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COMPLETELY CONTRACTIVE MAPS BETWEEN C^*-ALGEBRAS

W. T. SULAIMAN

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We give a simple proof that any completely contractive map between C^*-algebras is the top right hand corner of a two completely positive unital matrix operator. Some well-known results are deduced.

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1. Introduction. Let A and B be C^*-algebras, $S \subset A$ be a subspace, and $\phi : S \to B$ be a linear map. We define $\phi_n : M_n(S) \to M_n(B)$ by

$$\phi_n[a_{ij}] = [\phi(a_{ij})]. \tag{1.1}$$

We said that ϕ is n-positive if ϕ_n is positive and that ϕ is completely positive if ϕ_n is positive for all n. The map ϕ is said to be n-bounded (resp., n-contractive) if $\|\phi_n\| \le c$ (resp., $\|\phi_n\| \le 1$). The map ϕ is said to be completely bounded (resp., completely contractive) if $\|\phi\|_{cb} = \sup_n \|\phi\|_n < \infty$ (resp., $\|\phi\|_{cc} = \sup_n \|\phi_n\| \le 1$). n-positivity (resp., n-boundedness or n-contractivity) implies $(n-1)$-positivity (resp., $(n-1)$-boundedness or $(n-1)$-contractivity). The converse is not true in general.

For any C^*-algebra A, $M_n(M_p(A))$ is identified with $M_p(M_n(A))$ because there is a canonical isomorphism between $M_n(M_p(A))$ and $M_p(M_n(A))$ by the rearrangement of an $n \times n$ matrix of $p \times p$ blocks as a $p \times p$ matrix of $n \times n$ blocks with the (i,j)th entry of the (k,ℓ)-block becoming the (k,ℓ)th entry of the (i,j)th block. This rearrangement corresponds to a pre- and post-multiplying of a given matrix by a unitary and its adjoint.

2. Main results

Lemma 2.1 (see [1]). *Let A be a C^*-algebra, R, S, and $T \in A$ with T being positive and invertible. Then,*

$$\begin{pmatrix} T & S \\ S^* & R \end{pmatrix} \ge 0 \Longleftrightarrow R \ge S^* T^{-1} S. \tag{2.1}$$

Proof. The lemma follows from the identity

$$\begin{gathered} \begin{pmatrix} 0 & 0 \\ -S^*T^{-1} & I \end{pmatrix} \begin{pmatrix} T & S \\ S^* & R \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -S^*T^{-1} & I \end{pmatrix}^* \\ = \begin{pmatrix} 0 & 0 \\ 0 & R - S^*T^{-1}S \end{pmatrix} = \begin{pmatrix} T & S \\ S^* & R \end{pmatrix} - \begin{pmatrix} T^{1/2} & 0 \\ S^*T^{-1/2} & 0 \end{pmatrix} \begin{pmatrix} T^{1/2} & 0 \\ S^*T^{-1/2} & 0 \end{pmatrix}^* . \end{gathered} \tag{2.2}$$

Let $\phi : A \to B$ be a linear map. We denote by $\phi^* : A \to B$ the linear map defined by

$$\phi^*(a) = (\phi(a^*))^*. \tag{2.3}$$

Let $S(A)$ be the linear subspace of $M_2(A)$ given by

$$S(A) = \left\{ \begin{pmatrix} \lambda & a \\ b & \mu \end{pmatrix} : a, b \in A,\ \lambda, \mu \in \mathbb{C} \right\}. \tag{2.4}$$

Let $\Phi : S(A) \to S(B)$ be defined by

$$\Phi \begin{pmatrix} \lambda & a \\ b & \mu \end{pmatrix} = \begin{pmatrix} \lambda & \phi(a) \\ \phi^*(b) & \mu \end{pmatrix}. \tag{2.5}$$

□

Theorem 2.2. *The map ϕ is n-contractive which implies Φ is n-positive.*

Proof. Let

$$X = \left[\begin{pmatrix} \lambda_{ij} & a_{ij} \\ b_{ij} & \mu_{ij} \end{pmatrix} \right] \in M_n(S(A))^+. \tag{2.6}$$

We may identify X with

$$Y = \begin{pmatrix} [\lambda_{ij}] & [a_{ij}] \\ [b_{ij}] & [\mu_{ij}] \end{pmatrix}. \tag{2.7}$$

Therefore, Y is positive which implies $[\lambda_{ij}]$ and $[\mu_{ij}]$ are positive in $M_n(\mathbb{C})$. Since $Y \geq 0$ if and only if $Y + (1/m)I > 0$ for every $m \in I$, we may assume that $[\lambda_{ij}]$ and $[\mu_{ij}]$ are invertible. We have

$$\begin{aligned}
&\left[\begin{pmatrix} \lambda_{ij} & a_{ij} \\ a_{ji}^* & \mu_{ij} \end{pmatrix} \right] \geq 0 \\
&\Longleftrightarrow \left[\begin{pmatrix} [\lambda_{ij}] & [a_{ij}] \\ [a_{ij}]^* & [\mu_{ij}] \end{pmatrix} \right] \geq 0, \quad \text{via identification} \\
&\Longleftrightarrow [\mu_{ij}] \geq [a_{ij}]^* [\lambda_{ij}]^{-1} [a_{ij}], \quad \text{by Lemma 2.1} \\
&\Longleftrightarrow I \geq [\mu'_{ij}][a_{ij}][\lambda'_{ij}][\lambda'_{ij}][a_{ij}][\mu'_{ij}], \quad \text{where } [\mu'_{ij}] = [\mu_{ij}]^{-1/2}, \quad [\lambda'_{ij}] = [\lambda_{ij}]^{-1/2} \\
&\Longleftrightarrow I \geq \left([\lambda'_{ij}][a_{ij}][\mu'_{ij}] \right)^* \left([\lambda'_{ij}][a_{ij}][\mu'_{ij}] \right)
\end{aligned}$$

$$
\begin{aligned}
&\Longleftrightarrow 1 \geq \left\| \left([\lambda'_{ij}][a_{ij}][\mu'_{ij}]\right)^* \left([\lambda'_{ij}][a_{ij}][\mu'_{ij}]\right) \right\| \\
&\Longleftrightarrow 1 \geq 0\left\| \left([\lambda'_{ij}][a_{ij}][\mu'_{ij}]\right) \right\|^2 \\
&\Longleftrightarrow 1 \geq \left\| \left([\lambda_{ij}][a_{ij}][\mu'_{ij}]\right) \right\| \\
&\Longleftrightarrow 1 \geq 0\left\| \left[\sum_{s,t=1}^{n} \lambda'_{us} a_{st} \mu'_{tr} \right]_{u,r} \right\| \\
&\Longleftrightarrow 1 \geq \left\| \phi_n \left[\sum_{s,t=1}^{n} \lambda'_{us} a_{st} \mu'_{tr} \right]_{u,r} \right\| \\
&\Longleftrightarrow 1 \geq \left\| \left[\sum_{s,t=1}^{n} \lambda'_{us} \phi_n(a_{st}) \mu'_{tr} \right]_{u,r} \right\| \\
&\Longleftrightarrow 1 \geq \left\| [\lambda'_{ij}][\phi(a_{ij})][\mu'_{ij}] \right\| \\
&\Longleftrightarrow 1 \geq \left\| [\lambda'_{ij}][\phi(a_{ij})][\mu'_{ij}] \right\|^2 \\
&\Longleftrightarrow 1 \geq \left\| \left([\lambda'_{ij}][\phi(a_{ij})][\mu'_{ij}]\right)^* \left([\lambda'_{ij}][\phi(a_{ij})][\mu'_{ij}]\right) \right\| \\
&\Longleftrightarrow I \geq \left([\lambda'_{ij}][\phi(a_{ij})][\mu'_{ij}]\right)^* \left([\lambda'_{ij}][\phi(a_{ij})][\mu'_{ij}]\right) \\
&\Longleftrightarrow [\mu_{ij}] \geq [\phi(a_{ij})]^* [\lambda_{ij}]^{-1} [\phi(a_{ij})] \\
&\Longleftrightarrow \left[\begin{pmatrix} [\lambda_{ij}] & [\phi(a_{ij})] \\ [\phi(a_{ij})]^* & [\mu_{ij}] \end{pmatrix} \right] \geq 0 \\
&\Longleftrightarrow \left[\begin{pmatrix} \lambda_{ij} & \phi(a_{ij}) \\ \phi^*(a^*_{ji}) & \mu_{ij} \end{pmatrix} \right] \geq 0, \quad \text{via identification} \\
&\Longleftrightarrow \phi_n \left[\begin{pmatrix} \lambda_{ij} & a_{ij} \\ a^*_{ji} & \mu_{ij} \end{pmatrix} \right] \geq 0.
\end{aligned}
\tag{2.8}
$$

This completes the proof of the theorem. □

Theorem 2.3. *Let $\phi : E \to B$ be a map from a selfadjoint subspace E of a C^*-algebra A into a C^*-algebra B. Define a map $\Psi : S(E) \to B$ by*

$$
\Psi \begin{pmatrix} \lambda & a \\ b & \mu \end{pmatrix} = (\lambda + \mu)I + \phi(a) + \phi^*(b). \tag{2.9}
$$

(i) *The map ϕ is n-contractive $\Rightarrow \Psi$ is n-positive.*
(ii) *The map Ψ is $2n$-positive $\Rightarrow \phi$ is n-contractive.*
(iii) *The map ϕ is completely contractive $\Rightarrow \Psi$ is completely positive.*
(iv) *The map Ψ is n-positive $\Rightarrow \|\phi_n\| \leq 2$.*

Proof. (i) Define maps $\Phi : S(E) \to S(B)$, $\delta : M_2(B) \to B$ by

$$\Phi\begin{pmatrix} \lambda & a \\ b & \mu \end{pmatrix} = \begin{pmatrix} \lambda & \phi(a) \\ \phi^*(b) & \mu \end{pmatrix}, \qquad \delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a+b+c+d. \tag{2.10}$$

The map Φ is n-positive by Theorem 2.2. As δ is n-positive (in fact it is completely positive), then $\Psi = \delta \circ \Phi$ is n-positive.

(ii) There are two methods to prove (ii).

Method 1 (see [4]). Via identification, we have

$$\Psi_n\begin{pmatrix} [\lambda_{ij}] & [a_{ij}] \\ [a_{ij}]^* & [\mu_{ij}] \end{pmatrix} = ([\lambda_{ij}]+[\mu_{ij}])I + \phi_n[a_{ij}] + \phi_n^*[b_{ij}], \qquad [\lambda_{ij}],[\mu_{ij}] \in M_n(\mathbb{C}); \quad [a_{ij}],[b_{ij}] \in M_n(A). \tag{2.11}$$

Let $\|[a_{ij}]\| \le 1$, we want to show that $\|\phi_n[a_{ij}]\| \le 1$. Now

$$\begin{aligned}
\|[a_{ij}]\| \le 1 &\Longrightarrow \begin{pmatrix} I_n & [a_{ij}] \\ [a_{ij}]^* & I_n \end{pmatrix} \ge 0 \\
&\Longrightarrow \begin{pmatrix} I_n & 0 & 0 & [a_{ij}] \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ [a_{ij}]^* & 0 & 0 & I_n \end{pmatrix} \ge 0 \\
&\Longrightarrow \begin{pmatrix} \Psi_n\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} & \Psi_n\begin{pmatrix} 0 & [a_{ij}] \\ 0 & 0 \end{pmatrix} \\ \Psi_n\begin{pmatrix} 0 & 0 \\ [a_{ij}]^* & 0 \end{pmatrix} & \Psi_n\begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix} \end{pmatrix} \ge 0 \\
&\Longrightarrow \begin{pmatrix} I_n & \phi_n[a_{ij}] \\ \phi_n^*[a_{ij}]^* & I_n \end{pmatrix} \ge 0 \\
&\Longrightarrow \|\phi_n[a_{ij}]\| \le 1 \\
&\Longrightarrow \|\phi_n\| \le 1.
\end{aligned} \tag{2.12}$$

Method 2. Since $\Psi_{2n} = \delta_{2n} \circ \Phi_{2n}$ and Ψ_{2n}, δ_{2n} are both positive, then Φ_{2n} is also positive. As Φ_{2n} is unital, then $\|\Phi_{2n}\| \le 1$. Hence

$$||\Psi_{2n}|| \le ||\delta_{2n}||||\Phi_{2n}|| = 2 \cdot 1 = 2. \tag{2.13}$$

We identified

$$\begin{pmatrix} \lambda_{ij} & a_{ij} \\ b_{ij} & \mu_{ij} \end{pmatrix} \otimes M_n \quad \text{with} \begin{pmatrix} H & A \\ B & K \end{pmatrix}, \tag{2.14}$$

and write

$$\begin{pmatrix} \lambda_{ij} & a_{ij} \\ b_{ij} & \mu_{ij} \end{pmatrix} \otimes M_n \longleftrightarrow \begin{pmatrix} H & B \\ A & K \end{pmatrix}, \tag{2.15}$$

where

$$\begin{aligned}
&H = [\lambda_{ij}], \quad K = [\mu_{ij}], \quad A = [a_{ij}], \quad B = [b_{ij}], \quad H,K \in M_n,\ A,B \in M_n(A); \\
&\Psi_{2n}\left(\begin{pmatrix} H & A \\ B & K \end{pmatrix} \otimes M_2\right) = \delta_{2n}\left(\Phi_{2n}\begin{pmatrix} H & A \\ B & K \end{pmatrix} \otimes M_2\right) = \delta_{2n}\left(\begin{pmatrix} H & \phi(A) \\ \phi^*(B) & K \end{pmatrix} \otimes M_2\right) \\
&\qquad \longleftrightarrow \delta_{2n}\begin{pmatrix} H \otimes M_2 & \phi(A) \otimes M_2 \\ \phi(B) \otimes M_2 & K \otimes M_2 \end{pmatrix} = \begin{pmatrix} H & \phi(A) \\ \phi^*(B) & K \end{pmatrix} \times 4; \\
&||4\phi_n(A)|| = \left|\left| 4\begin{pmatrix} 0 & \phi_n(A) \\ 0 & 0 \end{pmatrix} \right|\right| = \left|\left| \Psi_{2n}\left(\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \otimes M_n\right) \right|\right| \\
&\qquad = ||\Psi_{2n}|| \left|\left| \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \otimes M_n \right|\right| = 4\|A\| \\
&\qquad \Longrightarrow ||\phi_n|| \le 1.
\end{aligned} \tag{2.16}$$

(iii) The proof of (iii) is obvious.
(iv) Via identification, we have

$$\begin{aligned}
&\Psi_{2n}\begin{pmatrix} H & A \\ B & K \end{pmatrix} = H + K + \phi_n(A) + \phi_n^*(B) \\
&||\phi_n(A)|| = \left|\left| \Psi_{2n}\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \right|\right| = \left|\left| \delta_n \circ \Phi_n \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \right|\right| \\
&\qquad = ||\delta_n||||\Phi_n|| \left|\left| \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \right|\right| \le 2\|A\| \\
&\qquad \Longrightarrow ||\phi_n|| \le 1.
\end{aligned} \tag{2.17}$$

□

3. Applications

Theorem 3.1 (see [2]). *Let E be a linear subspace of a C^*-algebra and let B be a commutative C^*-algebra. Let $\phi : E \to B$ be a linear map. If ϕ is contractive, then it is completely contractive.*

Proof. Define a map $\Psi : S(E) \to B$ by

$$\Psi\begin{pmatrix}\lambda & a\\ b & \mu\end{pmatrix} = (\lambda+\mu)I + \phi(a) + \phi^*(b),$$
$$\begin{aligned}\phi \text{ is contractive} &\Longrightarrow \Psi \text{ is positive (Theorem 2.3)}\\ &\Longrightarrow \Psi \text{ is completely positive [1, Proposition 1.2.2]}\\ &\Longrightarrow \phi \text{ is completely contractive (Theorem 2.3).}\end{aligned} \tag{3.1}$$

□

Theorem 3.2 (see [4, Theorem 1.1.13]). *Let E be a closed selfadjoint subspace of a C^*-algebra. Let B be a commutative C^*-algebra. Let $\Phi : E \to M_n(B)$ be a linear map. If Φ is n-positive then, it is completely positive.*

The following theorem is a generalization of Theorem 3.2.

Theorem 3.3. *Let E be a selfadjoint subspace of a C^*-algebra. Let B be a commutative C^*-algebra. Let $\phi : E \to M_n(B)$ be a linear map. If ϕ is n-contractive then, it is completely contractive.*

Proof. Define a map $\Psi : S(E) \to M_n(B)$ by

$$\Psi\begin{pmatrix}\lambda & a\\ b & \mu\end{pmatrix} = (\lambda+\mu)I + \phi(a) + \phi^*(b),$$
$$\begin{aligned}\phi \text{ is } n\text{-contractive} &\Longrightarrow \Psi \text{ is } n\text{-positive (Theorem 2.3)}\\ &\Longrightarrow \Psi \text{ is completely positive (Theorem 3.2)}\\ &\Longrightarrow \phi \text{ is completely contractive (Theorem 2.3).}\end{aligned} \tag{3.2}$$

□

Theorem 3.4 (see [4, Theorem 1.1.12]). *Let E be a closed selfadjoint subspace of a C^*-algebra, A containing the identity, and let $\phi : E \to M_n = B(\mathbb{C}^n)$ be n-positive map. Then, ϕ possesses a completely positive extension $\Psi : A \to M_n$ and therefore, ϕ is completely positive.*

In 1983 Smith [3] proved the following theorem.

Theorem 3.5 (see [3]). *Let $\phi : A \to M_n$ be bounded. Then $\|\phi\|_{cb} = \|\phi_n\|$.*

Here, we generalize Theorem 3.5 by giving the following theorem.

Theorem 3.6. *Let E be a closed selfadjoint subspace of a C^*-algebra. If $\phi : E \to M_n(\mathbb{C})$ is n-contractive, then ϕ is completely contractive.*

Proof. Define $\Psi : S(E) \to M_n(\mathbb{C})$ by

$$\Psi\begin{pmatrix}\lambda & a\\ b & \mu\end{pmatrix} = (\lambda+\mu)I + \phi(a) + \phi^*(b),$$
$$\begin{aligned}\phi \text{ is } n\text{-contractive} &\Longrightarrow \Psi \text{ is } n\text{-positive (Theorem 2.3)}\\ &\Longrightarrow \Psi \text{ is completely positive (Theorem 3.4)}\\ &\Longrightarrow \phi \text{ is completely contractive (Theorem 2.3).}\end{aligned} \tag{3.3}$$

□

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W. T. Sulaiman: College of Education, Ajman University, P.O. Box 346, Ajman, United Arab Emirates

E-mail address: waad@ajman.ac.ae