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K-SPACE FUNCTION SPACES

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ABSTRACT. A study is made of the properties on X which characterize when $C_\pi(X)$ is a k-space, where $C_\pi(X)$ is the space of real-valued continuous functions on X having the topology of pointwise convergence. Other properties related to the k-space property are also considered.

KEY WORDS AND PHRASES. Function spaces, k-spaces, Sequential spaces, Fréchet spaces, Countable tightness, k-countable, τ-countable.

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1. INTRODUCTION.

If X is a topological space, the notation C(X) is used for the space of all real-valued continuous functions on X. One of the natural topologies on C(X) is the topology of pointwise convergence, where subbasic open sets are those of the form

$$[\![x,V]\!] \equiv \{f \in C(X) | f(x) \in V\}$$

for $x \in X$ and V open in the space of real numbers, $\mathbb{R}$, with the usual topology. The space $C(X)$ with the topology of pointwise convergence will be denoted by $C_\pi(X)$.

For a completely regular space X, $C_\pi(X)$ is first countable, in fact metrizable, if and only if X is countable [2]. The purpose of this paper is to show to what extent this result can be extended to properties more general than first countability, such as that of being a k-space. *Throughout this paper all spaces will be assumed to be completely regular* T_1*-spaces*.

We first recall the definitions of certain generalizations of first countability. The space X is a *Fréchet space* if whenever $x \in \overline{A} \subseteq X$, there exists a sequence in A which converges to x. The space X is a *sequential space* if the open subsets of X are precisely those subsets U such that whenever a sequence converges to an element of U, the sequence is eventually in U. Also X is a *k-space* if the closed subsets of X are precisely those subsets A such that for every compact subspace $K \subseteq X$, $A \cap K$ is closed in K. Finally X has *countable tightness* if whenever $x \in \overline{A} \subseteq X$, there exists a countable subset $B \subseteq A$ such that $x \in \overline{B}$. The following diagram shows the implications between these properties.

first countable
↓
Fréchet ⟶ sequential ⟶ k-space
↓
countable tightness

We will show that the Fréchet space, sequential space, and k-space properties are equivalent for $C_\pi(X)$. In order to characterize these properties for $C_\pi(X)$ in terms of internal properties of X, we will need to make some additional definitions. Let $\mathcal{F}(X)$ be the set of all nonempty finite subsets of X. A collection $\mathcal{U}$ of open subsets of X is an *open cover for finite subsets of* X if for every $A \in \mathcal{F}(X)$, there exists a $U \in \mathcal{U}$ such that $A \subseteq U$. If $\{\mathcal{U}_n\}$ is a sequence of collections of subsets of X, a *string from* $\{\mathcal{U}_n\}$ is a sequence $\{U_n\}$ such that $U_n \in \mathcal{U}_n$

for every $n \in \mathbb{N}$ ($\mathbb{N}$ is the set of natural numbers). In addition, we will say that $\{U_n\}$ is <u>residually covering</u> if for every $x \in X$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x \in U_n$.

THEOREM 1. The following are equivalent.

(a) $C_\pi(X)$ is a Fréchet space.

(b) $C_\pi(X)$ is a sequential space.

(c) $C_\pi(X)$ is a k-space.

(d) Every sequence of open covers for finite subsets of X has a residually covering string.

PROOF. (d) ⇒ (a). Suppose that every sequence of open covers for finite subsets of X has a residually covering string. Let F be a subset of $C_\pi(X)$, and let f be an accumulation point of F in $C_\pi(X)$. Then for every $n \in \mathbb{N}$ and $A = \{x_1, \ldots, x_k\} \in \mathcal{F}(X)$, we may choose an

$$f_{n,A} \in F \cap [\![x_1, (f(x_1) - \tfrac{1}{n}, f(x_1) + \tfrac{1}{n})]\!] \cap \ldots \cap [\![x_k, (f(x_k) - \tfrac{1}{n}, f(x_k) + \tfrac{1}{n})]\!] .$$

Also define $U(n,A) = \{x \in X \mid |f_{n,A}(x) - f(x)| < \frac{1}{n}\}$, which is an open subset of X. Then for each $n \in \mathbb{N}$, define $\mathcal{U}_n = \{U(n,A) \mid A \in \mathcal{F}(X)\}$, which is an open cover for finite subsets of X. Now $\{\mathcal{U}_n\}$ has a residually covering string $\{U(n,A_n)\}$, so that for every $n \in \mathbb{N}$, we may define $f_n = f_{n,A_n}$.

We wish to establish that $\{f_n\}$ converges to f in $C_\pi(X)$. So let $x \in X$, and let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ with $N \geq \frac{1}{\varepsilon}$ such that for every $n \geq N$, $x \in U(n,A_n)$. But then if $n \geq N$,

$$|f_n(x) - f(x)| = |f_{n,A_n}(x) - f(x)| < \frac{1}{n} \leq \frac{1}{\mathbb{N}} \leq \varepsilon .$$

Therefore $\{f_n(x)\}$ converges to $f(x)$ for every $x \in X$, so that $\{f_n\}$ converges to f in $C_\pi(X)$. Hence $C_\pi(X)$ must be a Fréchet space.

(c) ⇒ (d). Suppose X has a sequence $\{\mathcal{U}_n\}$ of open covers for finite subsets such that no string from $\{\mathcal{U}_n\}$ is residually covering. Let $\mathcal{V}_1 = \mathcal{U}_1$, and for each $n > 1$, let $\mathcal{V}_n$ be an open cover for finite subsets of X which refines both V_{n-1}

and $\mathcal{U}_n$. For every $n \in \mathbb{N}$ and $A \in \mathcal{F}(X)$, let $U(n,A) \in \mathcal{V}_n$ such that $A \subseteq U(n,A)$, and let $f_{n,A} \in C(X)$ be such that $f_{n,A}(A) = \{\frac{1}{n}\}$, $f_{n,A}(X \backslash U(n,A)) = \{n\}$, and $f_{n,A}(X) \subseteq [\frac{1}{n}, n]$. Then define

$$F = \{f_{n,A} | n \in \mathbb{N} \text{ and } A \in \mathcal{F}(X)\},$$

and also define $F^* = \overline{F} \backslash \{c_o\}$ in $C_\pi(X)$, where c_o is the constant zero function.

First we establish that F^* is not closed in $C_\pi(X)$ by showing that c_o is an accumulation point of F in $C_\pi(X)$. To do this, let $W = [\![x_1, V_1]\!] \cap \ldots \cap [\![x_k, V_k]\!]$ be an arbitrary basic neighborhood of c_o in $C_\pi(X)$. If $A = \{x_1, \ldots, x_k\}$ and $n \in \mathbb{N}$ such that $\frac{1}{n} \in V_1 \cap \ldots \cap V_k$, then $f_{n,A} \in W \cap F$.

We will then obtain that $C_\pi(X)$ is not a k-space, as desired, if we can show that the intersection of F^* with each compact subspace of $C_\pi(X)$ is closed in that compact subspace. To this end, let K be an arbitrary compact subspace of $C_\pi(X)$. Then for every $x \in X$, the orbit $\{f(x) | f \in K\}$ is bounded in $\mathbb{R}$. For every $x \in X$, define $M(x) = \sup \{f(x) | f \in K\}$, and also for every $m \in \mathbb{N}$, define $X_m = \{x \in X | M(x) \leq m\}$. Note that $X = \bigcup \{X_m | m \in \mathbb{N}\}$, and that for every m, $X_m \subseteq X_{m+1}$.

Suppose, by way of contradiction, that for every $m, n \in \mathbb{N}$, there exists a $k \geq n$ and $V \in \mathcal{V}_k$ such that $X_m \subseteq V$. We define, by induction, a string $\{U_n\}$ from $\{\mathcal{U}_n\}$. First there exists a $k_1 \geq 1$ and $V_1 \in \mathcal{V}_{k_1}$ such that $X_1 \subseteq V_1$. For each $i = 1, \ldots, k_1$, choose $U_i \in \mathcal{U}_i$ so that $V_1 \subseteq U_i$. Now suppose k_m and $U_1, \ldots, U_{k_m}$ have been defined. Then there exists a $k_{m+1} \geq k_m + 1$ and $V_{m+1} \in \mathcal{V}_{k_{m+1}}$ such that $X_{m+1} \subseteq V_{m+1}$. For each $i = k_m + 1, \ldots, k_{m+1}$, choose $U_i \in \mathcal{U}_i$ so that $V_{m+1} \subseteq U_i$. This defines string $\{U_n\}$, which we know to not be residually covering. Let $x \in X$ be arbitrary. There is an $m \in \mathbb{N}$ such that $x \in X_m$. Let $n \geq k_m$. There is a $j \geq m$ such that $k_{j-1} + 1 \leq n \leq k_j$. Then $x \in X_m \subseteq X_j \subseteq V_j \subseteq U_n$. But this says that $\{U_n\}$ is residually covering, which is a contradiction.

We have just established that there exist $m, n \in \mathbb{N}$ such that for every $k \geq n$ and for every $V \in \mathcal{V}_k$, $X_m \not\subseteq V$. Then define $M = \max \{m,n\}$, let $x_o \in X$ be arbitrary, and define $W = [\![x_o, (-\frac{1}{M}, \frac{1}{M})]\!]$, which is a neighborhood of c_o in $C_\pi(X)$. Suppose $f \in W \cap F$. Then there exists a $k \in \mathbb{N}$ and $A \in \mathcal{F}(X)$ such that $f = f_{k,A}$. Since $\frac{1}{k} \le f(x_o) < \frac{1}{M}$, then $k > M \ge n$. Thus $X_m \not\subseteq U(k,A)$, so that there exists an $x_1 \in X_m \backslash U(k,A)$. But then $f(x_1) = k > M \ge m \ge M(x_1)$, so that $f \notin K$. Therefore $W \cap F \cap K = \emptyset$, so that c_o is not an accumulation point of $F^* \cap K$ in K. Hence $F^* \cap K$ must be closed in K. Since K was arbitrary, we obtain that $C_\pi(X)$ is not a k-space. □

THEOREM 2. $C_\pi(X)$ has countable tightness if and only if every open cover for finite subsets of X has a countable subcover for finite subsets of X.

PROOF. Suppose that every open cover for finite subsets of X has a countable subcover for finite subsets of X. Let F be a subset of $C_\pi(X)$, and let f be an accumulation point of F in $C_\pi(X)$. Then for each $n \in \mathbb{N}$ and $A = \{x_1, \ldots, x_k\} \in \mathcal{F}(X)$, choose

$$f_{n,A} \in F \cap [\![x_1, (f(x_1) - \tfrac{1}{n}, f(x_1) + \tfrac{1}{n})]\!] \cap \ldots \cap [\![x_k, (f(x_k) - \tfrac{1}{n}, f(x_k) + \tfrac{1}{n})]\!] .$$

Also let $U(n,A) = \{x \in X \mid |f_{n,A}(x) - f(x)| < \frac{1}{n}\}$, which is an open subset of X. Then for each $n \in \mathbb{N}$, $\{U(n,A) \mid A \in \mathcal{F}(X)\}$ is an open cover for finite subsets of X. So for each $n \in \mathbb{N}$, there exists a sequence $\{A(n,i) \mid i \in \mathbb{N}\}$ from $\mathcal{F}(X)$ such that $\{U(n,A(n,i)) \mid i \in \mathbb{N}\}$ is a cover for finite subsets of X. Then define $G = \{f_{n,A(n,i)} \mid n, i \in \mathbb{N}\}$.

To see that $f \in \overline{G}$, let $W = [\![x_1, V_1]\!] \cap \ldots \cap [\![x_k, V_k]\!]$ be a neighborhood of f in $C_\pi(X)$. Let $A = \{x_1, \ldots, x_k\}$, and choose $n \in \mathbb{N}$ so that $(f(x_j) - \frac{1}{n}, f(x_j) + \frac{1}{n}) \subseteq V_j$ for each $j = i, \ldots, k$. Then there is an $i \in \mathbb{N}$ such that $A \subseteq U(n,A(n,i))$. So for each $x \in A$, $|f_{n,A(n,i)}(x) - f(x)| < \frac{1}{n}$, and hence $f_{n,A(n,i)} \in W$.

Conversely, suppose that $C_\pi(X)$ has countable tightness, and let $\mathcal{U}$ be an open cover for finite subsets of X. For each $A \in \mathcal{F}(X)$, let $U(A) \in \mathcal{U}$ be such that $A \subseteq U(A)$. Also for each $n \in \mathbb{N}$ and $A \in \mathcal{F}(X)$, let $f_{n,A} \in C(X)$ be such that

$f_{n,A}(A) = \{\frac{1}{n}\}$, $f_{n,A}(X\backslash U(A)) = \{n\}$, and $f_{n,A}(X) \subseteq [\frac{1}{n},n]$. Then define $F = \{f_{n,A} | n \in \mathbb{N}$ and $A \in \mathcal{F}(X)\}$.

Since the constant zero function, c_0, is an accumulation point of F, then there is a countable subset G of F such that $c_0 \in \overline{G}$. There are sequences $\{n_i\} \subseteq \mathbb{N}$ and $\{A_i\} \subseteq \mathcal{F}(X)$ so that $G = \{f_{n_i,A_i} | i \in \mathbb{N}\}$.

To see that $\{U(A_i) | i \in \mathbb{N}\}$ is a cover for finite subsets of X, let $A = \{x_i, \ldots, x_k\} \in \mathcal{F}(X)$. Then there exists an $i \in \mathbb{N}$ such that $f_{n_i,A_i} \in [\![x_i, (-1,1)]\!] \cap \ldots \cap [\![x_k,(-1,1)]\!]$. But this means that $A \subseteq U(A_i)$, so that $\{U(A_i) | i \in \mathbb{N}\}$ is indeed a cover for finite subsets of X. □

Let us now give names to the two properties of X which are expressed in Theorems 1 and 2. We will call X <u>k-countable</u> whenever $C_\pi(X)$ is a k-space, and we will call X <u>τ-countable</u> whenever $C_\pi(X)$ has countable tightness. We state some immediate facts about these properties.

PROPOSITION 3. Every countable space is k-countable.

PROPOSITION 4. Every k-countable space is τ-countable.

PROPOSITION 5. Every τ-countable space is Lindelöf.

PROOF. Let X be τ-countable, and let $\mathcal{U}$ be an open cover of X. Let $\mathcal{V}$ be the family of all finite unions of members of $\mathcal{U}$. Then $\mathcal{V}$ is an open cover for finite subsets of X, so that it has a countable subcover $\mathcal{W}$ for finite subsets of X. Each member of $\mathcal{W}$ is a finite union of members of $\mathcal{U}$, so that since $\mathcal{W}$ covers X, then $\mathcal{U}$ has a countable subcover. □

This means that if $C_\pi(X)$ has countable tightness, X must be Lindelöf. In particular, $C_\pi(\Omega_0)$ does not have countable tightness, where Ω_0 is the space of countable ordinals with the order topology. This is in contrast to $C_\pi(\Omega)$, which we see from the next proposition has countable tightness, where $\Omega = \Omega_0 \cup \{\omega_1\}$.

PROPOSITION 6. If X^n is Lindelöf for every $n \in \mathbb{N}$, then X is τ-countable.

PROOF. Let X^n be Lindelöf for every $n \in \mathbb{N}$, and let $\mathcal{U}$ be an open cover for finite subsets of X. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{U^n \subseteq X^n | U \in \mathcal{U}\}$. Since $\mathcal{U}$ is an open cover for finite subsets of X, then each $\mathcal{U}_n$ is an open cover of X^n. So for each $n \in \mathbb{N}$, $\mathcal{U}$ has a countable subcollection $\mathcal{V}_n$ such that $\{U^n | U \in \mathcal{V}_n\}$ covers X^n. But then $\cup \{\mathcal{V}_n | n \in \mathbb{N}\}$ is a countable subcollection of $\mathcal{U}$ which is a cover for finite subsets of X. □

COROLLARY 7. Every compact space is τ-countable, and every separable metric space is τ-countable.

We now examine some properties of k-countable spaces.

PROPOSITION 8. Every closed subspace of a k-countable space is k-countable.

PROOF. Let X be a k-countable space, and let Y be a closed subspace of X. Let $\{\mathcal{V}_n\}$ be a sequence of open covers for finite subsets of Y. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{V \cup (X \setminus Y) | V \in \mathcal{V}_n\}$, which is an open cover for finite subsets of X. Now $\{\mathcal{U}_n\}$ has a residually covering string $\{V_n \cup (X \setminus Y)\}$, where each $V_n \in \mathcal{V}_n$. But then $\{V_n\}$ is a residually covering string from $\{\mathcal{V}_n\}$. □

PROPOSITION 9. Every continuous image of a k-countable space is k-countable.

PROOF. Let X be k-countable, and let $f: X \to Y$ be a continuous surjection. Let $\{\mathcal{V}_n\}$ be a sequence of open covers for finite subsets of Y. For each $n \in \mathbb{N}$, let $\mathcal{U}_n = \{f^{-1}(V) | V \in \mathcal{V}_n\}$, which is an open cover for finite subsets of X. Now $\{\mathcal{U}_n\}$ has a residually covering string $\{f^{-1}(V_n)\}$, where each $V_n \in \mathcal{V}_n$. But then $\{V_n\}$ is a residually covering string from $\{V_n\}$. □

In the next proposition, we use the term covering string, by which we mean a string which is itself a cover of the space.

PROPOSITION 10. If X is k-countable, then every sequence of open covers of X has a covering string.

PROOF. Let $\{\mathcal{U}_n\}$ be a sequence of open covers of X. For each $n \in \mathbb{N}$, let

$$\mathcal{V}_n = \{U_n \cup \cdots \cup U_{n+k+1} | k \in \mathbb{N} \text{ and each } U_i \in \mathcal{U}_i\},$$

which is an open cover for finite subsets of X. Thus $\{\mathcal{V}_n\}$ has a residually covering string $\{V_n\}$. Now $V_1 = U_1 \cup \cdots \cup U_{k_1}$ for some $k_1 \in \mathbb{N}$. Also $V_{k_1+1} =$

$U_{k_1+1}\cup\cdots\cup U_{k_2}$ for some $k_2 \in \mathbb{N}$ with $k_2 > k_1$. Continuing by induction, we can define an increasing sequence $\{k_i\}$ such that each $V_{k_i+1} = U_{k_i+1}\cup\cdots\cup U_{k_{i+1}}$. This defines U_n for each $n \in \mathbb{N}$. To see that $\{U_n\}$ is a covering string from $\{\mathcal{U}_n\}$ let $x \in X$. Then there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $x \in V_n$. Since $\{k_i\}$ is increasing, there is some i such that $k_i \geq N$. Then $x \in V_{k_i+1} = U_{k_i+1}\cup\cdots\cup U_{k_{i+1}}$, so that x is indeed in some U_n. □

We next give an important example of a space which is not k-countable.

EXAMPLE 11. The closed unit interval, I, is not k-countable.

PROOF. For each $n \in \mathbb{N}$, let $\mathcal{U}_n$ be the set of all open intervals in I having diameter less than $\frac{1}{2^n}$. Suppose $\{\mathcal{U}_n\}$ were to have a covering string $\{U_n\}$. Then since I is connected, there would be a simple chain $\{U_{n_1}, \ldots, U_{n_k}\}$ from 0 to 1. That is, $0 \in U_{n_1}$, $1 \in U_{n_k}$, and for each $1 \leq i \leq k-1$, there is a $t_i \in U_{n_i} \cap U_{n_{i+1}}$. But then

$$\begin{aligned} 1 &\leq |1 - t_{k-1}| + |t_{k-1} - t_{k-2}| + \cdots + |t_2 - t_1| + |t_1| \\ &< \frac{1}{2^{n_k}} + \frac{1}{2^{n_{k-1}}} + \cdots + \frac{1}{2^{n_2}} + \frac{1}{2^{n_1}} \\ &\leq \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^k} < 1. \end{aligned}$$

This is a contradiction, so that $\{\mathcal{U}_n\}$ cannot have a covering string. Therefore, by Proposition 10, I is not k-countable. □

The next three results are consequences of Example 11.

EXAMPLE 12. The Cantor set, $\mathbb{K}$, is not k-countable.

PROOF. Since there exists a continuous function from $\mathbb{K}$ onto I, then $\mathbb{K}$ cannot be k-countable because of Proposition 9 and Example 11. □

Our next proposition then follows from Example 12 and Proposition 8.

PROPOSITION 13. No k-countable space contains a Cantor set.

PROPOSITION 14. Every k-countable space is o-dimensional.

PROOF. Let X be k-countable, let $x \in X$, and let U be an open neighborhood of x in X. Since X is completely regular, there exists an $f \in C(X)$ such that $f(x) = 0$, $f(X\setminus U) = \{1\}$, and $f(X) \subseteq I$. Since I is not k-countable by Example 11, and since $f(X)$ is k-countable by Proposition 9, then there exists a $t \in I\setminus f(X)$. Thus $[0,t) \cap f(X)$ is both open and closed in $f(X)$, so that $f^{-1}([0,t))$ is an open and closed neighborhood of x contained in U. □

With all these necessary conditions which k-countable spaces must satisfy, one might wonder whether there exists an uncountable k-countable space. This is answered by the next two examples.

We will call a space X <u>virtually countable</u> if there exists a finite subset F of X such that for every open subset U of X with $F \subseteq U$, it is true that $X\setminus U$ is countable. Notice that a first countable virtually countable space is countable.

PROPOSITION 15. Every virtually countable space is k-countable.

PROOF. Let F be a finite subset of X such that every open U in X with $F \subseteq U$ has countable complement, and let $\{\mathcal{U}_n\}$ be a sequence of open covers for finite subsets of X. First let $U_1 \in \mathcal{U}_1$ be such that $F \subseteq U_1$. Then $X\setminus U_1$ is countable; say $X\setminus U_1 = \{x_{11}, x_{12}, x_{13}, \ldots\}$. Let $U_2 \in \mathcal{U}_2$ be such that $F \cup \{x_{11}\} \subseteq U_2$. Now $X\setminus U_2$ is also countable; say $X\setminus U_2 = \{x_{21}, x_{22}, x_{23}, \ldots\}$. Let $U_3 \in \mathcal{U}_3$ be such that $F \cup \{x_{11}, x_{12}, x_{21}\} \subseteq U_3$. Continuing by induction, we may define string $\{U_n\}$ from $\{\mathcal{U}_n\}$ such that for each n, $U_n = X\setminus\{x_{n1}, x_{n2}, x_{n3}, \ldots\}$ and

$$F \cup \{x_{11}, \ldots, x_{1n}, x_{21}, \ldots, x_{2,n-1}, \ldots, x_{n1}\} \subseteq U_{n+1}.$$

To see that every element of X is residually in $\{U_n\}$, let $x \in X$. If $x \in \bigcap_{n=1}^{\infty} U_n$, then x is residually in $\{U_n\}$. If $x \notin \bigcap_{n=1}^{\infty} U_n$, then let i be the first integer such that $x \notin U_i$. Then $x = x_{ij}$ for some j, so that for every $n \geq i + j$, $x \in U_n$. Therefore x is residually in $\{U_n\}$. □

EXAMPLE 16. The space of ordinals, Ω, which are less than or equal to the first uncountable ordinal is k-countable.

PROOF. It is easy to see that Ω is virtually countable. □

EXAMPLE 17. The Fortissimo space, $\mathbb{F}$, is k-countable, where $\mathbb{F}$ is $\mathbb{R}$ with the following topology: each $\{t\}$ is open for $t \neq 0$, and the open sets containing 0 are the sets containing 0 which have countable complements. Also $\mathbb{F}^2$ is not Lindelöf, which shows that the converse of Proposition 6 is not true.

PROOF. Obviously $\mathbb{F}$ is virtually countable. However, an alternate proof can be obtained from known properties of this space. In particular, it follows from [1] that $C_\pi(\mathbb{F})$ is homeomorphic to a Σ-product of copies of $\mathbb{R}$, and from [3] that a Σ-product of first countable spaces is a Fréchet space. □

The spaces in the previous two examples are not first countable. This raises the following question.

QUESTION 18. Is every first countable k-countable space countable?

One well studied example of an uncountable first countable space which is also a o-dimensional Lindelöf space and which does not contain a Cantor set is the Sorgenfrey line. However, in our last example we show that this space is not k-countable, and in fact is not even τ-countable.

EXAMPLE 19. The Sorgenfrey line, S, is not τ-countable. This shows that the converse of Proposition 5 is not true.

PROOF. For each $A \in \mathcal{F}(S)$, let $\delta(A) = \frac{1}{2} \min \{|a-a'| \mid a,a' \in A, \text{ with } a \neq a'\}$, and let $U(A) = \bigcup\{[a,a+\delta(A)) \mid a \in A\}$. Then define $\mathcal{U} = \{U(\hat{A}) \mid A \in \mathcal{F}(S)\}$, where $\hat{A} = A \cup \{-a \mid a \in A\}$. Clearly $\mathcal{U}$ is an open cover for finite subsets of S. Then $\{U^2 \mid U \in \mathcal{U}\}$ is an open cover of S^2. But each U^2, for $U \in \mathcal{U}$, intersects the set $\{(x,y) \in S^2 \mid x+y = 0\}$ on a finite set, so that $\{U^2 \mid U \in \mathcal{U}\}$ has no countable subcover of S^2. Therefore no countable subcollection of $\mathcal{U}$ can cover all doubleton subsets of S. □

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