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THE ORDER TOPOLOGY FOR FUNCTION LATTICES AND REALCOMPACTNESS

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ABSTRACT. A lattice K(X,Y) of continuous functions on space X is associated to each compactification Y of X. It is shown for K(X,Y) that the order topology is the topology of compact convergence on X if and only if X is realcompact in Y. This result is used to provide a representation of a class of vector lattices with the order topology as lattices of continuous functions with the topology of compact convergence. This class includes every C(X) and all countably universally complete function lattices with 1. It is shown that a choice of K(X,Y) endowed with a natural convergence structure serves as the convergence space completion of V with the relative uniform convergence.

KEY WORDS AND PHRASES. Order topology, relative uniform convergence, realcompactness, universally complete function lattice, convergence space completion.

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1. INTRODUCTION.

In this paper we will study a broad class of real function lattices which we call "2-universally complete." For this class we will show that the order topology T_o (also called the order bound topology and the relative uniform topology) is the topology of compact convergence in an appropriate representation (Theorem2). We will show (Proposition 3) that the 2-universally complete lattices include the lattices $C(X)$, all continuous real-valued functions on X, and all countably universally complete lattices containing 1. (An example of this latter type is discussed in Example 1.)

The proof of Theorem 2 requires a construction which is studied independently in § 1. In particular, sublattices $K(X,Y)$ of $C(X)$ for compactifications Y of X are investigated. Theorem 1 states that the order topology for $K(X,Y)$ is the topology of compact convergence on X if and only if X is realcompact in Y. (This concept of realcompactness was studied in [10].)

Since the order topology is the finest locally convex topology in which every relatively uniformly convergent net converges (see [5]), in § 3 we consider the 2-universally complete function lattice V with relative uniform convergence as a convergence function lattice V_ρ, without reference to its associated order topology. We show (Theorem 4) that $K(X,Y)$ endowed with a natural convergence structure serves as the completion in the convergence space sense of V_ρ.

We remark that (assuming without loss of generality that X is realcompact) it can be seen directly that the order topology is the topology of compact convergence for the lattice $C(X)$. This follows from [13, p. 124] since every positive linear functional is continuous with respect to the topology of compact convergence (see [6]) and since $C(X)$ with the topology of compact convergence is barrelled (see [11]).

2. THE ORDER TOPOLOGY FOR K(X,Y)

Let Y be a compact Hausdorff space and X a dense subspace. We denote by $F(X,Y)$ the set of all nonnegative extended real-valued continuous functions on Y which are finite on X. For f in $F(X,Y)$ we let Λ_f be the set in $Y \setminus X$ where f is infinite. We set

$$K(X,Y) = \bigcup \{C(Y \setminus \Lambda_f) : f \in F(X,Y)\}.$$

Since X is dense in each $Y \setminus \Lambda_f$, by restricting the functions in $K(X,Y)$ to X we can view $K(X,Y)$ as a sublattice (and also a subalgebra) of $C(X)$.

LEMMA 1. For each function g in $K(X,Y)$ there is a function f in $F(X,Y)$ such that $g \leq f$.

PROOF. Given g in $K(X,Y)$ there is a function $h \geq 0$ in $F(X,Y)$ such that g is in $C(Y \setminus \Lambda_h)$. We consider the compact subsets

$$A_n = h^{-1}\,[n-1,n]$$

and

$$B_n = h^{-1}\,[0,n-2] \cup h^{-1}[n+1,\,\infty]$$

of Y $(n = 1,2,\ldots)$ with the understanding that $B_1 = h^{-1}\,[2,\,\infty]$. Using separating functions on Y, one can construct for each n a continuous function f_n such that

$$f_n(x) = \sup\ \{g(z) : z \in A_n\} \quad \text{for } x \text{ in } A_n$$

and

$$f_n(x) = h(x) \quad \text{for } x \text{ in } B_n.$$

On $Y \setminus \Lambda_h$ the function f defined by

$$f(x) = \sup\ \{f_n(x) : n = 1,2,\ldots\}$$

is continuous, since at each point x in $Y \setminus \Lambda_h$ there is a neighborhood of x on which f is the supremum of finitely many functions f_n. Moreover, $f \geq h$ on $Y \setminus \Lambda_h$ and hence extends continuously to Y(i.e., $f(x) = \infty$ for x in Λ_h). Thus f is in $F(X,Y)$, and $g \leq f$.

For Y a compact Hausdorff space and X a dense subspace of Y, we will say that X is <u>realcompact in</u> Y if

$$X = \bigcap\ \{Y \setminus \Lambda_f :\ f \in F(X,Y)\}$$

This concept has been considered by Lorch in [10]. Where βX denotes the Stone-Čech compactification of X, we note that X is realcompact if and only if X is realcompact in βX. If X is realcompact in Y it follows that X is realcompact, since Y is a quotient of βX. On the other hand, the real line X with its discrete topology is realcompact but not realcompact in its one-point compactification Y: The set Λ_f is empty for each f in $F(X,Y)$ since X is not σ - compact, implying

$$\bigcap\ \{Y \setminus \Lambda_f :\ f \in F(X,Y)\} = Y.$$

We note that the following proposition is also a consequence of work done in [8].

PROPOSITION 1. A completely regular space X is realcompact in each of its compactifications if and only if it is Lindelöf.

PROOF. Suppose X is Lindelöf, Y is a compactification of X and $p \in Y \setminus X$. Arguing as in [9], for each x in X we define a Urysohn function h_x on Y such that $h_x(x) = 1$, $h_x(p) = 0$ and $0 \leq h_x \leq 1$. Then $\{h_x^{-1}(\frac{1}{2},\infty): x \in X\}$ is an open cover of X having a countable subcover corresponding to functions $\{h_n\}_{n=1}^{\infty}$. Let $h = \sum h_n/2^n$, a non-negative member of $C(Y)$ which is strictly positive on X and zero at p. Thus p is in $\Lambda_{1/h}$, showing that X is realcompact in Y. Conversely, if X is not Lindelöf, by [9] there is a compact set K in $\beta X \setminus X$ which is not contained in a zero set in $\beta X \setminus X$. Let Y be that quotient of βX obtained by identifying the points of K. Since the image of K in Y cannot be contained in a zero set in $Y \setminus X$, X is not realcompact in Y.

The subscript co will denote the topology of compact convergence and the subscript T_o will denote the order topology. For a completely regular space X with realcompactification νX, as noted in the introduction, $C_{T_o}(\nu X) = C_{co}(\nu X)$. Since $C_{co}(X) \neq C_{co}(\nu X)$ if X is not realcompact, we conclude that

$$C_{T_o}(X) = C_{co}(X)$$

if and only if X is realcompact (in βX). We provide the following generalization, noting that $K(X,\beta X)$ is $C(X)$.

THEOREM 1. Let Y be a compact Hausdorff space and X be a dense subspace of Y. Then in $K(X,Y)$ the order topology coincides with the topology of compact convergence on X if and only if X is realcompact in Y.

PROOF. Setting

$$Z = \bigcap\{Y \setminus \Lambda_f: \ f \in F(X,Y)\},$$

we note that

$$K(X,Y) = K(Z,Y).$$

We abbreviate $K(X,Y)$ and $F(X,Y)$ to K and F. The subscript ρ will denote the relative uniform convergence structure. To complete the proof, we show that the topology of K_{T_o} is the topology of compact convergence on Z. Let $\{f_\alpha\}$ be a net convergent to zero in K_ρ. (As noted in the Introduction, T_o is the finest locally convex topology in which every net convergent in K_ρ converges.) There is a $g \geq 0$ in K such that for all n,

$$|f_\alpha| \leq \frac{1}{n} \cdot g \quad (\alpha \geq \alpha_n).$$

Clearly $\{\frac{1}{n} \cdot g\}$ converges to zero in $C_{co}(Y \setminus \Lambda_g)$ and hence in $C_{co}(Z)$. Since $C_{co}(Z)$ is a topological vector lattice, $\{f_\alpha\}$ converges in $C_{co}(Z)$. Thus the map from K_{T_o} into $C_{co}(Z)$ is continuous. To show that T_o is coarser than the topology of compact convergence on Z, let U be a closed, absolutely convex, solid neighborhood of zero in K_{T_o}. We remark that for $f \in F$, the inclusion map from $C_{co}(Y \setminus \Lambda_f)$ into K_{T_o} is continuous. This follows from the fact that the map from $C_\rho(Y \setminus \Lambda_f)$ and hence from $C_{T_o}(Y \setminus \Lambda_f)$ into K_{T_o} is continuous. (As noted in the introduction, $C_{T_o}(Y \setminus \Lambda_f)$ is $C_{co}(Y \setminus \Lambda_f$.) Since $C_{co}(Y)$ is $C_{co}(Y \setminus \Lambda_1)$ and U is solid,

$$U \supseteq \{g \in K: ||g||_Z \leq \delta\}$$

where $||\cdot||_Z$ denotes the supremum on Z, and δ is a fixed positive number. The following argument uses techniques found in [11]. We will call a compact set G in Y a <u>support set for</u> U if for f in F, f is in U whenever its restriction to G vanishes. For example, Y is a support set for U and, assuming U does not contain F (if $U \supseteq F$ then $U = K$ by Lemma 1), the empty set is not a support set for U. We note several properties of support sets for U which will be needed.

(a) Let G be a support set for U. If f is in F with $||f||_G < \delta/2$ then f is in U.

To see this, consider the function $g = (f - \delta/2) \vee 0$. Although F is not a vector lattice, g is clearly in F. Since $||2g||_G = 0$, $2g$ is in U; since $||2(f-g)||_Z \leq \delta$, $2(f-g)$ is in U. Thus by the convexity of U, f is in U.

(b) Let G be a compact subset of Y. If for h in F, h is in U whenever h vanishes on a neighborhood of G, then G is a support set for U.

To see this, suppose $f \in F$ vanishes on such a set G. The function $g = (f - \delta/2) \vee 0$ vanishes on $f^{-1}(-\delta/2, \delta/2)$, a neighborhood of G. Thus $2g$ is in U. Since again $2(f - g)$ is in U we obtain $f \in U$.

(c) The intersection of two support sets for U is a support set for U.

To see this, let G and H be support sets for U and let f in F vanish on a neighborhood W of $G \cap H$. If f is bounded there is a g in $C(Y) \cap F$ such that $||g||_G = 0$ and $g(x) > f(x)$ for x in $H \setminus W$. Since $||g \wedge f||_G = 0$, $2(g \wedge f)$ is in U, and since $||(f - g) \vee 0||_H = 0$, $2[(f - g) \vee 0]$ is in U. Thus by the convexity of U, f is in U. Now suppose f is not bounded. Then f is the limit in K_{T_o} of the bounded functions $\{f \wedge n1\}$ since this sequence converges to f in $C_{co}(Y \setminus \Lambda_f)$. Each $f \wedge n1$ is in U and U is closed, so that f is in U.

(d) The intersection S of all support sets for U is a support set for U.

To see this, let f in F vanish on a neighborhood W of S. Since $Y \setminus W$ is compact it is covered by the complements of finitely many support sets for U. Thus f vanishes on the intersection of these finitely many support sets and is in U by (c).

We prove that the intersection S of all support sets for U is contained in Z. Let p be in $Y \setminus Z$. Then $h(p) = +\infty$ for some h in F. Since the inclusion map from $C_{co}(Y \setminus \Lambda_h)$ into K_{T_o} is continuous, there is a compact subset D of $Y \setminus \Lambda_h$ such that U contains $\{g \in C(Y \setminus \Lambda_h) : ||g||_D < \varepsilon\}$ for some $\varepsilon > 0$. Thus if g is in $C(Y \setminus \Lambda_h) \cap F$ and vanishes on D, then g is in U. For any f in F which vanishes on D, since $f \wedge n1$ is in $C(Y \setminus \Lambda_h)$ we have $f \wedge n1 \in U$. It follows from the fact that U is closed that f is in U. Thus D is a support set for U not containing p. We conclude that S is contained in Z. By (a) $U \supseteq \{f \in F : ||f||_S < \delta/2\}$.

Let g be any member of K having $||g||_S < \delta/2$. Then g is in $C(Y\setminus\Lambda_h)$ for some h in F. By Lemma 1 we can assume $h \geq |g|$. There is a number N large enough so that the closed set $h^{-1}[N,\infty)$ in $Y\setminus\Lambda_h$ is disjoint from S. Since $Y\setminus\Lambda_h$ is normal there is a function k in $C(Y\setminus\Lambda_h)$ which equals $|g|$ on S and h on $h^{-1}[N,\infty)$. Letting f be $k\vee|g|$ on $Y\setminus\Lambda_h$ and ∞ on Λ_h, we conclude that f is in F since $f \geq h$ on $h^{-1}[N,\infty)$. Since $f = |g|$ on S, f is in U. Since $f \geq |g|$ and U is solid, g is in U. Thus,

$$U \supseteq \{g \in K: ||g||_S < \delta/2\},$$

a neighborhood of zero in the topology of compact convergence on Z. This completes the proof.

3. THE ORDER TOPOLOGY FOR A 2-UNIVERSALLY COMPLETE LATTICE

We recall that an element e in a vector lattice V is said to be a weak order unit if for each v in V

$$v = \bigvee\{v \wedge ne: n = 1,2,3,\ldots\}.$$

We will assume that V is a vector lattice having a weak order unit e and that the real lattice homomorphisms on V separate the points of V. We let X be the set of lattice homomorphisms x on V such that $x(e) = 1$ with the topology of pointwise convergence. We map V into $C(X)$ by the usual Gelfand map $\hat{v}(x) = x(v)$ for all x in X. We will refer to X as the carrier space of V. The proof of the following proposition uses the techniques of Lemma 2 in [3].

PROPOSITION 2. The Gelfand mapping of V into $C(X)$ is injective.

PROOF. Suppose $\hat{v} = 0$ for v in V; thus, $x(v) = 0$ for all x in X. For each lattice homomorphism ϕ on V either $\phi(e) = 0$ or $\phi/\phi(e)$ is in X, so that

$$\phi(v\wedge ne) = \phi(v)\wedge n\phi(e) = 0.$$

Since the lattice homomorphisms separate X, $v\wedge ne = 0$ for all n. Thus

$$v = \bigvee\{v\wedge ne: n = 1,2,\ldots\} = 0.$$

We will henceforth identify V with its image in $C(X)$ and refer to it as a function lattice with 1 (the image of e). We also will not distinguish between functions in $C(X)$ and their extensions to functions from the

Stone-Čech compactification βX of X to the extended real numbers.

We will have need for an additional condition. In a function lattice V with 1 we will say that a collection $\{v_n\}_{n=1}^{\infty}$ is 2-disjoint if

1. For each n, $|v_n| \wedge |v_k| \neq 0$ for at most two indices k distinct from n, and
2. for each x in the carrier space X of V there is a v_n such that $v_n(x) \neq 0$.

We will say that V is 2-universally complete if each 2-disjoint collection has a supremum in V.

For what follows we recall that a countably universally complete lattice is one in which the supremum exists for each collection $\{v_n\}_{n=1}^{\infty}$ satisfying $|v_n| \wedge |v_j| = 0$ for $n \neq j$.

PROPOSITION 3. (a) The lattice $C(Y)$ for any completely regular space Y is 2-universally complete. (b) Each countably universally complete function lattice with 1 is 2-universally complete.

PROOF. For (a), since the carrier space of $C(Y)$ is the realcompactification of Y we may as well assume that Y is realcompact. Given a 2-disjoint collection $\{f_n\}$ in $C(Y)$, at any point y in Y there is a function f_n with $|f_n(x)| > 0$ for all x in a neighborhood N of y. The pointwise supremum of the collection $\{f_n\}$ on N thus involves at most three functions; one concludes that the pointwise supremum of $\{f_n\}$ is continuous, and thus the supremum of $\{f_n\}$ in $C(Y)$.

The proof of (b) follows from the observation that a 2-disjoint collection can be decomposed into three collections, each having a supremum by countable universal completeness.

We note that $C(\mathbb{R})$ ($\mathbb{R}$ the reals) is 2-universally complete but not countably universally complete: Letting f_n be a continuous function which vanishes off $[\frac{1}{n+1}, \frac{1}{n}]$ and has value 1 at some point x_n, we obtain a collection $\{f_n\}_{n=1}^{\infty}$ satisfying $|f_n| \wedge |f_j| = 0$ for $n \neq j$ whose supremum f would clearly vanish on $(-\infty, 0)$ and yet $f(0) = \lim_{n \to \infty} f(x_n) \geq f_n(x_n) = 1$.

For a function lattice V with 1, we let $\widetilde{\beta X}$ denote the quotient space of the Stone-Čech compactification βX of the carrier space X which is induced by the equivalance relation $p \sim q$ if $v(p) = v(q)$ for all v in V. The space $\widetilde{\beta X}$ is compact and (since V separates $\widetilde{\beta X}$) Hausdorff. Note that X is real-compact in $\widetilde{\beta X}$. Since V^+ is contained in $F(X,\widetilde{\beta X})$, V is a sublattice of $K(X,\widetilde{\beta X})$. Of course, if V separates βX then $K(X,\widetilde{\beta X}) = K(X,\beta X) = C(X)$.

We will provide an example of a 2-universally complete function lattice $\mathcal{L}$ with 1 which is not uniformly dense in any function space $C(S)$. For other such examples see [7]. Furthermore, for the carrier space X in this example, $\widetilde{\beta X} \neq \beta X$ and $K(X,\widetilde{\beta X}) = \mathcal{L}$. For this purpose we will need the following lemma.

We recall that a <u>ϕ-algebra</u> is an archimedean lattice-ordered algebra over the reals with identity 1 which is a weak order unit.

LEMMA 2. Let V be a ϕ-algebra. Then every lattice homomorphism on V is an algebra homomorphism.

PROOF. Let θ be a lattice homomorphism on V. Then θ is a lattice homomorphism on the order ideal $I(1)$ generated by 1, hence an extremal element in the continuous dual $I(1)'$ of $I(1)$ in the order unit topology. For g in $I(1)$, $0 \leq g \leq 1$, we define $\theta_g(f) = \theta(fg)$. Since for $f \geq 0$ in $I(1)$ we have $\theta_g(f) \leq \theta(fg) \leq \theta(f)$, it follows that $0 \leq \theta_g \leq \theta$. Thus $\theta_g = \lambda_g \theta$ for a scalar λ_g which can be easily evaluated to be $\theta(g)$, so that $\theta(fg) = \theta(f)\theta(g)$. This argument can be extended by standard means to show that θ is an algebra homomorphism on $I(1)$. We next consider nonnegative elements g in $I(1)$ and f in V. To facilitate computations if $\theta(g) = 0$, we let $g^* = g + 1$. Then

$$\begin{aligned}\theta(fg^*) &= \bigvee_n[\theta(fg^*) \wedge n\theta(g^*)] \\ &= \bigvee_n[\theta(f \wedge n1)g^*] \\ &= \bigvee_n[\theta(f \wedge n1)\theta(g^*)] \\ &= [\bigvee_n(\theta(f) \wedge n)]\,\theta(g^*) = \theta(f)\theta(g^*),\end{aligned}$$

the third step being valid because $f \wedge n1$ and g^* are in $I(1)$. Thus $\theta(fg) = \theta(f)\theta(g)$. The argument can now be repeated without the restriction that g is in $I(1)$. The standard extension establishes that θ is an algebra homomorphism on V.

EXAMPLE 1. The ϕ-algebra $\mathcal{L}$ of all real-valued measurable functions on [0,1] is countably universally complete (hence, 2-universally complete) and is complete in the uniform topology. It is known (see [7]) that $\mathcal{L}$ is not isomorphic as a ϕ-algebra to any function space C(S). It follows from Lemma 2 that $\mathcal{L}$ is not isomorphic as a lattice to any C(S). Thus $\mathcal{L}$ is not a uniformly dense sublattice of any C(S). Furthermore, if $\mathcal{L}$ separated the points in βX then the space of bounded functions in $\mathcal{L}$ would be $C(\beta X)$ by the Stone-Weierstrass theorem. But since $\mathcal{L}$ as a ϕ-algebra is closed under inversion, this would imply $\mathcal{L} = C(X)$. Thus $\widetilde{\beta X} \neq \beta X$. To show $\mathcal{L} = K(X,\widetilde{\beta X})$, consider f in $K(X,\widetilde{\beta X})$. Then f is in $C(\widetilde{\beta X} \setminus \Lambda_h)$ for some h in $F(X,\widetilde{\beta X})$. Since $\widetilde{\beta X} \setminus \Lambda_h$ is σ-compact, the topology of $C_{co}(\widetilde{\beta X} \setminus \Lambda_h)$ is metrizable. By the Stone-Weierstrass theorem there exists a sequence of functions in $\mathcal{L}$ convergent to f in $C_{co}(\widetilde{\beta X} \setminus \Lambda_h)$. Thus f on [0,1] is a pointwise limit of a sequence of measurable functions, and so in $\mathcal{L}$. We remark that X here is [0,1] in the discrete topology and $K(X,\widetilde{\beta X})$ consists of just those continuous extended real-valued functions on $\widetilde{\beta X}$ which are finite on a dense subset.

LEMMA 3. Let V be 2-universally complete. For each function g in $K(X,\widetilde{\beta X})$ there is a function f in V such that $g \leq f$.

PROOF. We begin by showing that for compact sets K_1 and K_2 in $\widetilde{\beta X}$ there is a function v in V which is zero on K_1, one on K_2 and satisfies $0 \leq v \leq 1$. Given p in K_1 and x in K_2, since V is a vector space and separates $\widetilde{\beta X}$ there is a function v_x in V such that

$$0 \leq v_x(p) < 1 < v_x(x).$$

Clearly,

$$0 \leq v_x(q) < 1 < v_x(y)$$

for all points q in some neighborhood U_x of p and all points y in some neighborhood N_x of x. Let v_p be the supremum of functions v_x corresponding to a finite subcover $\{N_x\}$ of K_2. Then

$$0 \leq v_p(q) < 1 < v_p(y)$$

for all y in K_2 and q in a neighborhood of W_p of p. Letting W be the infimum of functions v_p corresponding to a finite subcover $\{W_p\}$ of K_1 we obtain

$$0 \leq w(q) < 1 < \alpha < w(y)$$

for all q in K_1, y in K_2 and some real number α. The function

$$v = \{\frac{1}{\alpha-1}[(w-1)\vee 0]\} \wedge 1$$

has the desired properties. Now let g be a function in $K(X,\widetilde{\beta X})$. By Lemma 1 there is a function h in $F(X,\widetilde{\beta X})$ such that $h \geq g$. Consider the compact subsets

$$A_n = h^{-1}[2n-2,\ 2n]$$

and $$B_n = h^{-1}[0,\ 2n-3] \cup h^{-1}[2n+1,\ \infty]$$

of βX $(n = 1,2,\ldots)$ with the understanding that $B_1 = h^{-1}[3,\infty]$. It follows from the first part of this proof that there is a function v_n in V with $0 \leq v_n \leq 2n$ which has value $2n$ on A_n and is zero on B_n. Since $\{v_n\}$ is a 2-disjoint collection, its supremum is a function in V greater than or equal to h (and hence g).

Given u in V^+, we consider the ideal

$$[u]^V = \{v \in V\colon |v| \leq \lambda u \text{ for some } \lambda \text{ in } \mathbb{R}\}$$

and we set, for each v in $[u]$,

$$||v||_u = \inf\{\lambda > 0\colon |v| \leq \lambda u\}.$$

It is easy to verify that the normed spaces

$$\{([u]^V,\ ||\cdot||_u)\colon u \in V^+\}$$

form an inductive system ordered by inclusion, whose locally convex inductive limit is V_{T_o} (see, e.g., [13,p.122]).

PROPOSITION 4. Let B be a 2-universally complete function lattice with 1 and having carrier space X. Then

$$V_{T_o} = V \cap K_{T_o}(X,\ \widetilde{\beta X}).$$

PROOF. We recall that V_{T_o} is the locally convex inductive limit of the factors $\{([u]^V,\ ||\cdot||_u)\colon u \in V^+\}$, where $[u]^V$ is the ideal in V generated by u. It follows from Lemma 3 that $K_{T_o}(X,\ \widetilde{\beta X})$ is the locally convex inductive limit of the factors $\{([u]^K,\ ||\cdot||_u)\colon u \in V^+\}$, where $[u]^K$ is the ideal in $K(X,\widetilde{\beta X})$ generated by u. It is easy to verify that the topology of V_{T_o} is finer than that of $V \cap K_{T_o}(X,\widetilde{\beta X})$. For the converse, let U be a solid neighborhood of zero in V_{T_o}. Then for some collection $\{\alpha_u : u \in V\}$ of positive scalars, U contains the convex hull of $\bigcup\{\alpha_u[-u,u]^V : u \in V\}$, where $[-u,u]^V$ denotes an order interval in V. Denoting by $[-u,u]^K$ the order interval in $K(X,\widetilde{\beta X})$, we let v be in the intersection with V of the convex hull of $\bigcup\{\alpha_u[-u,u]^K : u \in V\}$. Since $v = \sum_{i=1}^{n}\lambda_i f_i$ for scalars λ_i satisfying

$$\sum_{i=1}^{n} |\lambda_i| \leq 1 \text{ and } f_i \text{ in } \alpha_{u_i}[-u_i,u_i]^K, \text{ then}$$

$$|v| \leq \sum_{i=1}^{n} |\lambda_i| \cdot |\alpha_{u_i} u_i| \in U.$$

By the solidness of U, v is in U.

The next theorem is a consequence of Proposition 4 and Theorem 1.

THEOREM 2. Let V be a 2-universally complete function lattice with 1. The order topology T_o on V is the topology of compact convergence on the carrier space of V.

4. CONVERGENCE STRUCTURES RELATED TO UNIFORM CONVERGENCE

In this section we will be using the ideas of convergence space theory (see, e.g., [1]). We will consider convergence structures on $K(X,Y)$, where Y is compact and Hausdorff and X is realcompact in Y. We let $K_\sigma(X,Y)$ denote the convergence space inductive limit of the system

$$\{C_{co}(Y\setminus\Lambda_f) : f \in F(X,Y)\},$$

together with the continuous inclusion maps. Thus a net $\{g_\alpha\}$ converges to g in $K_\sigma(X,Y)$ if and only if g_α is in a factor $C(Y\setminus\Lambda_f)$ for all α beyond some α_0 and $\{g_\alpha\}_{\alpha \geq \alpha_0}$ converges to g in $C_{co}(Y\setminus\Lambda_f)$; equivalently, a filter Θ converges to g in $K_\sigma(X,Y)$ if and only if Θ contains the neighborhood filter at g in some factor $C_{co}(Y\setminus\Lambda_f)$. We note that for X realcompact, $K_\sigma(X,\beta X)$ is the convergence space $C_{I'}(X)$ studied in [2].

A set A in a vector lattice W is <u>bounded</u> if there is an element w in W such that $|a| \leq w$ for all a in A. A filter Θ is <u>bounded</u> if some set $A \in \Theta$ is bounded. It is easy to verify that if W_δ is a convergence vector lattice then the space $W_{\delta b}$ containing only the bounded filters from W_δ is also a convergence vector lattice. Thus $K_{\sigma b}(X,Y)$ is a convergence vector lattice. (A net $\{f_\alpha\}$ converges to a function f in $K_{\sigma b}(X,Y)$ if and only if it converges in $K_\sigma(X,Y)$ and is bounded - i.e., there exists a function g in $K(X,Y)$ and an index α_0 such that $|f_\alpha| \leq g$ for all $\alpha \geq \alpha_0$.)

We note that relative uniform convergence on a vector lattice is a convergence vector lattice structure.

THEOREM 3. Let V be a function lattice with 1. If V is 2-universally complete, the identity map from V_ρ onto its image in $K_{\sigma b}(X,\widetilde{\beta X})$ is bicontinuous.

PROOF. Let net $\{v_\alpha\}$ converge to zero in V_ρ: For some u in V and $n = 1,2,\ldots,$

$$|v_\alpha| \leq \frac{1}{n} u \text{ for } \alpha \geq \alpha_n.$$

Thus $\{v_\alpha\}$ converges to zero in $C_\rho(\widetilde{\beta X} \setminus \Lambda_u)$, where X is the carrier space of V. It follows that $\{v_\alpha\}$ converges to zero in $C_{co}(\widetilde{\beta X} \setminus \Lambda_u)$ and hence in $K_{\sigma b}(X,\widetilde{\beta X})$. Conversely, let net $\{v_\alpha\}$ in V converge to zero in $K_{\sigma b}(X,\widetilde{\beta X})$. For some g in $K(X,\widetilde{\beta X})$ and α_0.

$$|v_\alpha| \leq g \quad \text{for } \alpha \geq \alpha_0.$$

By Lemma 3 we can assume that g is in V; we can also assume $g \geq 1$ and $\{v_\alpha\}$ converges to zero in $C_{co}(\widetilde{\beta X} \setminus \Lambda_g)$. Thus, given n, there is an α_n such that for x in $g^{-1}[1,n]$

$$|v_\alpha(x)| \leq \frac{1}{n} \leq \frac{1}{n} g^2(x) \quad \text{for } \alpha \geq \alpha_n.$$

For x not in $g^{-1}[1,n]$, since $g^2(x) > n\, g(x)$,

$$|v_\alpha(x)| \leq g(x) < \frac{1}{n} g^2(x) \quad \text{for } \alpha \geq \alpha_0 .$$

By Lemma 3 there is a w in V such that $w \geq g^2$. Thus for α beyond α_0 and α_n,

$$|v_\alpha| \leq \frac{1}{n} g^2 \leq \frac{1}{n} w.$$

We conclude that $\{v_\alpha\}$ converges to zero in V_ρ.

COROLLARY 1. For realcompact X,

$$C_\rho(X) = K_{\sigma b}(X,\beta X) = C_{I'b}(X).$$

We recall that a set A in a convergence vector lattice W_δ is dense in W if every element of W is the limit in δ of a net in A. The space W_δ is complete if every Cauchy net (filter) converges. If W_δ is complete, it follows readily that $W_{\delta b}$ is complete.

THEOREM 4. Let V be a 2-universally complete function lattice with 1 having carrier space X. Then $K_{\sigma b}(X,\widetilde{\beta X})$ is complete and contains V as a dense subspace. Moreover, V_ρ is complete if and only if it equals $K_{\sigma b}(X,\widetilde{\beta X})$.

PROOF. The space $K_\sigma(X,\widetilde{\beta X})$, being an inductive limit of complete factors $C_{co}(\widetilde{\beta X}\setminus\Lambda_f)$, is easily seen to be complete; thus $K_{\sigma b}(X,\widetilde{\beta X})$ is complete. Given f in $K(X,\widetilde{\beta X})$, f is in $C(\widetilde{\beta X}\setminus\Lambda_g)$ for some g in $F(X,\widetilde{\beta X})$. Since $V\cap C(\widetilde{\beta X}\setminus\Lambda_g)$ is a sublattice of $C(\widetilde{\beta X}\setminus\Lambda_g)$ containing the constant functions and separating the points of $\widetilde{\beta X}\setminus\Lambda_g$, there is a net $\{v_\alpha\}$ in V converging to f in $C_{co}(\widetilde{\beta X}\setminus\Lambda_g)$ by the Stone-Weierstrass Theorem. By Lemma 3 there is a w in V with $w\geq|f|$; $\{(v_\alpha\wedge w)\vee(-w)\}$ converges to f in $K_{\sigma b}(X,\widetilde{\beta X})$. The last statement of the theorem is now a consequence of Theorem 3.

It follows from Theorem 4 that if a 2-universally complete function lattice V with 1 separates βX and if V_ρ is complete, then $V = C(X)$. If, moreover, X is σ-compact and locally compact then $K_\sigma(X,\beta X) = C_{co}(X)$ since $\beta X\setminus X = \Lambda_f$ for some f in $F(X,\beta X) = C(X)$, implying $V_\rho = C_{cob}(X)$.

COROLLARY 2. The space $\mathcal{L}_\rho$ of all real-valued measurable functions on [0,1] with the relative uniform convergence structure is complete.

PROOF. It was shown in Example 1 that $\mathcal{L} = K(X,\widetilde{\beta X})$.

We cite two examples to show that "relatively uniformly complete" and "2-universally complete" are independent concepts.

EXAMPLE 2. Let V be the space of continuous functions f on the real line such that the restriction of f to any compact set consists of finitely many line segments. Clearly, V is a function lattice containing 1. To see that V is 2-universally complete, let G be a 2-disjoint collection of functions in V and let K be a compact subset of $\mathbb{R}$. For each y in K there is a function f_y in G such that $|f_y|(y)>0$, since y is in the carrier space X of V. Thus, $|f_y|(z) > 0$ for all z in some neighborhood of y. Since finitely many such neighborhoods cover K there are finitely many functions $|f_y|$ whose supremum f is positive on K. It follows that $|g|\wedge f\neq 0$ for at most finitely many g in G; i.e., $g(K) = 0$ for all but finitely many g in G. Thus, since on any compact set the pointwise supremum of G is a supremum of finitely many functions, G has a supremum in V. However, any continuous function which vanishes outside the interval $(0,1)$ is a relative uniform limit of functions in V; thus V is not relatively uniformly complete.

EXAMPLE 3. We will call a function on the reals $\mathbb{R}$ "ultimately a polynomial" if it is continuous and is equal to a polynomial on the complement of some interval $[-n,n]$ and we let V be the solid hull in $C(\mathbb{R})$ of the set of functions which are ultimately polynomials. We will argue that the carrier space X of V is $\mathbb{R}$. Where $C^o(\mathbb{R})$ is the space of bounded continuous functions on $\mathbb{R}$,

$$C^o(\mathbb{R}) \subseteq V \subseteq C(\mathbb{R})$$

Since V separates $\mathbb{R}$ and $V \cap C^o(\mathbb{R})$ separates X, we can assume (examining the adjoint maps)

$$\beta\mathbb{R} \supseteq X \supseteq \mathbb{R}.$$

Letting f denote the extended real-valued function on $\beta\mathbb{R}$ whose restriction to $\mathbb{R}$ is $f(x) = x$, we will prove that f is infinite on $\beta\mathbb{R} \setminus \mathbb{R}$. If so, then $X = \mathbb{R}$ since f must be finite on X. Let p be a point in $\beta\mathbb{R} \setminus \mathbb{R}$ and $\{r_\alpha\}$ a net in $\mathbb{R}$ convergent to p. Then $\{f(r_\alpha)\}$ converges to $f(p)$. If $f(p)$ were real, we would conclude that $f(p) = p$ by uniqueness of the limit of $\{r_\alpha\}$ in $\beta\mathbb{R}$, a contradiction. We can now show that V is not 2-universally complete. Let $\{f_n\}_{n=-\infty}^{\infty}$ be a collection of continuous functions on $\mathbb{R}$ chosen so that $f_n(x)$ is zero outside the interval $(2n-3,2n+1)$ and equal to e^x on the interval $[2n-2,2n]$. Clearly, $\{f_n\}$ is a 2-disjoint collection in V with no supremum in V. On the other hand, if $\{v_\alpha\}$ is a relatively uniformly Cauchy net in V, there is a strictly positive function w in V such that for $\alpha,\beta \geq \gamma_n$ $(n=1,2,\ldots)$

$$|v_\alpha - v_\beta| \leq \frac{1}{n} w.$$

Clearly $\{v_\alpha\}$ is bounded in V, and since w is bounded on each compact set, $\{v_\alpha\}$ is Cauchy in $C_{co}(\mathbb{R})$. Thus $\{v_\alpha\}$ converges in $C_{co}(\mathbb{R})$ to some function f in $C(\mathbb{R})$. It follows that for all $\alpha \geq \gamma_n$

$$|v_\alpha - f| \leq \frac{1}{n} w.$$

Thus $|f|$ is bounded by the function $|v_{\gamma_1}| + w$ (which is in V) so that f is in V, and $\{v_\alpha\}$ converges relatively uniformly to f in V. Hence, V is relatively uniformly complete.

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