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RESEARCH NOTES

A CHARACTERIZATION OF PSEUDOCOMPACTNESS

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ABSTRACT. It is proved here that a completely regular Hausdorff space X is pseudocompact if and only if for any continuous function f from X to a pseudocompact space (or a compact space) Y, $f^*\phi$ is z-ultrafilter whenever ϕ is a z-ultrafilter on X.

KEY WORDS AND PHRASES. Pseudocompact, βX, z-filter, z-ultra function.

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1. INTRODUCTION.

For notations and basic results one is referred to [1]. We only consider here completely regular Hausdorff spaces.

Let f be continuous from X to Y. Let ϕ be a z-ultrafilter on X, then $f^*\phi$ denotes the z-filter $\{B \varepsilon Z(Y): f^{-1}(B)\varepsilon\phi\}$ on Y and is known to be prime. We further know that a prime z-filter is contained in a unique z-ultrafilter. Let $\Delta(f)\phi$ denote the z-ultrafilter containing $f^*\phi$. Thus we have a function $\Delta(f)$ from βX to βY sending ϕ to $\Delta(f)\phi$. The function f is called z-ultra if $f^*\phi = \Delta(f)\phi$ for every z-ultrafilter ϕ on X.

2. MAIN RESULTS

PROPOSITION. A continuous function f from X to Y is z-ultra if and only if for every zero-set B in Y, $\Delta(f)^{-1}(\overline{B}^{\beta Y}) = \overline{f^{-1}(B)}^{\beta X}$.

PROOF. Let f be z-ultra. Then, $\phi \in \Delta(f)^{-1}(\overline{B}^{\beta Y})$ if and only if $\Delta(f)\phi = f^{*}\phi \in \overline{B}^{\beta Y}$. But this is equivalent to $B \in f^{*}\phi$ or to $f^{-1}(B) \in \phi$, which happens if and only if $\phi \in \overline{f^{-1}(B)}^{\beta X}$.

Conversely, $B \in f^{*}\phi$ if and only if $\phi \in \overline{f^{-1}(B)}^{\beta X}$, i.e. $\Delta(f)\phi \in \overline{B}^{\beta Y}$, since $\overline{f^{-1}(B)}^{\beta X} = \Delta(f)^{-1}(\overline{B}^{\beta Y})$. But $\Delta(f)\phi \in \overline{B}^{\beta Y}$ is equivalent to saying that $B \in \Delta(f)\phi$. We see that $f^{*}\phi = \Delta(f)\phi$.

In order to prove the main theorem of the paper we need the following observations for pseudocompact spaces. If X is pseudocompact, then a subset of βX is a zero-set if and only if it is closure of a zero-set in X and conversely, a subset of X is a zero-set in X if and only if its closure is so in βX.

THEOREM. If a space X is pseudocompact then any continuous function f from X to any pseudocompact space Y is z-ultra. Conversely, if the inclusion of X in βX is z-ultra, then X is pseudocompact.

PROOF. Let B be a zero-set in Y. Since $\overline{B}^{\beta Y}$ is a zero-set in βY as Y is pseudocompact, $\Delta(f)^{-1}(\overline{B}^{\beta Y})$ is a zero-set in βX. Pseudocompactness of X implies that $\Delta(f)^{-1}(\overline{B}^{\beta Y}) = \overline{A}^{\beta X}$ for some zero-set A in X. We show that $A = f^{-1}(B)$. Since $\Delta(f)/X = f$, we observe that $\Delta(f)^{-1}(B) \cap X = f^{-1}(B)$. Clearly, $\Delta(f)^{-1}(\overline{B}^{\beta Y}) \cap X = \Delta(f)^{-1}(B) \cap X = f^{-1}(B)$. Next, $\Delta(f)^{-1}(\overline{B}^{\beta Y}) \cap X = \overline{A}^{\beta X} \cap X = A$.. Hence $f^{-1}(B) = A$, and we have f to be z-ultra.

Conversely, let i be the inclusion of X in βX. Since $\Delta(i)/X = i$, $\Delta(i)$ is the identity on βX. Let B be a nonempty zero-set in βX. Since i is z-ultra, from the above proposition we have that $B = \Delta(i)^{-1}(B) = \overline{i^{-1}(B)}^{\beta X} = \overline{B \cap X}^{\beta X}$ and [1,6I.1] shows that X is pseudocompact.

As an application of our theorem we prove the following well known theorem due to Glicksberg [2].

THEOREM. If X is pseudocompact and Y is compact, then X x Y is pseudocompact.

PROOF. Let f: X x Y $\to$ Z be a continuous function, Z some pseudocompact space. Consider a z-ultrafilter ϕ on X x Y. Let π_2: X x Y $\to$ Y denote the projection on the second coordinate. Since Y is compact and $\pi^*_2\phi$ is a z-filter, it is fixed as well. Let $y_o \in \bigcap \pi^*_2\phi$. Hence ϕ_1, the restriction of ϕ to the subspace X x $\{y_o\}$ is a z-ultrafilter on X x $\{y_o\}$. Let f_1 denote the restriction of f to the subspace X x $\{y_o\}$. Since X is pseudocompact, f_1 is z-ultra. Clearly, $f^*\phi \subseteq f^*_1\phi_1$. Next, let B $\in f^*_1\phi_1$. Hence $f_1^{-1}(B) \in \phi_1$. Since $f^{-1}(B)$ contains $f_1^{-1}(B)$, $f^{-1}(B)$ intersects every member of ϕ. Thus $f^{-1}(B) \in \phi$ as it is a z-ultrafilter. We get that B $\in f^*\phi$. Hence $f^*\phi = f^*_1\phi_1$ and it follows that f is z-ultra.

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