Internat. J. Math. & Math. Sci.
Vol. 3 No. 3 (1980) 505-520



# A NON-LINEAR HYPERBOLIC EQUATION

**ELIANA HENRIQUES de BRITO**

Instituto de Matemática - UFRJ
Caixa Postal 1835, ZC-00
Rio de Janeiro - RJ
BRASIL

(Received January 10, 1979)

ABSTRACT. In this paper the following Cauchy problem, in a Hilbert space H, is considered:

$$(I + \lambda A)u'' + A^2u + [\alpha + M(|A^{\frac{1}{2}}u|^2)]Au = f$$
$$u(0) = u_o$$
$$u'(0) = u_1$$

M and f are given functions, A an operator in H, satisfying convenient hypothesis, $\lambda \geq 0$ and $\alpha$ is a real number.

For $u_o$ in the domain of A and $u_1$ in the domain of $A^{\frac{1}{2}}$, if $\lambda > 0$, and $u_1$ in H, when $\lambda = 0$, a theorem of existence and uniqueness of weak solution is proved.

KEY WORDS AND PHRASES. *Nonlinear Wave Equation, Cauchy Problem, Existence and Uniqueness.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. *34G20, 35L05.*

## 1. INTRODUCTION.

The physical origin of the problem here considered lies in the theory of vibrations of an extensible beam of length $\ell$, whose ends are held a fixed distance apart, hinged or clamped, and is either stretched or compressed by an axial force, taking into account the fact that, during vibration, the elements of a beam perform not only a translatory motion, but also rotate; see Timoshenko [9].

A mathematical model for this problem is an initial-boundary value problem for the non-linear hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - \lambda \frac{\partial^4 u}{\partial t^2 \partial x^2} + \frac{\partial^4 u}{\partial x^4} - [\alpha + \int_0^\ell [\frac{\partial u}{\partial s}(s,t)]^2 \, ds] \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

where $u(x,t)$ is the deflection of point $x$ at time $t$, $\alpha$ is a real constant, proportional to the axial force acting on the beam when it is constrained to lie along the $x$ axis, and $\lambda$ is a nonnegative constant ($\lambda = 0$ means neglecting the rotatory inertia, while $\lambda > 0$ means considering it). The non-linearity of the equation is due to considering the extensibility of the beam.

This model, when $\lambda = 0$, was treated by Dickey [2], Ball [1] and, in a Hilbert space formulation, by Medeiros [5]. For related problems, see Pohozaev [7], Lions [4], Menzala [6] and Rivera [8].

In this paper, a theorem of existence and uniqueness of weak solution for a Cauchy problem in a Hilbert space H, is proved for the equation

$$(I + \lambda A)u'' + A^2 u + [\alpha + M(|A^{\frac{1}{2}}u|^2)] \, Au = f, \tag{1.2}$$

with suitable conditions on the operator A and the given functions M and f.

This paper is divided in three parts. In Part 1, the theorem is stated and existence of a weak solution is proved. In Part 2, its uniqueness is established. Finally, an application is given, in Part 3, when H is $L^2(\Omega)$, $\Omega$ a bounded open set with regular boundary in $R^n$, and A is the Laplace operator $-\Delta$.

## 2. EXISTENCE OF WEAK SOLUTION.

Let $H$ be a real Hilbert space, with inner product $( \, , \, )$ and norm $| \; |$.

Let $A$ be a linear operator in $H$, with domain $D(A) = V$ dense in $H$. With the graph norm of $A$, denoted $\| \; \|$, i.e.

$$\| v \|^2 = |v|^2 + |Av|^2 \text{ , for } v \text{ in } V,$$

$V$ is a real Hilbert space and its injection in $H$ is continuous. We assume this injection compact.

Suppose $A$ self-adjoint and positive, i.e., there is a constant $k > 0$ such that

$$(Av,v) \geq k|v|^2 \text{ , for } v \text{ in } V. \tag{2.1}$$

Let $V'$ be the dual of $V$ and $<,>$ denote the pairing between $V'$ and $V$. Identifying $H$ and $H'$, it follows that $V \subset H \subset V'$. Injections being continuous and dense, it is known that, for $h$ in $H$ and $v$ in $V$, $<h,v> = (h,v)$.

Define $A^2$: $V \to V'$ by

$$< A^2u, v > = (Au,Av), \text{ for } u, v \text{ in } V. \tag{2.2}$$

It follows that $A^2$ is a bounded linear operator from $V$ into $V'$.

Let $a(u,v)$ denote the bilinear form in $D(A^{\frac{1}{2}})$ associated to $A$, i.e.,

$$a(u,v) = (A^{\frac{1}{2}}u, A^{\frac{1}{2}}v), \text{ for } u, v \text{ in } D(A^{\frac{1}{2}})$$

$a(u)$ means $a(u,u)$.

Given $\lambda \geq 0$, consider in $W = D((\lambda A)^{\frac{1}{2}})$ the graph norm of $(\lambda A)^{\frac{1}{2}}$, denoted $| \; |_\lambda$, i.e.,

$$|w|_\lambda^2 = |w|^2 + \lambda|A^{\frac{1}{2}}w|^2, \text{ for } w \text{ in } W$$

Note that $W = H$, if $\lambda = 0$, and $W = D(A^{\frac{1}{2}})$, if $\lambda > 0$; hence $V$ is dense in $W$.

Let $\alpha$ be a real number, $M$ a real $C^1$ function, with $M'(s) \geq 0$, for $s \geq 0$. Assume the existence of positive constants $m_o$ and $m_1$ such that $M(s) \geq m_o + m_1 s$, for $s \geq 0$. Notice that, should M be the identity function, replacement of $\alpha + s$ by $(\alpha - m_o) + (m_o + s)$, with arbitrary $m_o > 0$, ensures the fulfilment of the above condition on M.

The theorem can now be stated.

THEOREM. Given f in $L^2(0,T;H)$, $u_o$ in V, $u_1$ in W, there is a unique function $u = u(t)$, $0 \leq t < T$, such that:

a) $u \in L^\infty(0,T;V)$

b) $u' \in L^\infty(0,T;W)$

c) u is a weak solution of

$$(I + \lambda A)u'' + A^2u + [\alpha + M(|A^{\frac{1}{2}}u|^2)]\, Au = f, \tag{2.3a}$$

i.e., for every v in V, u satisfies in $\mathcal{D}'(0,T)$:

$$\frac{d}{dt}[(u'(t),v) + \lambda a(u'(t),v)] + (Au(t),Av) +$$

$$+ [\alpha + M(a(u(t)))]\; a(u(t),v) = (f(t),v)\;, \tag{2.3b}$$

d) The following initial conditions hold:

$$u(0) = u_o\;,\quad u'(0) = u_1 \tag{2.4ab}$$

Before proving the theorem, some remarks are pertinent.

Equation (2.3a) makes sense, because (a) and (b) above imply that u, $A^{\frac{1}{2}}u$, Au, u', $(\lambda A)^{\frac{1}{2}}u'$ belong to $L^\infty(0,T;H)$.

Initial condition (2.4a) makes sense, because it is known, (see Lions [3]) that if u and u' are in $L^\infty(0,T;H)$, then

$$u \text{ belongs to } C^o(0,T;H), \tag{2.5}$$

Now, initial condition (2.4b) must be understood.

Remember $u' \in L^\infty(0,T;W)$ implies that $(I+\lambda A)u' \in L^\infty(0,T;V')$, because

$$\langle (I + \lambda A)u',v\rangle = (u',v) + \lambda a(u',v),\ \text{for v in V}$$

From (2.3a), it follows that $(I + \lambda A)u'' \in L^2(0,T;V')$. The fact that both $(I + \lambda A)u'$ and $(I + \lambda A)u''$ belong to $L^2(0,T;V')$ ensures that

$$(I + \lambda A)u' \in C^o(0,T;V') \tag{2.6}$$

Therefore $(I + \lambda A)u'(0)$ is defined. Given $u_1$ in W, set $(I + \lambda A)u'(0) = (I + \lambda A)u_1$, in V'. It follows that $u'(0) = u_1$, because, it will be proved below,

$$(I + \lambda A)w = 0, \text{ for } w \text{ in } W, \text{ implies } w = 0. \tag{2.7}$$

Indeed, V being dense in W, there is a sequence $(v_j)_{j \in N}$ in V that converges to w in W, i.e., as $j \to \infty$,

$$|w - v_j|_\lambda^2 = |w - v_j|^2 + \lambda a(w - v_j) \longrightarrow 0$$

and

$$0 = \langle(I + \lambda A)w,v_j\rangle = (w,v_j) + \lambda a(w,v_j)$$

tends to

$$(w,w) + \lambda a(w) = |w|_\lambda^2$$

Hence $w = 0$.

<u>Proof of Existence</u>:

It will follow Galerkin method. Suppose, for simplicity, v separable.

Let, then, $(w_j)_{j \in N}$ be a sequence in V such that, for each m, the set $w_1,\dots,w_m$ is linearly independent and the finite linear combinations of $w_1, w_2,\dots$ are dense in V. Let $V_m$ denote the finite subspace of V, spanned by $w_1,\dots,w_m$.

(i) <u>Approximate Solutions</u>

Search for $u_m(t) = \sum_{j=1}^{m} g_{jm}(t)w_j$ in $V_m$, such that, for all v in $V_m$,

$$((I + \lambda A)u_m''(t),v) + (Au_m(t),Av) + [\alpha+M(a(u_m(t))](Au_m(t),v)=(f(t),v) \tag{2.8}$$

$$u_m(0) = u_{om},\ u_m'(0) = u_{1m}, \tag{2.9}$$

where $u_{om}$ converges to $u_o$ in V and $u_{1m}$ to $u_1$ in W.

This system of ordinary differential equations with initial conditions has a solution $u_m(t)$, defined for $0 \le t < t_m \le T$. It is convenient to emphasize that the matrix $((I+\lambda A)w_j,w_i)$, $i,j=1,\ldots,m$, is invertible, for otherwise the homogeneous system of linear equations

$$\sum_{j=1}^{m} ((I + \lambda A)w_j,w_i)x_j = 0 \quad , \quad i = 1,\ldots,m,$$

would have a non-trivial solution $\alpha_1,\ldots,\alpha_m$, hence

$$\Big|\sum_{j=1}^{m} \alpha_j w_j\Big|_\lambda^2 = ((I + \lambda A)\sum_{j=1}^{m} \alpha_j w_j, \sum_{i=1}^{m} \alpha_i w_i) = 0,$$

a contradiction to the linear independence of $w_1,\ldots,w_m$.

(ii) <u>A Priori Estimates</u>

For $v = 2u'_m(t)$, (2.8) becomes:

$$\frac{d}{dt}|u'_m(t)|^2 + \lambda \frac{d}{dt} a(u'_m(t)) + \frac{d}{dt}|Au_m(t)|^2 + \alpha \frac{d}{dt} a(u_m(t)) +$$

$$+ M(a(u_m(t))) \frac{d}{dt} a(u_m(t)) = 2(f(t),u'_m(t)) \tag{2.10}$$

Set $\bar{M}(\sigma) = \int_o^\sigma M(s)ds$.

We integrate (2.10) from 0 to $t < t_m$ and obtain:

$$|u'_m(t)|^2 + \lambda a(u'_m(t)) + |Au_m(t)|^2 + \bar{M}(a(u_m(t)))$$

$$\le K_m + |\alpha|a(u_m(t)) + \int_o^t |u'_m(s)|^2\, ds, \tag{2.11}$$

where $K_m = |u_{1m}|^2 + \lambda a(u_{1m}) + |Au_{om}|^2 + M(a(u_{om})) + \int_o^T |f(s)|^2 ds.$

By choice, $u_{om}$ and $u_{1m}$ converge respectively to $u_o$ in V and to $u_1$ in W (remember that $|u_{1m}|_\lambda^2 = |u_{1m}|^2 + \lambda a(u_{1m})$).

Therefore, there is a constant $C_o > 0$, independent of m and greater than $K_m$ such that (2.11) still holds, with $K_m$ replaced by $C_o$.

Now, $M(s) \geq m_o + m_1 s$ implies $\quad \bar{M}(\sigma) \geq m_o\sigma + \frac{m_1}{2}\sigma^2$ (2.12)

For $\sigma = a(u_m(t))$, from (2.11), (2.12) and $\quad |\alpha|\sigma \leq \frac{|\alpha|}{2m_1} + \frac{m_1}{2}\sigma^2$

one obtains:

$$|u_m'(t)|^2 + \lambda a(u_m'(t)) + |Au_m(t)|^2 + m_o a(u_m(t)) \leq C + \int_o^t |u_m'(s)|^2 ds, \tag{2.13}$$

where $C = C_o + \frac{|\alpha|}{2m_1}$, a constant independent of m.

In particular,

$$|u_m'(t)|^2 \leq C + \int_o^t |u_m'(s)|^2 ds.$$

Hence, applying Gronwall inequality

$$|u_m'(t)|^2 \leq C\, e^T, \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$|u_m'(t)|^2 + \lambda a(u_m'(t)) + |Au_m(t)|^2 + m_o a(u_m(t)) \leq K, \tag{2.15}$$

where $K = C(1 + Te^T)$, for all t in $[0,t_m]$ and all m.

In particular, as $k|u_m(t)|^2 \leq a(u_m(t))$, it follows that $u_m(t)$ remains bounded; hence it can be extended to $[0,T]$. Therefore, (2.15) holds, in fact, for all m and t in $[0,T]$.

(iii) <u>Passage to the Limit</u>

It follows that there is a sub-sequence of $(u_m)$, still dentoed $(u_m)$, for which, as $m \to \infty$, the following is true, in the weak star convergence of $L^\infty(0,T;H)$:

$$u_m \rightharpoonup u, \tag{2.16}$$

$$a(u_m) \rightharpoonup a(u), \tag{2.17}$$

$$Au_m \longrightarrow Au, \tag{2.18}$$

$$u'_m \longrightarrow u', \tag{2.19}$$

$$\lambda a(u'_m) \longrightarrow \lambda a(u'), \tag{2.20}$$

$$M(a(u_m))Au_m \longrightarrow \psi \tag{2.21}$$

It must still be proved that, in fact

$$\psi = M(a(u))Au. \tag{2.22}$$

(2.22) will be shown to follow from the Lemma below, whose proof, here reproduced, was given by J.L. Lions [3] and [4].

LEMMA. The mapping $v \longrightarrow M(a(v))Av$ from V into H is monotonic.

PROOF. The function $\bar{M}(\sigma) = \int_0^\sigma M(s)ds$ is non-decreasing (because $\bar{M}'(\sigma) = M(\sigma) \geq 0$) and convex (because $\bar{M}''(\sigma) = M'(\sigma) \geq 0$).

Take

$$\phi(v) = \bar{M}(a(v)), \text{ for } v \text{ in } V$$

It is easy to see that $\phi$ has a Gateau derivative,

$$\phi'(v) = 2M(a(v))Av, \text{ for } v \text{ in } V,$$

and that $\phi$ is convex, i.e., for $0 \leq \rho \leq 1$,

$$\phi(\rho v + (1-\rho)w) \leq \rho\phi(v) + (1-\rho)\phi(w), \text{ for } v, w \text{ in } V.$$

This inequality can be written in the form

$$\frac{\phi(w + \rho(v-w)) - \phi(w)}{\rho} \leq \phi(v) - \phi(w)$$

Passing to the limit, as $\rho \to 0$ it follows that

$$(\phi'(w), v-w) \leq \phi(v) - \phi(w)$$

and, interchanging the roles of v and w,

$$(\phi'(v), w-v) \leq \phi(w) - \phi(v)$$

Adding the two inequalities above, one obtains:

$$(\phi'(w) - \phi'(v), w-v) \geq 0,$$

This proves the Lemma.

It can now be shown that (2.22) holds.

Indeed, because of the Lemma, for all v in $L^2(0,T;V)$, it is true that

$$\int_0^T (M(a(u_m))Au_m - M(a(v))Av),\ u_m - v)dt \geq 0$$

Because $(u_m)$ is bounded in $L^\infty(0,T;V)$ and $(u_m')$ in $L^\infty(0,T;H)$ and the injection of V in H is compact, $(u_m)$ can, further, be supposed to converge to u strongly in $L^2(0,T;H)$. Hence, as $m \to \infty$:

$$\int_0^T (\psi - M(a(v))Av,\ u - v)dt \geq 0$$

Set $u - v = \rho w$, $\rho \geq 0$, divide the inequality by $\rho$ and let $\rho \to 0$, to obtain:

$$\int_0^T (\psi - M(a(u))Au,\ w)dt \geq 0$$

This holds for all w in $L^\infty(0,T;V)$, hence $\psi = M(a(u))Au$.

In the following, let k be fixed, $k < m$; take v in $V_k$ and let $m \to \infty$. (2.19) and (2.20) imply that, in $\mathcal{D}'(0,T)$,

$$\frac{d}{dt}(u_m'(t),v) \quad \to \quad \frac{d}{dt}(u'(t),v)\ , \tag{2.23}$$

$$\lambda \frac{d}{dt}\, a(u_m'(t),v) \to \quad \lambda \frac{d}{dt}\, a(u'(t),v) \tag{2.24}$$

Passing to the limit in (2.8), then (2.23) and (2.24), with (2.17), (2.18), (2.21) and (2.22) ensure that

$$\frac{d}{dt}[(u'(t),v) + \lambda a(u'(t),v)] + (Au(t),Av) +$$

$$+ [\alpha + M(a(u(t)))]\ a(u(t),v) = (f(t),v), \tag{2.25}$$

holds in $\mathcal{D}'(0,T)$, for all v in $V_k$. By density, (2.25) holds in $\mathcal{D}'(0,T)$, for all v in V.

Therefore, u is, indeed, a weak solution of (2.3a).

It must still be shown, in order to complete the proof of existence, that u satisfies (2.4ab).

(iv) Initial Conditions

(2.19) means that, for v in V and $\theta$ in $C'(0,T)$ such that $\theta(0) = 1$ and $\theta(T) = 0$, as $m \to \infty$

$$\int_0^T (u_m'(t),v)\theta(t)dt \to \int_0^T (u'(t),v)\theta(t)dt \tag{2.26}$$

Because of (2.5) and (2.16), integrating (2.26) by parts, it follows that

$$(u_{om},v) \to (u(0),v), \text{ for } v \text{ in } V. \tag{2.27}$$

But $u_{om} \to u_o$ in V; hence (2.27) yields

$$(u_o,v) = (u(0),v), \text{ for } v \text{ in } V, \text{ i.e., } u \text{ satisfies (2.4a).}$$

To show that u satisfies (2.4b), consider equations (2.3b) and (2.8) for $v = w_j$, $j = 1,2\ldots.$ It follows, using (2.17)-(2.18), (2.21) and (2.22), that, as $m \to \infty$

$$\frac{d}{dt}[(u_m'(t), w_j) + \lambda a(u_m'(t),w_j)] \tag{2.28}$$

converges to $\frac{d}{dt}[(u'(t),w_j) + \lambda a(u'(t),w_j)]$ weak star in $L^\infty(0,T)$.

(2.28) means that for v in V, $\theta$ in $C^1(0,T)$ such that $\theta(0)=1$, $\theta(T) = 0$,

$$\int_0^T \frac{d}{dt}[(u_m'(t),w_j) + \lambda a(u_m'(t),w_j)]\, \theta(t)dt$$

$$\longrightarrow \int_0^T \frac{d}{dt}[(u'(t),w_j) + \lambda a(u'(t),w_j)]\, \theta(t)dt. \tag{2.29}$$

Because of (2.6), (2.19) and (2.20), integrating (2.29) by parts, it follows that

$$(u_{1m},w_j) + \lambda a(u_{1m},w_j) \longrightarrow (u'(0),w_j) + \lambda a(u'(0),w_j), \tag{2.30}$$

But $u_{1m} \longrightarrow u_1$ in W, hence the left-hand side of (2.30) converges also to $(u_1,w_j) + \lambda a(u_1,w_j)$. Therefore

$$(u'(0),w_j) + \lambda a(u'(0),w_j) = (u_1,w_j) + \lambda a(u_1,w_j) \tag{2.31}$$

As (2.31) holds for $j = 1,2,\ldots$, it follows that, in fact, for all v in V:

$$(u'(0),v) + \lambda a(u'(0),v) = (u_1,v) + \lambda a(u_1,v).$$

In other words

$$(I + \lambda A)u'(0) = (I + \lambda A)u_1 \quad \text{in } V'.$$

But this implies, (see [6]) $u'(0) = u_1$; i.e. u satisfies (2.4b).

## 3. UNIQUENESS

Let u and $\bar{u}$ be two solutions of (2.3a) with the same initial conditions (2.4ab). Thus $w = u - \bar{u}$ satisfies

$$(I + \lambda A)w'' + A^2w + \alpha Aw + M(a(u))w + [M(a(u)) - M(a(\bar{u}))] A\bar{u} = 0, \tag{3.1}$$

$$w(0) = 0, \quad w'(0) = 0. \tag{3.2ab}$$

The standard energy method cannot be used to prove uniqueness, because, while the left-hand side of (3.1) belongs to $L^2(0,T;V')$, u' belongs to $L^\infty(0,T;W)$ and not to $L^\infty(0,T;V)$. A modification has to be made; this procedure can be found in Lions [3].

Consider:

$$z(t) = \begin{bmatrix} -\int_t^s w(\xi)d\xi & \text{for } t \le s \\ & \\ 0 & \text{for } t > s \end{bmatrix} \tag{3.3}$$

and

$$w_1(t) = \int_0^t w(\xi)d\xi, \tag{3.4}$$

Then
$$z(t) = w_1(t) - w_1(s), \tag{3.5}$$
$$z(0) = - w_1(s) , \tag{3.6}$$
$$z(s) = 0 , \tag{3.7}$$
and
$$z'(t) = w(t) . \tag{3.8}$$

As $w \in L^\infty(0,T;V)$, it is clear [see (3.3) and (3.8)] that $z$ and $z'$ are in $L^1(0,T;V)$. Hence, it follows from (3.1) that

$$\int_0^s < (I + \lambda A)w''(t), z(t) > dt + \int_0^s (Aw(t), Az(t))dt +$$
$$+ \alpha \int_0^s (Aw(t),z(t))dt + \int_0^s M(a(u(t)))(Aw(t),z(t))dt +$$
$$+ \int_0^s [M(a(u(t))) - M(a(\bar{u}(t)))](A\bar{u}(t),z(t))dt = 0 . \tag{3.9}$$

But, [see (3.8)]

$$< (I+\lambda A)w''(t),z(t) > = \frac{d}{dt}((I+\lambda A)w'(t),z(t)) - ((I+\lambda A)w'(t),z'(t))$$
$$= \frac{d}{dt}((I+\lambda A)w'(t),z(t)) - \frac{1}{2}\frac{d}{dt}((I+\lambda A)w(t),w(t))$$

Therefore, using (3.2ab) and (3.7), it follows that (remember $|w|_\lambda^2 = |w|_\lambda^2 + \lambda a(w)$)

$$\int_0^s < (I + \lambda A)w''(t),z(t) > dt = - \frac{1}{2} |w(s)|_\lambda^2 . \tag{3.10}$$

Now, [see (3.8)]

$$(Aw(t), Az(t)) = (Az'(t), Az(t)) = \frac{1}{2} \frac{d}{dt} |Az(t)|^2$$

Thus, [see (3.6) and (3.7)]

$$\int_0^s (Aw(t), Az(t))dt = - \frac{1}{2}|Aw_1(s)|^2 \tag{3.11}$$

As $|w|_\lambda \geq |w|$, (3.9), (3.10) and (3.11) yield

$$|w(s)|^2 + |Aw_1(s)|^2 \leq 2|\alpha| \int_0^s |(w(t),\ Az(t))|dt$$

$$+ 2\int_0^s M(a(u(t)))|(w(t),\ Az(t))|dt$$

$$+ 2\int_0^s |M(a(u(t))) - M(a(\bar{u}(t)))|\ |(\bar{u}(t),\ Az(t))|dt \tag{3.12}$$

As u, $\bar{u} \in L^\infty(0,T;V)$ and, for s ≥ 0, M ≥ 0 is a $C^1$ function, with $M^1 \geq 0$, there is a constant C > 0 such that

$$2\int_0^s M(a(u(t)))|(w(t),\ Az(t))|dt \leq 2C_o \int_0^s |w(t)|\ |Az(t)|dt \tag{3.13}$$

And

$$2\int_0^s |M(a(u(t)) - M(a(\bar{u}(t)))|\ |(\bar{u}(t),\ Az(t))|dt$$

$$\leq 2C_o \int_0^s |a(u(t)) - a(\bar{u}(t))|\ |\bar{u}(t)|\ |Az(t)|dt$$

$$\leq 2C_o^2 \int_0^s |(A(u(t) + \bar{u}(t)),\ w(t))|\ |Az(t)|dt$$

$$\leq 2C_o^3 \int_0^s |w(t)|\ |Az(t)|dt \tag{3.14}$$

Notice that,[see (3.5) ]

$$2|w(t)|\ |Az(t)| \leq 2[|w(t)|^2 + |Aw_1(t)|^2] + |Aw_1(s)|^2 . \tag{3.15}$$

Hence, it follows from (3.15) that

$$2|\alpha| \int_0^s |(w(t),Az(t))|dt \leq 2|\alpha| \int_0^t [|w(t)|^2 + |Aw_1(t)|^2]dt + |\alpha|s|Aw_1(s)|^2; \tag{3.16}$$

Hence (3.13) and (3.15) give

$$2\int_0^s M(a(u(t)))|(w(t),\ Az(t))|dt$$

$$\leq 2C_o \int_0^s [|w(t)|^2 + |Aw_1(t)|^2]dt + C_o\ s|Aw_1(s)|^2 \tag{3.17}$$

Now (3.14) and (3.15) give

$$2\int_0^s |M(a(u(t))) - M(a(\bar{u}(t)))|\ |(\bar{u}(t),\ Az(t))|dt$$

$$\leq 2C_o^3 \int_0^s [|w(t)|^2 + |Aw_1(t)|^2]dt + C_o^3\ s|Aw_1(s)|^2 \tag{3.18}$$

It now follows from (3.12), with (3.16) - (3.17) that

$$|w(s)|^2 + (1 - Cs)|Aw_1(s)|^2 \leq 2C\int_0^s [|w(t)|^2 + |Aw_1(t)|^2]dt, \tag{3.19}$$

where $C = |\alpha| + C_o + C_o^3$.

Take $s_o$ such that, for $0 \leq s \leq s_o$, $\frac{1}{2} \leq 1 - Cs \leq 1$. Hence (3.19) yields for $0 \leq s \leq s_o$:

$$|w(s)|^2 + \frac{1}{2}\ |Aw_1(s)|^2 \leq 2C\int_0^s [|w(t)|^2 + |Aw_1(t)|^2]dt.$$

A fortiori, for $0 \leq s \leq s_o'$,

$$|w(s)|^2 + |Aw_1(s)|^2 \leq 4C\int_0^s [|w(t)|^2 + |Aw_1(t)|^2]dt.$$

Applying Gronwall inequality, it then follows that

$$w(s) = 0, \text{ for } 0 \leq s \leq s_o\ .$$

Similarly, it is proved that $w(s) = 0$, for $s_o \leq s \leq s_o + \tau$, with $\tau > 0$. It then follows that, in fact, $w(s) = 0$, for $0 \leq s < T$.

The proof of uniqueness is complete.

## 4. APPLICATION

For $\Omega$ a bounded open set in $R^n$, with regular boundary, consider

$$H = L^2(\Omega), \quad V = H^1_o(\Omega) \cap H^2(\Omega)$$

Let $\Delta$ be the Laplace and $\nabla$ the gradient operators in $R^n$ respectively. Take $A = -\Delta$, hence $A^{\frac{1}{2}} = \nabla$. Hypothesis on A are satisfied. Notice that, in this case, the condition $(Av,v) \geq k|v|^2$, for v in V, is the Friedrichs - Poincaré inequality; the compactness of the injection of V in H is the Rellich theorem.

It is clear that

$$W = L^2(\Omega), \text{ if } \lambda = 0$$

$$W = H^1(\Omega), \text{ if } \lambda > 0$$

Now (,) and | | are respectively the inner product and the norm in $L^2(\Omega)$.

Given

$$u_o \in H^1_o(\Omega) \quad H^2(\Omega)$$

$$u_1 \in L^2(\Omega), \text{ if } \lambda = 0; \; u_1 \in H^1(\Omega), \text{ if } \lambda > 0,$$

$$f \in L^2(0,T; L^2(\Omega)),$$

the theorem proved above ensures existence and uniqueness of weak solution for the non-linear hyperbolic equation

$$(I - \lambda\Delta)u + \Delta^2 u - [\alpha \; M(|\nabla u|^2)] \; \Delta u = f,$$

satisfying $u(0) = u_o$, $u'(0) = u_1$.

ACKNOWLEDGEMENT. This research was supported by FIMEP and CEPG - UFRJ. Special acknowledgement is due to Professor L.A. Medeiros, who drew my attention to this problem, for his generous help and encouragement.

## REFERENCES

[1] Ball, J.M. Initial Boundary Value Problems for an Extensible Beam, Journal of Functional Analysis and Applications 42, (1973) 61-90.

[2] Dickey, R.W. Free Vibrations and Dynamic Buckling of the Extensible Beam, J. Math. Anal. and Appl. 29, (1970), 443-454.

[3] Lions, J.L. Quelques Méthodes de Résolution des Problémes aux Limites Non-Linéaires, Dunod, Gauthier-Villars, Paris, 1969.

[4] Lions, J.L. On Some Questions in Boundary Value Problems of Mathematical Physics, Lecture Notes, Instituto de Matemática, Universidade Federal do Rio de Janeiro, 1977.

[5] Medeiros, L.A. On a New Class of Non-Linear Wave Equations, to appear in in Journal of Mathematical Analysis and Applications.

[6] Menzala, G.P. Une Solution d'une Equation Non-Linéaire d'Evolution, C.R. Acad. Sc. Paris, t. 286, (1978), 273-275.

[7] Pohozaev, S.I. On a Class of Quasi Linear Hyperbolic Equations, Mat. USSR Sbornik, 25, (1975) 1, 145-158.

[8] Rivera, P.H. On a Non-Linear Hyperbolic Equation in Hilbert Spaces, Anais Acad. Bras. Ciẽncias, Vol. 50, No. 2, 1978.

[9] Timoshenko, S. D.H. Young, and W. Weaver. Vibration Problems in Engineering, John Wiley, 1974.