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# CONTRIBUTIONS TO THE THEORY OF HERMITIAN SERIES
# III. MEANVALUES

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ABSTRACT. Let f(z) be holomorphic in the strip $-\sigma < y < \sigma < \infty$ and satisfy the conditions for having an expansion in an Hermitian series

$$f(z) = \sum_{n=0}^{\infty} f_n h_n(z), \quad h_n(z) = (\pi^{\frac{1}{2}} 2^n n!)^{-\frac{1}{2}} e^{-\frac{1}{2}z^2} H_n(z),$$

absolutely convergent in the strip. Two meanvalues

$$\mathfrak{M}_k (f; y) = \{\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-kx^2} |f(x+iy)|^2 dz\}^{\frac{1}{2}}, \quad k = 0, 1.$$

are discussed, directly using the condition on f(z) or via the Hermitian series. Integrals involving products $h_m(x+iy)\, h_n(x-iy)$ are discussed. They lead to expansions of the mean squared in terms of Laguerre functions of $y^2$ when k = 0 and in terms of Hermite functions $h_n(2^{\frac{1}{2}}iy)$ when k = 1. The sumfunctions are holomorphic in y. They are strictly increasing when $|y|$

increases.

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1. BASIC FORMULAS.

The functions

$$h_n(z) = (-1)^n \, (\pi^{\frac{1}{2}} \, 2^n \, n!)^{-\frac{1}{2}} \, e^{\frac{1}{2}z^2} \frac{d^n}{dz^n}(e^{-z^2}) \tag{1.1}$$

form a closed orthonormal system in $L_2(-\infty, \infty)$. If $x \mapsto f(x)$ is such that $e^{-\gamma x^2} \in L_2(-\infty, \infty)$ for some $\gamma$, $0 \leqq \gamma < \frac{1}{2}$, then the Fourier-Hermite coefficients exist

$$f_n = \int_{-\infty}^{\infty} f(s) \, h_n(s) \, ds \tag{1.2}$$

and $f(x)$ has a formal Hermitian series

$$f(x) \sim \sum_{n=0}^{\infty} f_n \, h_n(x). \tag{1.3}$$

Here we consider only the case where $f(x+iy)$ is a holomorphic function of $z = x+iy$ in a strip $-\sigma < y < \sigma < \infty$ and the series is absolutely convergent. The author has shown [1] that this will be the case if for every $\beta$, $0 < \beta < \sigma$, there is a finite $B(\beta)$ such that

$$|f(x+iy)| < B(\beta) \quad \exp[-|x|(\beta^2 - y^2)^{\frac{1}{2}}] \tag{1.4}$$

for $-\infty < x < \infty$, $0 \leqq |y| < \beta$. This implies and is implied by

$$|f_n| < A(\alpha) \, \exp[-\alpha(2n+1)^{\frac{1}{2}}] \tag{1.5}$$

for $0 < \alpha < \sigma$ and a finite positive $A(\alpha)$.

The function $z \mapsto h_n(z)$ is a solution of the Hermite-Weber differential equation

$$w'' + [2n + 1 - z^2]\, w = 0 \tag{1.6}$$

also known as the equation of the parabolic cylinder which is moreover satisfied by $h_{-n-1}(iz)$ and $h_{-n-1}(-iz)$. For further properties of these functions see E. Hille [1, 2, 3, 4].

We shall also encounter Laguerre polynomials in the discussion. Here the polynomial $L_n^{(\alpha)}(u)$ of order $\alpha$ and degree n is defined by

$$e^{-u}\, u^{\alpha}\, L_n^{(\alpha)}(u) = \frac{1}{n!} \frac{d^n}{du^n} [e^{-u}\, u^{n+\alpha}] \tag{1.7}$$

or explicitly

$$L_n^{(\alpha)}(u) = \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-u)^j}{j!} . \tag{1.8}$$

See G. Szegö [5, pp. 96-98, 102]. We note that

$$H_{2n}(u) = (-4)^n\, n!\, L_n^{(-\frac{1}{2})}(u^2). \tag{1.9}$$

Here we set $u = i\, 2^{\frac{1}{2}}\, y$ and use (1.1) to get

$$(-1)^n\, h_{2n}(i\, 2^{\frac{1}{2}}\, y) = \left[\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\right]^{\frac{1}{2}} L_n^{(-\frac{1}{2})}(-2y^2)\, e^{y^2}. \tag{1.10}$$

For $u < 0$, $-1 < \alpha$ the function $u \mapsto L_n^{(\alpha)}(u)$ is positive, strictly decreasing and its graph is concave upwards. Formula (1.10) shows that $h_{2n}(i\, 2^{\frac{1}{2}}\, y)$ has similar properties for $y > 0$.

Both the Hermite and the Laguerre functions are essentially special cases of confluent hypergeometric functions, that is solutions of a second order linear differential equation of the form

$$z\, w'' + (c - z)\, w' - a\, w = 0. \tag{1.11}$$

The equation is satisfied by the entire function

$${}_1F_1(a, c; z) = 1 + \sum_{j=1}^{\infty} \frac{a(a+1) \ldots (a+j-1)}{j!\, c(c+1) \ldots (c+j-1)}\, z^j. \tag{1.12}$$

For $a = -n$, $c = 1+\alpha$ we get

$$ {}_1F_1(-n,\ 1+\alpha;\ z) = \binom{n+\alpha}{n} L_n^{(\alpha)}(z). \tag{1.13}$$

The Hermite polynomials are constant multiples of

$$ {}_1F_1(-n,\ \tfrac{1}{2};\ z^2) \quad \text{or of} \quad z\,{}_1F_1(-n,\ \tfrac{3}{2};\ z^2) \tag{1.14}$$

according as n is even or odd.

We shall need the asymptotic behavior of these functions for large values of n and fixed, non-real negative, values of z. Here the basic results are due to Oskar Perron [6. p.72].

LEMMA 1: For large positive values of n and complex, non-real negative values of z, we have the asymptotic expansion

$$ {}_1F_1(a+n,\ c;\ z) \sim \frac{\Gamma(c)}{2\pi^{\frac{1}{2}}}\ e^{\frac{1}{2}z}\ (zn)^{\frac{1}{4}-\frac{1}{2}c}\ e^{2(zn)^{\frac{1}{2}}}\left[1 + \sum_{j=1}^{\infty} p_j\ n^{-\frac{1}{2}j}\right], \tag{1.15}$$

where the powers and roots take their principal values. If the series is replaced by its $m^{\text{th}}$ partial sum the error committed is of the order of magnitude of the last term.

Here $n \to +\infty$. . In the Laguerre case $n \to -\infty$ instead and we get (see G. Szegö [5], p. 193):

LEMMA 2: Let $\alpha$ be an arbitrary real number. Then

$$ L_n^{(\alpha)}(z) = \tfrac{1}{2}\pi^{-\frac{1}{2}}\ e^{\frac{1}{2}z}(-z)^{-\frac{1}{4}-\frac{1}{2}\alpha}n^{\frac{1}{2}\alpha-\frac{1}{4}}e^{2(-nz)^{\frac{1}{2}}}\left[\sum_{j=0}^{p-1} C_j(z)n^{-\frac{1}{2}j}+O(n^{-\frac{1}{2}p})\right] \tag{1.16}$$

Here $C_o(z) = 1$ and $C_j$'s are independent of n and are holomorphic in the plane cut along the positive real axis. The powers must be taken real positive for z real negative. The bound for the remainder holds uniformly in any closed region having no point in common with $x \geqq 0$.

The Hermitian case is discussed in Section 5. We shall use (1.10) a lot and this makes it desirable to have comparison formulas for $L_n^{(k)}(-u)$ with $L_n^{(-\frac{1}{2})}(-u)$.

LEMMA 3: For all non-negative integers k and n and all u > 0

$$L_n^{(k)}(-u) \leqq \frac{\Gamma(\frac{1}{2})\,\Gamma(k+n+1)}{\Gamma(k+1)\,\Gamma(n+\frac{1}{2})} L_n^{(-\frac{1}{2})}(-u). \tag{1.17}$$

PROOF. Formula (1.8) shows that inequality is trivially true if for $j = 0, 1, 2, \ldots.\ n$ we have

$$\binom{n+k}{n-j} \leqq Q \binom{n-\frac{1}{2}}{n-j} \text{ with } Q = \frac{\Gamma(\frac{1}{2})\,\Gamma(k+n+1)}{\Gamma(k+1)\,\Gamma(n+\frac{1}{2})} \tag{1.18}$$

or equivalently

$$(n+k)(n+k-1)\ldots(k+j+1) \leqq Q\ (n-\tfrac{1}{2})(n-\tfrac{3}{2})\ldots(\tfrac{1}{2}+j)$$

This is true with equality for $j = 0$ by the choice of Q. If this equality is divided, the left member by

$$(k+1)(k+2)\ \ldots.\ (k+j+2)$$

and the right member by the smaller quantity

$$\tfrac{1}{2}(\tfrac{1}{2}+1)\ \ldots\ (\tfrac{1}{2}+j+1)$$

the inequality results.

## 2. FORMULAS OF FELDHEIM.

In 1940 Ervin Feldheim produced a paper [7] which has an important bearing on our problem. As a matter of fact it serves as the point of departure of this work. Several of his formulas figure in earlier papers of mine, but I can lay no claim to his formulas (20), (20') and (51) which are basic for the following. Formula (20') on p. 244 reads for $m > n$

$$\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-t^2} H_m(t+v)\, H_n(t-v)\, dt = 2^m\, n!\, v^{m-n} L_n^{(m-n)}(2v^2). \tag{2.1}$$

Here we set $t = x$, $v = iy$ and express the upper case H's in terms of the lower case h's to obtain

$$\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} h_m(x+iy)\, h_n(x-iy)\, dx = \pi^{-\frac{1}{2}}\, 2^{\frac{1}{2}(m-n)} \left[\frac{n!}{m!}\right]^{\frac{1}{2}} (iy)^{m-n}\, L_n^{(m-n)}(-2y^2)\, e^{y^2}. \quad (2.2)$$

Feldheim's formula (51) on p. 243 is quite complicated. It involves two positive parameters, $\lambda$ and $\mu$, an arbitrary variable v and the variable of integration t. Here we set

$$\lambda = \mu = 1, \; t = 2^{\frac{1}{2}}x, \quad v = 2^{\frac{1}{2}}iy \quad (2.3)$$

and introduce the h's to obtain

$$\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-x^2}\, h_m(x+iy)\, h_n(x-iy)\, dx$$

$$=(-1)^n\, \pi^{-\frac{1}{4}} \left[\frac{(m+n)!}{m!n!}\right]^{\frac{1}{2}} 2^{-\frac{1}{2}(m+n+1)}\, h_{m+n}(2^{\frac{1}{2}}iy). \quad (2.4)$$

It should be noted that the right member of this formula is positive whenever $m+n$ is even, in particular for $m = n$.

## 3. THE MEANVALUES.

Let us form

$$\mathfrak{M}_k(f;\, y) = \{\pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-kx^2}\, |f(x+iy)|^2\, dx\}^{\frac{1}{2}}, \; k = 0,\, 1 \quad (3.1)$$

where $z \mapsto f(z) = f(x+iy)$ is supposed to satisfy condition (1.4) so that $f(z)$ can be expanded in an Hermitian series absolutely convergent in the strip $-\sigma < y < \sigma < \infty$. Formula (1.4) ensures the existence of the meanvalues and for $k = 0$ gives the estimate

$$\mathfrak{M}_0(f;\, y) < B(\beta)\, [R_\beta(y)]^{-\frac{1}{2}} \quad (3.2)$$

with

$$R_\beta(y) = (\beta^2 - y^2)^{\frac{1}{2}} \quad (3.3)$$

for every y, $0 \leqq |y| < \beta$ and $\beta$, $0 < \beta < \sigma$. For every fixed admissible $\beta$ this is obviously an increasing function of $|y|$. That the bound is increasing evidently does not imply that the meanvalue is increasing. This question will be examined later.

$\mathfrak{M}_1(f;\, y)$ is more complicated. Here (1.4) implies that

$$[\mathfrak{M}_1(f;\, y)]^2 < 2\,[B(\beta)]^2 \int_0^\infty \exp\,[-\,x^2 - 2\,B_\beta(y)\,x]\,dx \tag{3.4}$$

The right member is obviously a bound decreasing function of $|y|$ for fixed $\beta$. We can expand the integrand in powers of $R_\beta(y)$. The integral is an entire function of $R_\beta(y)$; as a matter of fact the Mittag-Leffler function

$$E_{\frac{1}{2}}(w) = \sum_{n=0}^{\infty} \frac{w^n}{\Gamma(\frac{1}{2}n + 1)}\,, \quad w = -\,2\,R_\beta(y). \tag{3.5}$$

Here $E_{\frac{1}{2}}(w)$ is of order 2 and normal type.

4. <u>THE FUNCTION</u> $h_n(iv)$.

In the first paper of this series [2] it was shown that $h_n(z)$ is the unique solution of the Volterra integral equation

$$w(z) = c_n(z) + N^{-1} \int_0^z t^2 \sin\,[N(z-t)]\,w(t)\,dt \tag{4.1}$$

where

$$N = (2n+1)^{\frac{1}{2}},\quad c_n(z) = C(n)\cos\,(Nz - \tfrac{1}{2}\,n\pi), \tag{4.2}$$

while $C(n)$ depends on the parity of $n$ so that

$$C(2k) = |H_{2k}(0)|\,[\pi^{\frac{1}{2}}\,2^{2k}\,(2k)!]^{-\frac{1}{2}} = \pi^{-\frac{1}{4}}\left[\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}\right]^{\frac{1}{2}}. \tag{4.3}$$

$$C(2k+1) = |H'_{2k+1}(0)|\,(4k+3)^{-\frac{1}{2}}\,[\pi^{\frac{1}{2}}2^{2k+1}(2k+1)!]^{-\frac{1}{2}} \sim \pi^{-\frac{1}{4}}\left[\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}\right]^{\frac{1}{2}}. \tag{4.4}$$

In both cases we have

$$C(n) \sim n^{-\frac{1}{4}} \tag{4.5}$$

The method of successive approximations applies to (4.1) and leads to rapidly convergent series of the form

$$h_n(z) = c_n(z) \sum_{m=0}^{\infty} P_m(Nz)\ N^{-4m} + s_n(z) \sum_{m=0}^{\infty} Q_m(Nz)\ N^{-4m} \tag{4.6}$$

where

$$s_n(z) = C(n) \sin (Nz - \tfrac{1}{2} n\pi).$$

The factors $P_m$ and $Q_m$ are polynomials in z with rational coefficients. $P_m$ is even and $Q_m$ is odd and they are of degree 3m or 3m-1 whichever has the right parity. We have

$$P_o(z) = 1,\ P_1(z) = \tfrac{1}{2} z^2 - 1,\ Q_o(z) = 0,\ Q_1(z) = \frac{1}{6} z^3 - \frac{1}{4} z$$

For large values of $|z|$ and n we have

$$h_n(z) \sim c_n(z). \tag{4.7}$$

Suppose now that $z = iv$ with $v > 0$. We have then (see [3] pp. 81-82)

$$0 < i^{-n} h_n(iv) - g_n(v) < C(n)\ e^{Nv} [\exp (y^3/6N) - 1] \tag{4.8}$$

where

$$g_n(v) = \tfrac{1}{2} C(n) [e^{Nv} + (-1)^n e^{-Nv}]. \tag{4.9}$$

If $v = o(n^{\frac{1}{6}})$ it is seen that the quantity within brackets in (4.8) is $o(1)$. For our purposes a weaker result suffices:

LEMMA 4. There is a finite quantity $M(\sigma)$, depending only on $\sigma$, such that for all n and $0 \leqq |y| \leqq \sigma$

$$0 < i^{-n} h_n(iv) < M(\sigma)\ n^{-\frac{1}{4}} e^{Nv}. \tag{4.10}$$

Combining Lemma 3 and 4 with formulas (1.10) and (2.2) we obtain

LEMMA 5. For all non-negative integers k and n and $0 \leqq y < \sigma$

$$L_n^{(k)}(-2y^2)\ e^{y^2} < M(\sigma) \frac{n^{-\frac{1}{4}}\Gamma(k + n + 1)}{\Gamma(k+1)[\Gamma(n + \frac{1}{2})\Gamma(n+1)]^{\frac{1}{2}}}\ e^{(8n+2)^{\frac{1}{2}}|y|}. \tag{4.11}$$

We note also (see Theorem 1.4, p. 883 of [3])

LEMMA 6. For $y \neq 0$

$$e^{\pm N\pi x} \frac{h_n(x+iy)}{h_n(iy)} = 1 + O(1/N). \tag{4.12}$$

Here the sign in the exponent is the same as that of $y$. The relation holds uniformly with respect to $x$ and $y$ in the regions $-1/\varepsilon \leqq x \leqq 1/\varepsilon$, $0 < \varepsilon \leqq |y| \leqq 1/\varepsilon$.

5. ESTIMATE OF $[\mu_o(f;\, y)]^2$.

We have

$$\begin{aligned} [\mu_o(f;\, y)]^2 &= \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} \{\sum_{m=0}^{\infty} f_m\, h_m(x+iy)\}\{\sum_{n=0}^{\infty} \bar{f}_n\, h_n(x-iy)\}\, dx \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f_m\, \bar{f}_n\, \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} h_m(x+iy)\, h_n(x-iy)\, dx. \end{aligned} \tag{5.1}$$

We split this double series into three parts. $S_o$ involving the terms where $m = n$, $S_1$ the terms where $m > n$ and $S_{-1}$ those where $n > m$. A moment's reflection shows that $S_{-1}$ must be the complex conjugate of $S_1$ so no separate estimate is needed.

Now by (2.2)

$$S_o = \sum_{n=0}^{\infty} |f_n|^2\, L_n^{(o)}(-2y^2)\, e^{y^2}. \tag{5.2}$$

Here we use (1.4) and Lemma 5 to obtain

$$S_o < M(\alpha) \sum_{n=0}^{\infty} \exp[-\, (\alpha-|y|)(8n+2)^{\frac{1}{2}}] \tag{5.3}$$

which converges for $|y| < \alpha$. The sum of the majorant series is

$$O[(\alpha-|y|)^{-2}]. \tag{5.4}$$

This is a rather poor estimate. Formulas (3.2) and (3.3) suggest that the exponent should be $-\frac{1}{2}$ rather than $-2$.

For $|y| > \varepsilon > 0$ Lemma 2 gives a slightly better result. We set $k = 0$ and get

$$L_n^{(0)}(-2y^2)\ e^{y^2} = C\ y^{-\frac{1}{2}}\ n^{-\frac{1}{4}}\ e^{(8n)^{\frac{1}{2}}y}[1 + O(n^{-\frac{1}{2}})]. \tag{5.5}$$

This introduces the factor $y^{-\frac{1}{2}}\ n^{-\frac{1}{4}}$ in (5.3) and gives

$$S_o < C(\alpha,\ \varepsilon)\ y^{-\frac{1}{2}} \sum_{n=0}^{\infty} n^{-\frac{1}{4}} \exp[-(\alpha-|y|)(8n+2)^{\frac{1}{2}}] \tag{5.6}$$

and the estimate

$$S_o = O\,[y^{-\frac{1}{2}}\ (\alpha-|y|)^{-\frac{3}{2}}]. \tag{5.7}$$

The series $S_1$ presents greater difficulties. Here we set $m = n+k$ and note that

$$S_1 = \sum_{n=0}^{\infty} f_n\ n! \sum_{k=1}^{\infty} \bar{f}_{n+k}\ 2^{n+k}\ (iy)^k\ L_n^{(k)}(-2y^2)\ e^{y^2}. \tag{5.8}$$

Here we naturally think of Lemma 5. It turns out to be too weak to give absolute convergence of the series (5.8) in the whole strip $|y| < \alpha < \sigma$. Using it and later also Lemma 1 we can prove convergence in a strip of width about $\alpha$ but omitting the real axis. This result clearly indicates that the argument has to be based on the asymptotic formula of Lemma 2. The neighborhood of the origin is not accessible by this method but will be taken care of by considerations of analyticity. We use formula (1.5) to obtain

$$|S_1| < A^2(\alpha) \sum_{n=0}^{\infty} (\pi^{\frac{1}{2}}2^n\ n!)^{-\frac{1}{2}}\ n!\ e^{-\alpha(2n+1)^{\frac{1}{2}}} \times$$

$$\times \sum_{k=1}^{\infty} [\pi^{\frac{1}{2}}\ 2^{k+n}\ (k+n)!]^{-\frac{1}{2}}\ e^{-\alpha(2k+2n+1)^{\frac{1}{2}}}\ 2^{k+n}\quad |y|^k\ L_n^{(k)}(-2y^2)\ e^{y^2}. \tag{5.9}$$

This we simplify and use formula (1.16). Thus

$$|S_1| < B(\alpha,\ \tau)\ y^{-\frac{1}{2}} \sum_{n=1}^{\infty} n^{-\frac{1}{4}}\ e^{-2\alpha(2n+1)^{\frac{1}{2}} + |y|(8n)^{\frac{1}{2}}}\ T_n \tag{5.10}$$

where

$$T_n = \sum_{k=1}^{\infty} 2^{-k} \left[\frac{n!}{(k+n)!}\right]^{\frac{1}{2}} n^{\frac{1}{2}k} e^{-\alpha[(2k+2n+1)^{\frac{1}{2}} - (2n+1)^{\frac{1}{2}}]}. \tag{5.11}$$

Here the last factor is < 1 and may be neglected. Further

$$n^{\frac{1}{2}k}\left[\frac{n!}{(k+n)!}\right]^{\frac{1}{2}} = \left[\frac{n \quad n \ \dots \ n}{(n+1)(n+2)\dots(n+k)}\right]^{\frac{1}{2}} < 1 \tag{5.12}$$

so that

$$T_n < \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Hence

$$|S_1| < B(a, \tau)\ |y|^{-\frac{1}{2}} \sum_{n=1}^{\infty} n^{-\frac{1}{4}} \exp\,[(\alpha-|y|)(8n+2)^{\frac{1}{2}}] \tag{5.13}$$

which converges for $0 < |y| < \alpha$ and has a sum satisfying (5.7). Thus

$$h_0(f;\ y) = O\,[\,|y|^{-\frac{1}{4}}\ (\alpha-|y|)^{-\frac{3}{4}}]. \tag{5.14}$$

which holds for every $\alpha$, $0 < \alpha < \sigma$ and for every $y$, $0 < \varepsilon \leqq |y| < \alpha < \sigma < \infty$. The constant implied by the order symbol of course depends on $\alpha$ and $\varepsilon$.

6. ESTIMATE OF $[h_1(f;\ y)]^2$.

With the same assumptions of $f(z)$ as above we have by formula (2.4)

$$\begin{aligned}[h_1(f;\ y)]^2 &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n f_m \bar{f}_n\, 2^{-\frac{1}{2}(m+n+1)}\, h_{m+n}(i2^{\frac{1}{2}}y) \\ &= \sum_{k=0}^{\infty}\Big[\sum_{m+n=k}(-1)^n f_m \bar{f}_n\Big]2^{-\frac{1}{2}(k+1)}\, h_k(i2^{\frac{1}{2}}y).\end{aligned} \tag{6.1}$$

This series will be shown to be absolutely convergent for $0 \leqq y < \alpha < \sigma$. Let us first estimate the finite sum where $m+n = k$. By (1.5)

$$
\begin{aligned}
|S_k| = \sum_{m+n=k} (-1)^n f_m \bar{f}_n &\leqq A^2(\alpha)\pi^{-\frac{1}{2}} \sum_{m+n=k} (2^{m+n}\, m!\, n!)^{-\frac{1}{2}} \times \\
&\times \exp \{ -\alpha [ (2m+1)^{\frac{1}{2}} + (2n+1)^{\frac{1}{2}} ] \} \qquad (6.2) \\
&= A^2(\alpha)\ 2^{-\frac{1}{2}k} \sum_{m=0}^{k} \left[ \frac{k!}{m!\ (k-m)!} \right]^{\frac{1}{2}} T_{m,n}
\end{aligned}
$$

where

$$T_{m,n} = \exp \{ -\alpha [ (2m+1)^{\frac{1}{2}} + (2n+1)^{\frac{1}{2}} ] \} \tag{6.3}$$

We now apply Cauchy's inequality and find that

$$\left\{ \sum_{m=0}^{k} \left[ \frac{k!}{m!\ (k-m)!} \right]^{\frac{1}{2}} T_{m,n} \right\}^2 \leqq \sum_{m=0}^{k} \frac{k!}{m!\ (k-m)!} \sum_{m+n=k} T_{m,n}^2 . \tag{6.4}$$

Here the first sum equals $2^k$. In the second sum the largest term corresponds to $m \sim \frac{1}{2}k$ and this term is approximately

$$\exp [ -2\alpha(4k+2)^{\frac{1}{2}} ] \tag{6.5}$$

so that the second sum is at most

$$(k+1) \exp [ -2\alpha(4k+2)^{\frac{1}{2}} ] \tag{6.6}$$

Hence

$$|S_k| < A^2(\alpha) \quad 2^{-\frac{1}{2}k} [ \Gamma(k+1) ]^{-\frac{1}{2}} (k+1)^{\frac{1}{2}} \exp [ -\alpha(4k+2)^{\frac{1}{2}} ] \tag{6.7}$$

Thus by (6.1), (6.7) and Lemma 4

$$[\mathfrak{M}_1(f;\ y)]^2 < A^2(\alpha)\ (2\pi)^{-\frac{1}{2}} \sum_{k=1}^{\infty} (k+1)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2}k+\frac{1}{2})}{[ \Gamma(k+1) ]^{\frac{1}{2}}} \quad \exp [ -(\alpha-|y|)(4k+2)^{\frac{1}{2}} ] . \tag{6.8}$$

Using the duplication formula for the Gamma function we see that this simplifies to an expression of the form

$$\sum_{k=0}^{\infty} \frac{2^{-k}}{\Gamma(\frac{1}{2}k + 1)} \ \exp [ -\ (\alpha-|y|)(4k+2)^{\frac{1}{2}} ] \tag{6.9}$$

Here

$$G(s) \equiv \sum_{k=0}^{\infty} \frac{2^{-k}}{\Gamma(\frac{1}{2}k + 1)} e^{s(4k+2)^{\frac{1}{2}}} \tag{6.10}$$

is an entire function of s of order 2 and minimal type. The Dirichlet series (6.10) converges for all finite values of s. Compare formula (3.5) where the majorant is of order 2 and normal type.

## 7. ANALYTICITY OF THE SQUARED MEANS.

The functions

$$y \mapsto [\mathfrak{M}_o(f;\, y)]^2 \;,\; y \mapsto [\mathfrak{M}_1(f;\, y)]^2 \tag{7.1}$$

are actually analytic functions of y. To see this we consider the corresponding Laguerre series in the first case, the Hermitian series in the second. In the first case we consider the series

$$F(z) \equiv \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} [f_{k+n}\, \bar{f}_n + (-1)^k\, \bar{f}_{k+n}\, f_n]\, z^k\, L_n^{(k)}(2z^2)\, e^{-z^2} \tag{7.2}$$

which for $z = iy$ reduces to $[\mathfrak{M}_o(f;\, y)]^2$ and represents the analytic continuation of the mean square function in any bounded domain in the strip $-\sigma < y < \sigma$ in which the series converges uniformly with respect to z. The variable parts of the two series are

$$(iy)^k\, L_n^{(k)}(-2y^2)\, e^{y^2} \quad \text{and} \quad z^k\, L_n^{(k)}(2z^2)\, e^{-z^2} \;, \tag{7.3}$$

respectively. Now Lemma 2 shows that for fixed k and n the ratio of the absolute values of the two expressions tend to the limit $|y/z|^{\frac{1}{2}}$ which is bounded away from 0 and $\infty$ if z is restricted to a finite domain D in the strip $-\sigma < y < \sigma < \infty$ having a positive distance from the lines $y = -\sigma, 0, \sigma$. It follows that the series (7.2) converges uniformly in D and thus represents a holomorphic function in D.

Actually F(z) is holomorphic in the whole strip $0 \leqq |y| < \sigma$. This may be concluded from an application of formula (2.1) where we set $v = z$. A simple calculation gives

$$F(z) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} f(t + z)\, f(t - z)\, dt \tag{7.4}$$

valid for any z in the strip $0 \leqq |y| < \sigma < \infty$. It follows, in particular, that the maximum modulus of $F(z)$ for the substrip $|y| \leqq \alpha < \sigma$ is an increasing function of $\alpha$.

Similar considerations apply to the square of the second mean as we see from formulas (2.3) and (2.4) where we replace iy by z and set

$$G(z) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-t^2} f(t + z)\, f(t - z)\, dt \tag{7.5}$$

which for $z = iy$ reduces to $[\mathfrak{M}_1(f;\, y)]^2$. While the latter function of y has an expansion

$$[\mathfrak{M}_1(f;\, y)]^2 = \sum_{n=0}^{\infty} g_n\, h_n(i\, 2^{\frac{1}{2}} y) \tag{7.6}$$

we have

$$G(z) = \sum_{n=0}^{\infty} g_n\, h_n(2^{\frac{1}{2}} z). \tag{7.7}$$

By the virtue of Lemma 6 the absolute convergence of the series (7.6) for a $y = \alpha$, $\alpha < \sigma$, implies the absolute convergence of (7.7) for any z with $\mathrm{Im}\,(z) = \alpha$ and the convergence is uniform in every finite subinterval of the line. Thus $G(z)$ is holomorphic in the basic strip and is actually an entire function of z. The properties of the maximum modulus are the same as in the first case, that is an increasing function of $\alpha$.

## REFERENCES

[1] Hille, E. Contributions to the Theory of Hermitian Series. II. The Representation Problem. *Trans. Amer. Math. Soc. 47* (1940) 80-94.

[2] Hille, E. A Class of Reciprocal Functions. *Ann. Math.* (2) 27 (1926) 427-464.

[3] Hille, E. Contributions to the Theory of Hermitian Series. *Duke Math. J.*, 5 (1939) 875-956.

[4] Hille, E. Sur les fonctions analytiques définies par des séries d'Hermite. *J. Math. Pures Appl.* (9) 40 (1961) 335-342.

[5] Szegö, G. Orthogonal Polynomials. *Amer. Math. Soc., Colloquium Publications. Vol.* 23, 1939.

[6] Perron, O. Über das Verhalten einer ausgearteten hypergeometrischen Reihe bei unbegrenztem Wachstum eines Parameters. *J. Reine Angew. Math.* 151 (1921) 68-78.

[7] Feldheim, E. Dévelopments en série de polynomes d'Hermite et de Laguerre à l'aide des transformations de Gauss et de Hankel. *Nederl. Akad. Wetensch. Proc.* 43 (1940) 224-248.