ON A CLASS OF CONTINUOUS FLOW

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ABSTRACT. This paper involves characterizations of a class of continuous flows. The flows considered are those in which the positive prolongation of each point coincides with the trajectory through the point. The characterizations are based on the theory of prolongation.

KEY WORDS AND PHRASES. *Flow, prolongation, trajectory, periodic, critical.*

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1.   INTRODUCTION.

The purpose of this paper is to study dynamical systems in which the positive prolongation of each point coincides with the trajectory through the point. Ahmad [1] studied similar flows--flows in which the positive prolongation of each point coincides with the closure of the positive semi-trajectory through   the point. He referred to such flows as flows of characteristic 0^+. Other authors

have studied these flows or similar ones (see [1], [2], [3], [5], [6]). Knight

[6], for example, gave characterizations of flows of characteristic 0, i. e.,

flows in which the prolongation of each point coincides with the closure of the

trajectory through the point. The author in [7] characterized flows in which

the positive prolongation of each point coincides with the positive semi-trajec-

tory through the point.

In this paper we give several characterizations of flows where the posi-

tive prolongation of each point coincides with the trajectory through the point.

We show that several seemingly different flows are all equivalent, thus making

it unnecessary to study all such flows. We also give a simple characterization

of our flows for the case where the phase space is restricted to the two dimen-

sional space R^2.

2.   DEFINITIONS AND NOTATIONS.

We shall let R, R^+, R^- represent the real numbers, the non-negative real

numbers, and the non-positive real numbers, respectively. By a dynamical system

or a continuous flow is meant a pair (X,π), where X is a topological space, re-

ferred to as the phase space, and π: X×R → X is a mapping satisfying the follow-

ing three axioms:

(1) $\pi(x,0) = x$,

(2) $\pi(\pi(x,t),s) = \pi(x,t+s)$,

(3) π is continuous.

For convenience, we shall let $\pi(x,t)$ be denoted by x·t, or simply xt. For each

x ε X, C(x) = x·R, ($C^+(x) = x·R^+$), ($C^-(x) = x·R^-$) represents the trajectory (pos-

itive semitrajectory) (negative semitrajectory) through x. A point x in X   is

called a critical point if xt = x for all t in R. A non-critical point is called

a periodic point if there exists a number t, t $>$ 0, in R satisfying xt = x.   A

subset M of X is said to be _positively invariant (negatively invariant)_ _(in-variant)_ if $C^+(M) = M(C^-(M) = M)$ $(C(M) = M)$. For each x in X, we let $K(x) = \overline{C(x)}$, $K^+(x) = \overline{C^+(x)}$, and $K^-(x) = \overline{C^-(x)}$. The _positive limit set_ of a point x in X is denoted by $L^+(x)$, i. e., $L^+(x) = \{y \in X: xt_i \to y$ for some net (t_i) of R^+ with $t_i \to +\infty\}$. Similarly, the _negative limit set_ and the _limit set_ of x are denoted by $L^-(x)$ and $L(x)$, respectively. The _positive prolongation_ of a point x of X is denoted by $D^+(x)$, i.e., $D^+(x) = \{y \in X:$ there exist nets (x_i) of X and (t_i) of R^+ such that $x_i \to x$, $x_i t_i \to y\}$. Similarly, $D^-(x)$ and $D(x)$ denote the _negative prolongation_ and the _prolongation_ of x, respectively. The _positive prolongational limit set_ of a point x in X is denoted by $J^+(x)$, i.e., $J^+(x) = \{y \in X:$ there exist nets (x_i) of X and (t_i) of R^+ such that $x_i \to x$, $t_i \to +\infty$, $x_i t_i \to y\}$. Similarly, $J^-(x)$ and $J(x)$ denote the _negative prolongational limit set_ and the _prolongational limit set_, respectively. A point x in X is said to be _positively Poisson stable_ if $x \in L^+(x)$. The negative and bilateral versions are defined similarly. A point x in X is said to be _nonwandering_ if $x \in J^+(x)$. A point x in X is said to be _positively dispersive_ if $J^+(x) = \emptyset$. The negative and bilateral versions are defined similarly.

For more information on the above concepts and related notions pertinent to this paper one is referred to the bibliography.

Throughout this paper, the phase space is always assumed to be Hausdorff.

3. CHARACTERIZATIONS OF FLOWS SATISFYING $D^+(x) = C(x)$.

THEOREM 3.1. Let (X,π) be any continuous flow. Then the following condi-tions are equivalent:

(1) $D^+(x) = C(x)$ for all x in X.

(2) $J^+(x) = C(x)$ for all x in X.

(3) $J^-(x) = C(x)$ for all x in X.

(4) $J(x) = C(x)$ for all x in X.

(5) $D(x) = C(x)$ for all x in X, and there are no positively dispersive points.

(6) $D^-(x) = C(x)$ for all x in X.

PROOF. Assume that (1) holds. Let x ε X. Then $J^+(x) \neq \emptyset$. For, $J^+(x) = \emptyset$ implies that $D^+(x) = C^+(x) \cup J^+(x) = C^+(x)$. It follows from (1) that $C^+(x) = C(x)$. Therefore, $x \cdot (-1) = x \cdot t$ for some $t \in R^+$, and hence $x \cdot (t+1) = x$.   This shows that x is either a critical or a periodic point; in either case $L^+(x) = C^+(x) \neq \emptyset$ implies that $J^+(x) \neq \emptyset$, which contradicts the assumption that $J^+(x) = \emptyset$. We note that $J^+(x) \subset D^+(x) = C(x)$. But, since $J^+(x)$ is an invariant set, we must also have $C(x) \subset J^+(x)$. Hence, $J^+(x) = C(x)$, and (2) holds.

Assume that (2) holds. Let x ε X. Then $x \in J^+(x)$. Hence, $x \in J^-(x)$ since, in general, $x \in J^+(y)$ implies $y \in J^-(x)$ for any two points x, y in X.   Since $J^-(x)$ is invariant, we have $C(x) \subset J^-(x)$. Now, let $y \in J^-(x)$. Then $x \in J^+(y) = C(y)$, and hence $y \in C(x)$. This shows that $J^-(x) \subset C(x)$. Therefore, $J^-(x) = C(x)$, and (3) holds.

Assume that (3) holds. Let x ε X. Then $C(x) = J^-(x) \subset J(x)$. Now, let $y \in J^+(x)$. Then, $x \in J^-(y) = C(y)$, and hence $y \in C(x)$. This shows that $J^+(x) \subset C(x)$. Hence, we have $J(x) = J^+(x) \cup J^-(x) \subset C(x)$. Consequently, $J(x) = C(x)$, and (4) holds.

Assume that (4) holds. Let x ε X. Obviously, $D(x) = C(x)$, since $D(x) = C(x) \cup J(x)$. To see that x is not positively dispersive, i. e., $J^+(x) \neq \emptyset$, we note that by (4), $J^+(x) = \emptyset$ implies that $x \in J^-(x)$. But $x \in J^-(x)$ implies that $x \in J^+(x)$, contradicting $J^+(x) = \emptyset$. Thus, (5) holds.

Assume that (5) holds. Let x ε X. Then, $\emptyset \neq J^+(x) \subset D(x) = C(x)$. Since $J^+(x)$ is a nonempty invariant subset of $C(x)$, we must have $C(x) \subset J^+(x)$. Hence,

$x \in J^+(x)$, and, consequently, $x \in J^-(x)$. This shows that $C(x) \subset J^-(x)$. There-

fore, $C(x) \subset D^-(x)$ since $J^-(x) \subset D^-(x)$. It is obvious from (5) that $D^-(x) \subset$

$C(x)$. Thus, we have $D^-(x) = C(x)$, and (6) holds.

Finally, assume that (6) holds. Let $x \in X$. We note that $J^-(x) = \emptyset$ would

imply that $C^-(x) = C(x)$ since $D^-(x) = C^-(x) \cup J^-(x)$. But, as was shown earlier,

$C^-(x) = C(x)$ would imply that x is a critical or a periodic point, thus contra-

dicting $J^-(x) = \emptyset$. Thus, we have $\emptyset \neq J^-(x) \subset D^-(x) = C(x)$. Again, since $J^-(x)$

is invariant, we must have $C(x) \subset J^-(x)$. This implies that $x \in J^-(x)$, and hence,

$x \in J^+(x)$. Therefore, $C(x) \subset J^+(x) \subset D^+(x)$. Now, let $y \in D^+(x)$. Then,

$x \in D^-(y) = C(y)$. But, $x \in C(y)$ implies that $y \in C(x)$. This shows that $D^+(x)$

$\subset C(x)$. Therefore, $D^+(x) = C(x)$, and hence (1) holds. This completes the proof

of the theorem.

We note that if $D(x) = C(x)$, then $D(x) = K(x)$ since $C(x) \subset K(x) \subset D(x)$.

Thus, the equivalence of statements (1) and (5) in the preceding theorem shows

that the class of flows that we are studying here form a subclass of flows of

characteristic 0 (recall [6] that a flow is said to have characteristic 0 if

$D(x) = K(x)$ for all x in the phase space). In addition, we showed in the pre-

ceding theorem that $D^+(x) = C(x)$ implies that $J^+(x) \neq \emptyset$. Since $J^+(x)$ is in-

variant, we must have $C(x) \subset J^+(x)$. Therefore, we have

COROLLARY 3.2. If $D^+(x) = C(x)$ for each x in X, then the flow is a non-

wandering flow of characteristic 0.

EXAMPLE 3.3. Consider the flow (R^2, π) defined by the system of differ-

ential equations

$$\dot{x} = 1$$

$$\dot{y} = 0$$

It is easy to verify that this flow satisfies the condition $D(x) = C(x)$ for

all x in R^2 and is of characteristic 0. But any point x has the property $D^+(x)$ $\neq$ C(x). This shows that our flows form a proper subclass of flows of characteristic 0.

REMARK 3.4. A trivial example of a flow of characteristic 0 that satisfies the condition $D^+(x)$ = C(x) is a global Poincaré center.

THEOREM 3.5. Let (X,π) be a dynamical system where the phase space X is either a locally compact metric space or a complete metric space. Further, assume that $D^+(x)$ = C(x) for each x in X. Then the set of periodic and critical points is dense in X.

PROOF. Obviously, if x is either periodic or critical, then x ϵ C(x) = $L^+(x)$ = $L^-(x)$. On the other hand, suppose that there is a point x, which is Poisson stable but neither periodic nor critical. If x ϵ $L^+(x)$, then since $L^+(x)$ is invariant, we have C(x) $\subset$ $L^+(x)$ $\subset$ $D^+(x)$ = C(x). Therefore, $L^+(x)$ = C(x). But this contradicts a known result (see, e. g., [4]) that if X is either locally compact and metric or complete metric, then every point that is neither periodic nor critical satisfies $\overline{L^+(x) - C(x)}$ = $L^+(x)$. A similar argument shows that if x ϵ $L^-(x)$, then x is either periodic or critical. Thus, we have shown that the set of points that are either periodic or critical coincides with the set of Poisson stable points. We further note that by Corollary 3.2, all points of X are non-wandering. The proof of our theorem now follows from a known result (see, e. g., [4]) that if the phase space X is either locally compact and metric or complete metric, and if all points are nonwandering, then the set of Poisson stable points is dense in X.

THEOREM 3.6. Let (X,π) be any dynamical system, where, $X \subset R^2$.   Then $D^+(x) = C(x)$ for all x in X if and only if the flow has characteristic 0 and there are no positively dispersive points.

PROOF. The "only if" part follows from Corollary 3.2. Now, suppose that the flow has characteristic 0 and $J^+(x) \neq \emptyset$ for each x in X. We have $J^+(x) \subset D(x) = K(x)$. First let us assume that $L(x) = \emptyset$. Then, $K(x) = C(x)$, and hence $J^+(x) \subset C(x)$. But $\emptyset \neq J^+(x) \subset C(x)$ implies that $C(x) \subset J^+(x)$. Hence, $J^+(x) = C(x)$, which implies that $D^+(x) = C(x)$. Now, let us assume that $L(x) \neq \emptyset$. Let $y \in L(x)$. Then $y \in J(x)$ and, hence, $x \in J(y) \subset D(y) = K(y)$. But, since $L(x)$ is a closed invariant set, we must have $K(y) \subset L(x)$. Therefore, $x \in L(x)$. Hence x is either a critical or a periodic point. This implies that $D(x) = K(x) = C(x)$. Thus, $D^+(x) \subset C(x) = L^+(x) \subset D^+(x)$. This shows that $D^+(x) = C(x)$, and the proof is complete.

NOTATION. Let (R^2,π) be any continuous flow and S be the set of critical points. For any $s \in S$ we shall hence forth let

$$N_s = \{x \in R^2 : x = s \text{ or } x \text{ is periodic and } S \cap \text{int } C(x) = \{s\}\}$$

REMARK 3.7. Knight showed in [5] that there are six basic types of planar flows having characteristic 0. These are

(1) parallelizable flows,

(2) flows having a global Poincaré center,

(3) flows where S consists of one local Poincaré center s, N_s is un-
    bounded and ∂N_s is a single trajectory. The restriction of the flow
    to $R^2 - N_s$ is parallelizable,

(4) flows similar to Example 3 of [5],

(5) flows similar to Example 3 of [5], except that $\partial N_{s_1} = \partial N_{s_2}$ where

$S = \{s_1, s_2\}$, and

(6) flows having only critical points.

From this and from theorem 3.6, it follows that there are three basic
types of flows (R^2, π) satisfying $D^+(x) = C(x)$ for each $x \in R^2$. These are
the flows in (2), (5) and (6).

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