. Let F be a finite subgroup of H. The following two conditions are equivalent:*

(1) *given any embedding $\phi : H \to H$ such that $[H : \phi(H)]$ is relatively prime to n, $\phi(F) = F$;*

(2) *if L is any group for which $L \times \mathbb{Z} \cong H \times \mathbb{Z}$, and β_1 and β_2 are any two embeddings of L onto subgroups K_1 and K_2, respectively, of H, with both $[H : K_1]$ and $[H : K_2]$ relatively prime to n, then $\beta_1^{-1}(F) = \beta_2^{-1}(F)$.*

Proof. Assume that condition (1) holds and suppose that we are given L, β_1, and β_2 as in (2). Then F is contained in both K_1 and K_2. In order to prove (2), it suffices to show that, given any isomorphism $\beta : K_1 \to K_2$, $\beta(F) = F$. By [9, Theorem 4.2] it follows that there is an embedding $\gamma : H \to K_1$ such that $[K_1 : \gamma(H)]$ is relatively prime to n (note that $n(H) = n(K_1)$). Let $\epsilon : K_1 \to H$ and $\delta : K_2 \to H$ be the inclusions. Then we have embeddings $\epsilon \circ \gamma$ and $\delta \circ \beta \circ \gamma$ of H into H. By (1), it follows that $\epsilon \circ \gamma(F) = F$ and $\delta \circ \beta \circ \gamma(F) = F$. Moreover, $\epsilon(F) = F$ and $\delta(F) = F$, and consequently we have $\beta(F) = F$. So we have proved that (1) implies (2).

The converse implication is clear. □

Remark 3. Notice that for any infinite $\mathcal{X}_0$-group H and any group L for which $L \times \mathbb{Z} \cong H \times Z$, L is an $\mathcal{X}_0$-group and $n(L) = n(H)$. It is then not hard to see that conditions (1) and (2) of Theorem 2 are equivalent to the following condition:

(3) if β_1 and β_2 are any two embeddings of H onto subgroups K_1 and K_2, respectively, of L, with $[L : K_1]$ and $[L : K_2]$ relatively prime to n, then $\beta_1(F) = \beta_2(F)$.

We are now able to state and prove a significant result on induced morphisms.

Theorem 4. *Let H be an $\mathscr{X}_0$-group, and let $n = n(H)$. Let F be a finite subgroup of H with the property that, given any embedding $\phi : H \to H$ such that $[H : \phi(H)]$ is relatively prime to n, $\phi(F) = F$. Then, for subgroups K of H with $[H : K]$ relatively prime to n, the association $K \mapsto K/F$ defines an epimorphism $\eta : \chi(H) \to \chi(H/F)$.*

Proof. We first note that, by implication, F must be a normal subgroup of H. By the equivalence of (1) and (2) in Theorem 2, it follows that η is well defined. The proof is completed in a way similar to the proof of [7, Theorem 2.1] using [9, Proposition 6.1]. □

For an $\mathscr{X}_0$-group H, T_H has finite characteristic subgroups $[T_H, T_H]$ and ZT_H to which [7, Theorem 2.1] applies. We point out some other subgroups to which the more general Theorem 4 is applicable.

Theorem 5. *Let H be an infinite $\mathscr{X}_0$-group. Let $F = [H,H] \cap T_H$. Then H, together with F, satisfies condition (1) of Theorem 2.*

Proof. Let $\phi : H \to H$ be any embedding such that $[H : \phi(H)]$ is relatively prime to n. Then $\phi[H,H] = [\phi H, \phi H] < [H,H]$. Also $\phi(T_H) < T_H$. Thus $\phi(F) < F$. Since F is finite, it follows that $\phi(F) = F$. □

Theorem 6. *Let H be an infinite $\mathscr{X}_0$-group. Let $F = ZH \cap T_H$. Then H together with F satisfies condition (1) of Theorem 2.*

Proof. Let $\phi : H \to H$ be any embedding such that $[H : \phi(H)]$ is relatively prime to n. Then ϕ can be extended to an isomorphism $\psi : H \times \mathbb{Z}^k \to H \times \mathbb{Z}^k$ for some $k \in \mathbb{N}$ (see the proof of [9, Theorem 4.1]). Now $Z(H \times \mathbb{Z}^k) = (ZH) \times \mathbb{Z}^k$. Since the isomorphism ψ preserves centers and preserves torsion, it follows that $\psi(F) = F$. Since the induced homomorphism ϕ maps T_H isomorphically onto T_H, it follows that $\phi(F) = F$. □

The following result offers an alternative approach to [2, Theorem 3.1], or to a generalization of it.

Proposition 7. *Let $n \in \mathbb{N}$, and let*

$$T = \langle x, y, z \mid x^2 = y^2 = z^{2n} = 1,\ [x,y] = z^n,\ [x,z] = 1 = [y,z] \rangle. \tag{1}$$

Then the subgroup $F = \langle x, y, z^n \rangle$ of T is a characteristic subgroup of T.

Proof. We note that F is generated by elements of order 2 and every element of order 2 in T is contained in F. Therefore F is a characteristic subgroup of T. □

Proposition 8. *Let $n, u \in \mathbb{N}$ be such that u is relatively prime to $2n$. Let t be the multiplicative order of $u \bmod 2n$, and let $\tilde{t}$ be the multiplicative order of $u \bmod n$. Let T and F be the groups of Proposition 7, and let ζ be the action of $\mathbb{Z}$ on T defined (for $a \in \mathbb{Z}$) by*

$$(a, z) \longmapsto z^{(u^a)}, \qquad (a, x) \longmapsto x, \qquad (a, y) \longmapsto y. \tag{2}$$

Then, for the group $H = T \rtimes_\zeta \mathbb{Z}$, $F \lhd H$ *and we have an epimorphism* $\chi(H) \to \chi(H/F) = (\mathbb{Z}_{\tilde{t}})^*/\{1,-1\}$.

In particular, if $\tilde{t} = t$, *then* $\chi(H) \simeq \chi(H/F)$.

Proof. Our conditions ensure that indeed ζ is an action. By Proposition 7, F is a characteristic subgroup of T, and thus by Theorem 4, there is an epimorphism $\chi(H) \to \chi(H/F)$. The group H/F is isomorphic to the group

$$\langle a, b \mid a^n = 1,\ bab^{-1} = a^u \rangle \tag{3}$$

and therefore by [5, Theorem 3.8] we have $\chi(H/F) = (\mathbb{Z}_{\tilde{t}})^*/\{1,-1\}$.

By [8, Theorem 2.6] there is an epimorphism

$$(\mathbb{Z}_t)^*/\{1,-1\} \longrightarrow \chi(H), \tag{4}$$

and so, if $\tilde{t} = t$, then $\chi(H) \simeq \chi(H/F)$. □

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