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STURMIAN THEOREMS FOR SYSTEMS OF DIFFERENTIAL EQUATIONS

C.Y. CHAN

Department of Mathematics
Florida State University
Tallahassee, Florida 32306

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ABSTRACT. Two Sturmian theorems are established for second order linear nonhomogeneous systems of two differential equations with the use of a maximum principle. The results also hold for homogeneous systems. For illustration, an example is given.

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1. INTRODUCTION.

Sturmian theorems for second order linear homogeneous systems of n differential equations with coefficient matrices having nonnegative off-diagonal elements were given by Ahmad and Lazer [2] with the use of an extremal characterization of the smallest positive eigenvalue. The main purpose here is to establish two Sturmian theorems for second order linear nonhomogeneous systems of two differential equations by using a maximum principle. The results also hold for homogeneous systems. The method used is different from the above-mentioned paper, and we do not require all the off-diagonal elements of the coefficient matrices to be nonnegative; furthermore, we allow the systems to involve first order derivative terms.

Using a maximum principle for a scalar equation, we give some sufficient conditions for one component of a solution of a nonhomogeneous system to be greater than or equal to the other. In particular, Theorems 1 and 4 are useful towards the hypotheses concerning the inequalities of the components of the solutions in the two Sturmian results. As an illustration, an example is given. With the use of a maximum principle for a linear system, two Sturmian theorems for nonhomogeneous systems are established. As an illustration, another example is given.

2. Comparison theorems. Let us consider the nonhomogeneous system

$$Lu_i + \sum_{j=1}^{2} h_{ij}(x)u_j = f_i(x),\ i = 1, 2, \tag{2.1}$$

where $Lu_i \equiv u_i'' + g(x)u_i'$, $u = (u_1(x), u_2(x))$ is a real 2-vector solution, and the coefficients g and h_{ij} are bounded. Such type of systems with $g \equiv 0$ and $f_i \equiv 0$ represents the motion of a particle of unit mass subject to horizontal and vertical forces in the (u_1,u_2)-plane with x denoting the time.

The next four comparison results give us some sufficient conditions for one component of a solution of a boundary value problem involving (2.1) to be greater than or equal to the other component.

THEOREM 1. Let the boundary conditions for the system (2.1) be

$$u_i(a) = r,\ u_i(b) = s,\ i = 1, 2, \tag{2.2}$$

where r and s are given constants. If $f_1 \geq f_2$,

$$h_{11} \leq h_{21}, \tag{2.3}$$

$$\sum_{j=1}^{2} (h_{1j} - h_{2j}) \leq 0, \tag{2.4}$$

then every solution u with $u_2 \geq 0$ for $a < x < b$ satisfies the inequality

$$u_1 \leq u_2 \text{ for } a < x < b.$$

PROOF. Let $v \equiv u_1 - u_2$. Since $f_1 \geq f_2$,

$$Lv + \sum_{j=1}^{2} (h_{1j} - h_{2j})u_j \geq 0 \text{ for } a < x < b.$$

It follows from (2.4) and $u_2 \geq 0$ that

$$(L + h_{11} - h_{21})v \geq 0 \quad \text{for } a < x < b.$$

At the end-points a and b, $v=0$. If $v > 0$ at some interior point of the interval [a, b] , then it attains its positive maximum M at some point in the interior of the interval. By the strong maximum principle (cf. Protter and Weinberger [3, p. 6]) for a scalar equation and the continuity of v, we have $v \equiv M$ for $a \leq x \leq b$. This contradicts $v=0$ at the end-points a and b. Thus

$$u_1 \leq u_2 \text{ for } a < x < b.$$

An argument analogous to the above proof of Theorem 1 gives the following result.

THEOREM 2. If $f_1 \leq f_2$, (2.3) and (2.4) hold, then every solution of the boundary value problem (2.1) and (2.2) with $u_2 \leq 0$ for $a < x < b$ satisfies the inequality $u_1 \geq u_2$ for $a < x < b$.

Remark 1. If $h_{12} \le h_{22}$, and (2.3) holds, then we have (2.4).

To illustrate the above theorems, let us consider the following example.

EXAMPLE 1. Let the boundary value problem for u be given by

$$u_1'' + u_2/2 = 0 \quad \text{for } 0 < x < \pi, \tag{2.5}$$

$$u_2'' + u_2 = 0 \quad \text{for } 0 < x < \pi, \tag{2.6}$$

$$u_i(0) = 0 = u_i(\pi), \; i = 1,2. \tag{2.7}$$

Solving (2.6) and (2.7), we have $u_2 = 2p \sin x$, where p is an arbitrary constant. If $p \geq 0$, then the hypotheses of Theorem 1 are satisfied, and hence for $0 < x < \pi$, $u_1 \le u_2$. If $p < 0$, then the hypotheses of Theorem 2 are satisfied, and hence for $0 < x < \pi$, $u_1 \ge u_2$. In fact, with $u_2 = 2p\sin x$, it follows from (2.5) and (2.7) that $u_1 = p\sin x$.

A proof similar to that of Theorem 1 gives the following two comparison results.

THEOREM 3. If $f_1 \ge f_2$, (2.3) holds, and

$$\sum_{j=1}^{2} (h_{1j} - h_{2j}) \ge 0, \tag{2.8}$$

then every solution of the boundary value problem (2.1) and (2.2) with $u_2 \le 0$ for $a < x < b$ satisfies the inequality $u_1 \le u_2$ for $a < x < b$.

THEOREM 4. If $f_1 \le f_2$, (2.3) and (2.8) hold, then every solution of the boundary value problem (2.1) and (2.2) with $u_2 \geq 0$ for $a < x < b$ satisfies the inequality $u_1 \ge u_2$ for $a < x < b$.

In particular, Theorem 4 gives a criterion for one nonnegative component of a solution to be less than or equal to the other component. This criterion may be used when such inequalities of components are made in the hypotheses of Theorems 5 and 6. In establishing the Sturmian theorems, we need the following strong maximum principle, which follows from the corresponding result for a coupled elliptic system (cf. Protter and Weinberger [3,p. 192]).

LEMMA 1. If $Lu_i + \sum_{j=1}^{2} c_{ij}(x)u_j \geq 0$ for $a < x < b$, $i = 1, 2$, where the coefficients c_{ij} are bounded on the interval $a \leq x \leq b$,

$$c_{ij} \geq 0 \quad \text{for } i \neq j, \text{ and } \sum_{j=1}^{2} c_{ij} \leq 0, \tag{2.9}$$

and if M is a nonnegative constant such that $u_i \leq M$ at $x = a$ and $x = b$ for $i=1, 2$, then $u_i \leq M$ in the interval $a < x < b$.

Remark 2. Condition (2.9) implies that $c_{ii} \leq 0$.

Let

$$LU_i + \sum_{j=1}^{2} H_{ij}(x)U_j = F_i(x), \; i = 1, 2, \tag{2.10}$$

where the coefficients H_{ij} are bounded, and $F_i \leq 0$. Let us consider nontrivial nonnegative solutions of (2.1) and (2.10) respectively. These correspond respectively to trajectories lying in the first quadrant of the (u_1, u_2)-plane and (U_1, U_2)-plane (cf. Cheng [1]). We shall also need the following condition; (I) there does not exist an interval where u vanishes identically.

The following result gives a Sturmian theorem for u satisfying (2.1) and U satisfying (2.10) respectively.

THEOREM 5. If $f_i \geq 0$, $h_{ij} \leq H_{ij}$ for $i, j = 1, 2$, $H_{12} \geq 0$, and one of the two conditions $h_{21} \geq 0$ and $h_{21} \leq 0$ holds, then between any two consecutive zeros of U satisfying (2.10) such that $0 < U_1 \leq U_2$ between the zeros, there exists at most one zero of any solution u of (2.1) satisfying $u_1 \geq u_2 \geq 0$ and condition (I).

PROOF. Between two consecutive zeros of U satisfying (2.10), let $w_i = u_i/U_i$, $i = 1, 2$. Then $w_1 \geq w_2 \geq 0$, and (2.1) gives

$$Lw_i + \frac{2U_i'}{U_i} w_i' + \frac{1}{U_i} \left[\sum_{j=1}^{2} U_j (h_{ij} w_j - H_{ij} w_i) \right] \geq 0. \tag{2.11}$$

For $i = 1$, the last term of the left-hand side of (2.11) is given by

$$(h_{11} - H_{11})w_1 + [(h_{12} - H_{12})w_2 + H_{12}(w_2 - w_1)]U_2/U_1 \le (h_{11} - H_{11})w_1$$

since $w_1 \ge w_2 \ge 0$, $h_{12} \le H_{12}$, and $H_{12} \ge 0$. For $i = 2$, it follows from $w_1 \ge w_2 \ge 0$, $h_{21} \le H_{21}$, and $U_1 \le U_2$ that the last term of the left-hand side of (2.11) is given by

$$[h_{21}(w_1 - w_2) + (h_{21} - H_{21})w_2]U_1/U_2 + (h_{22} - H_{22})w_2$$

$$\le \begin{cases} h_{21}w_1 + (h_{22} - H_{22} - h_{21})w_2 & \text{if } h_{21} \ge 0, \\ (h_{22} - H_{22})w_2 & \text{if } h_{21} \le 0. \end{cases}$$

Thus (2.11) gives rise to the following system

$$Lw_1 + \frac{2U_1'}{U_1} w_1' + (h_{11} - H_{11})w_1 \ge 0, \tag{2.12}$$

$$Lw_2 + \frac{2U_2'}{U_2} w_2' + h_{21}w_1 + (h_{22} - H_{22} - h_{21})w_2 \ge 0 \quad \text{if } h_{21} \ge 0, \tag{2.13}$$

$$Lw_2 + \frac{2U_2'}{U_2} w_2' + (h_{22} - H_{22})w_2 \ge 0 \quad \text{if } h_{21} \le 0. \tag{2.14}$$

If between two consecutive zeros of U, u has two zeros, then $w = (w_1(x), w_2(x))$ also has. Since w determined by (2.12) and (2.13), or by (2.12) and (2.14) satisfies the hypotheses of Lemma 1, it follows that $w \le 0$ between these points. This in turn implies that $u \le 0$ between its two zeros, and we have a contradiction since u is nonnegative and satisfies condition (I). Thus between any two consecutive zeros of U, there exists at most one zero of u.

Let us construct an example to illustrate Theorem 5.

EXAMPLE 2. Let U be the solution of the boundary value problem (2.5), (2.6) and (2.7) given in Example 1 with $p > 0$. Then $0 < U_1 < U_2$ for $0 < x < \pi$. Let us consider nontrivial nonnegative solutions u of the following problem

$$u_1'' + u_2/2 = 0 \quad \text{for } 0 < x, \tag{2.15}$$

$$u_2'' + u_2/4 = 0 \text{ for } 0 < x, \tag{2.16}$$

$$u_1(0) = 0 = u_2(0). \tag{2.17}$$

Solving (2.16) and (2.17), we obtain $u_2 = q\sin(x/2)$, where q is an arbitrary nonnegative constant. Using this, we obtain from (2.15) and (2.17) that

$$u_1 = 2q\sin(x/2) + kx,$$

where k is an arbitrary nonnegative constant. The hypotheses of Theorem 5 are satisfied with $h_{21} \equiv 0$. It follows that there exists at most one zero of u in the interval $0 < x < \pi$. In fact, we see explicitly that the nontrivial nonnegative solutions u determined above do not have a zero in the interval $0 < x < \pi$.

Another Sturmian theorem is as follows.

THEOREM 6. If $f_i \geq 0$ for $i = 1, 2$, $h_{12} \leq H_{12}$, $H_{12} \geq 0$, $h_{21} \geq H_{21}$, $h_{11} \leq H_{11}$, $\sum_{j=1}^{2} (h_{2j} - H_{2j}) \leq 0$, and one of the two conditions $h_{21} \geq 0$ and $h_{21} \leq 0$ holds,

then between any two consecutive zeros of U satisfying (2.10) such that $0 < U_1 \leq U_2$ between the zeros, there exists at most one zero of any solution u of (2.1) satisfying $u_1 \geq u_2 \geq 0$ and condition (I).

PROOF. From (2.11), we obtain the following system (2.12) and

$$Lw_2 + \frac{2U_2'}{U_2} w_2' + h_{21}w_1 + (h_{22} - H_{22} - H_{21})w_2 \geq 0 \quad \text{if } h_{21} \geq 0,$$

$$Lw_2 + \frac{2U_2'}{U_2} w_2' + \sum_{j=1}^{2} (h_{2j} - H_{2j})w_2 \geq 0 \quad \text{if } h_{21} \leq 0.$$

The theorem follows from an argument similar to that in the proof of Theorem 5.

Remark 3. If $f_1 > 0$, then u need not satisfy condition (I) in Theorems 5 and 6. This is because when $f_1 > 0$, then (2.12) becomes a strict inequality, and w_1 cannot be identically zero on an interval. Hence in the proofs of Theorems 5 and 6, $w \leq 0$ between the two zeros of u implies that $w_1 < 0$ somewhere there. This in turn gives $u_1 < 0$ somewhere between the two zeros of u, and we have the desired contradiction since u is nonnegative.

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