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A NOTE ON POWER INVARIANT RINGS

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ABSTRACT. Let R be a commutative ring with identity and $R^{((n))} = R[[X_1,\dots,X_n]]$ the power series ring in n independent indeterminates $X_1,\dots,X_n$ over R. R is called power invariant if whenever S is a ring such that $R[[X_1]] \cong S[[X_1]]$, then $R \cong S$. R is said to be forever-power-invariant if S is a ring and n is any positive integer such that $R^{((n))} \cong S^{((n))}$, then $R \cong S$. Let $I_c(R)$ denote the set of all $a \in R$ such that there is R - homomorphism σ: $R[[X]] \to R$ with $\sigma(X) = a$. Then $I_c(R)$ is an ideal of R. It is shown that if $I_c(R)$ is nil, R is forever-power-invariant.

KEY WORDS AND PHRASES. Power series ring, Power invariant ring, Forever-power-invariant, Ideal-adic topology.

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1. INTRODUCTION.

In this paper all rings are assumed to be commutative and to have identity elements. Throughout this paper the symbol ω and ω_o are used to denote the sets of positive and negative integers, respectively. Let $R^{((n))} = R[[X_1,\dots,X_n]]$ be the formal power series ring in n indeterminates $X_1,\dots,X_n$ over a ring R and let $\alpha_1,\dots,\alpha_n$ be elements of $R^{((n))}$. Let $(R^{((n))}, (\alpha_1,\dots,\alpha_n))$ denote the topological ring $R^{((n))}$ with the $(\alpha_1,\dots,\alpha_n)$ - adic topology where $(\alpha_1,\dots,\alpha_n)$ is the ideal of $R^{((n))}$ generated by $\alpha_1,\dots,\alpha_n$. It is well known that $(R^{((n))}, (\alpha_1,\dots,\alpha_n))$ is Hausdorff if and only if $\cap_{j\in\omega}(\alpha_1,\dots,\alpha_n)^j = (0)$. In this case, the topological ring $R^{((n))}$ is metrizable, and we say that $(R^{((n))}, (\alpha_1,\dots,\alpha_n))$ is complete if each Cauchy sequence of $R^{((n))}$ converges in $R^{((n))}$. Clearly, $(R^{((n))}, (X_1,\dots,X_n))$ is a complete Hausdorff space. If $\alpha\in R^{((n))}$, then α is uniquely expressible in the form $\Sigma_{j=0}^{\infty} \alpha_j$, where $\alpha_j \in R[X_1,\dots,X_n]$ for each $j\in\omega_0$ such that α_j is 0 or a homogenous polynomial (that is form) of degree j in $X_1,\dots,X_n$ over R. We call $\Sigma_{j=0}^{\infty} \alpha_j$ the homogenous decomposition of α, and for each $j\in\omega_0$, α_j is called the j-th homogenous component of α.

Coleman and Enochs [3] raised the following question: Can there be non-isomorphic rings R and S whose polynomial rings R[X] and S[X] are isomorphic? Hochster [8] answered the question in the affirmative. The analogous question about formal power series rings was raised by O'Malley [13]: If $R[[X]] \cong S[[X]]$, must $A \cong B$? Hermann [7] showed that there are non-isomorphic rings R and S whose formal power series ring R[[X]] and S[[X]] are isomorphic. Then what is necessary and sufficient conditions on a ring R in order that whenever S is a ring such that $R[[X]] \cong S[[X]]$, then $R \cong S$? Several authors [7,10,13] investigated sufficient conditions on R so that R should be power invariant, but we do not know the necessary conditions on R. The fact that rings with nilpotent Jacobson radical are power invariant is known in [10] and Hamann [7] proved that a ring R is power invariant, if J(R), the Jacobson radical of R, is nil. In this paper we impose more relaxed condition on J(R) so that R should be power invariant and forever-power-invariant. Let $I_c(R)$ denote the set of all $a\in R$ such that there is an R-homomorphism σ: $R[[X]] \to R$ with $\sigma(X) = a$. Then $I_c(R)$ is an ideal of R contained in J(R) and contains the nil-radical of R (by Theorem E, [4]). Then $I_c(R)$ may be properly contained in J(R) and it may properly contain the nil-radical of R. For example, if $A = {}^{Z}/_{(4)}[X]$, then $M = (2,X)$ is a maximal ideal of A. Let $R = A_M[[Y]]$, then the nil-radical of R is 2R and $I_c(R) = (2,Y)$ and $J(R) = (2,X,Y)$. Also it is easy to see that the nil-radical of A_M is (2) and $I_c(A_M) = (2)$ and $J(A_M) = (2,X)$. This shows that for some ring R, $I_c(R)$ is nil, but J(R) is not nil. It is well known that $J(R^{((n))}) = J(R) + \Sigma_{i=1}^{n} X_i R^{((n))}$. Analogously, the following relation was proved in [6]: $I_c(R^{((n))}) = I_c(R) + \Sigma_{i=1}^{n} X_i R^{((n))}$; therefore, for any ring R and any positive integer n, $I_c(R^{((n))})$ can not be nil.

2. SOME POWER INVARIANT RINGS.

Let $\alpha = \sum_{i=0}^{\infty} a_i X_i \in R[[X]]$. If $\bigcap_{n=1}^{\infty} (a_0{}^n) = (0)$ (or $\bigcap_{n=1}^{\infty} (\alpha^n) = (0)$) and R is complete with respect to the (a_0)-adic topology (or R[[X]] is complete with respect to the (α)-adic topology), then there is an R-endomorphism ϕ of R[[X]] such that $\phi(X) = \alpha$, ([14] and [15]).

The following theorem from [15] will be needed for our main results.

THEOREM 1. Let $\alpha = \sum_{i=0}^{\infty} a_i X^i \in R[[X]]$. Then there exists an R-automorphism ϕ of R[[X]] such that $\phi(X) = \alpha$ if and only if the following conditions are satisfied:

(1) $(R[[X]], (\alpha))$ is a complete Hausdorff space;

(2) a_1 is a unit of R.

The next theorem (Theorem 5.6, [5]) is the more generalized form of Theorem 1.

THEOREM 2. Let $\alpha_i = \sum_{j=0}^{\infty} \alpha_j^{(i)} \in R^{((n))}$ for i=1,...,n, be homogeneous decompositions of elements of $R^{((n))}$. There exists an R-automorphism ϕ of $R^{((n))}$ such that $\phi(X_i) = \alpha_i$ for each i if and only if the following conditions are satisfied:

(1) $(R^{((n))}, (\alpha_1,\ldots,\alpha_n))$ is a complete Hausdorff space;

(2) $R\,\alpha_1^{(1)} + \ldots + R\,\alpha_1^{(n)} = R\,X_1 + \ldots + R\,X_n$.

Moreover, if such an automorphism ϕ exists, then it is unique.

Also, we need the following proposition:

PROPOSITION 3. Let M be a unitary free R-module of finite rank n and let $\{x_i\}_{i=1}^{n}$ be a free basis for M. Let $M_n(R)$ denote the ring of n x n matrices over R, and let $z_1,\ldots,z_n$ be elements of M such that $z_i = \sum_{j=1}^{n} a_{ij}\, x_j$ for each i=1,...,n where $a_{ij} \in R$ for each i and j. Then the following conditions are equivalent:

(1) $R\,z_1 + \ldots + R\,z_n = R\,x_1 + \ldots + R\,x_n$

(2) det (A), the determinant of A, is a unit of R where $A = (a_{ij})$ is the n x n matrix.

(3) $\{z_i\}_{i=1}^{n}$ is a free basis for M.

The proof of the proposition is straightforward so we omit its proof.

Finally, we list the theorem from [4] which plays a particularly important role in this paper.

THEOREM 4. Let

$I_1 = \{a \in R|$ there exists an R-automorphism σ: $R[[X]] \to R[[X]]$ with $\sigma(x) = X + a$ and $I_2 = \{a \in R|$ there exists an R-homomorphism σ: $R[[X_1,\dots,X_n]] \to R[[Y_1,\dots,Y_m]]$ such that $\sigma(X_i) = a + f$ for some X_i and $f \in \Sigma_{j=1}^m Y_j\, R[[Y_1,\dots,Y_m]]\}$.

Then $I_c(R) = I_1 = I_2$.

Now we are ready for our first result.

THEOREM 5. If R is a ring such that $I_c(R)$ is nil, then R is power invariant.

PROOF. Suppose that $I_c(R)$ is nil. Let ϕ be an isomorphism of $R[[X]]$ onto $S[[X]]$. Then $\phi(R)[[\phi(X)]] = S[[X]]$; therefore, in order to show power invariance of R, it suffices to show that $R[[X]] = S[[Y]]$ implies $R \cong S$, where Y is an indeterminate over a ring S. Let $W = R[[X]] = S[[Y]]$ and let $Y = a_0 + XU$ and $X = b_0 + YV$ where $a_0 \in R$, $b_0 \in S$ and $U,V \in W$. Clearly $(W,(Y))$ is a complete Hausdorff space; therefore, there is an R-endomorphism σ of $R[[X]]$ such that $\sigma(X) = Y = a_0 + XU$. Then by Theorem 4, $a_0 \in I_c(R)$ and so a_0 is a nilpotent element of R. Let $a_0 = \sum_{i=0}^{\infty} c_i Y^i$ where $c_i \in S$ for each $i \in \omega_0$, then c_i is nilpotent for each $i \in \omega_0$ and we have

$$Y = \sum_{i=0}^{\infty} c_i Y^i + b_0U + YVU \qquad (1)$$

The Y coefficients in both sides of (1) yields $1 = c_1 + b_0u_1 + v_0u_0$ where u_0 and v_0 are constant terms of U and V considered as elements of $S[[Y]]$, respectively and u_1 is the Y coefficient of U considered as an element of $S[[Y]]$. Since X is an element of $J(R[[X]]) = J(W)$, $b_0 + YV$ is an element of $J(S[[Y]])$ and so b_0 is an element of $J(S)$. Recall that c_1 is a nilpotent element of S, then $c_1 + b_0u_1 \in J(S)$; therefore, $v_0u_0 = 1 - c_1 - b_0u_1$ is a unit of S. This forces U and V to be units of $W = S[[Y]]$. If we consider U as an element of $R[[X]]$ and let $U = \sum_{i=0}^{\infty} a_{i+1} X^i$, $a_{i+1} \in R$ for each $i \in \omega_0$, then the constant term a_1 is a unit of R. Then $Y = \sum_{i=0}^{\infty} a_i X^i$ where a_1, the X coefficient, is a unit of R, and $(W,(Y))$ is a complete Hausdorff Space. Then by Theorem 1, there exists an R-automorphism ψ of $R[[X]]$ which maps X onto $Y = a_0 + XU = \sum_{i=0}^{\infty} a_i X^i$.

Then $R \cong R[[X]]/(X) \cong W/(a_0 + XU) = W/(Y) \cong S$. This completes the proof.

Let $R[t]$ be the polynomial ring in an indeterminate t over a ring R, then $J(R[t])$ coincides with the nil-radical of $R[t]$; therefore, $I_c(R[t])$ is a nil ideal of $(R[t])$ and by Theorem 5, $R[t]$ is power invariant. Similarly, if $R[t_1,\ldots,t_n]$ is the polynomial ring in n indeterminates $t_1,\ldots,t_n$ over R, then $R[t_1,\ldots,t_n]$ is power invariant.

It is natural to raise the following question: For what kind of ring R, is R isomorphic to S whenever $R[[X_1,\ldots,X_n]]$ and $S[[X_1,\ldots,X_n]]$ are isomorphic for some positive integer n? To wit, we give the following definition.

DEFINITION. A ring R is said to be forever-power-invariant provided R is isomorphic to S whenever there is a ring S and a positive integer n such that $R[[X_1,\ldots,X_n]]$ and $S[[X_1,\ldots,X_n]]$ are isomorphic where $X_1,\ldots,X_n$ are independent indeterminates over R and S.

EXAMPLE. If R is a quasi-local ring then so is $R[[X_1,\ldots,X_n]]$for any positive integer n. Since any quasi-local ring is power invariant [7], $R[[X_1,\ldots,X_n]]$ is power invariant if R is a quasi-local ring. Then clearly every quasi-local ring is forever-power-invariant.

THEOREM 6. If R is a ring such that $I_c(R)$ is nil, then R is forever-power-invariant.

PROOF. Suppose that R is a ring such that $I_c(R)$ is nil. Let $W = R[[X_1,\ldots,X_n]] = S[[Y_1,\ldots,Y_n]]$. To prove this theorem, it suffices to show that R and S are isomorphic. Let $Y_i = a_0^{(i)} + X_1U_1^{(i)} + \ldots + X_nU_n^{(i)}$ and $X_i = b_0^{(i)} + Y_1V_1^{(i)} + \ldots + Y_nV_n^{(i)}$ for each $i = 1,\ldots,n$ where $U_k^{(i)}$ and $V_k^{(i)}$ are elements of W for each $i = 1,\ldots,n$ and $k = 1,\ldots,n$ and $a_0^{(i)} \in R$, $b_0^{(i)} \in S$ for each $i = 1,\ldots,n$. Since $(W,(Y_i))$ is a complete Hausdorff space, there is a R-homomorphism ϕ of $R[[X_1]]$ into $R[[X_1,\ldots,X_n]]$ such that $\phi(X_1) = (Y_i) = a_0^{(i)} + X_1U_1^{(i)} + \ldots + X_nU_n^{(i)}$. Then by Theorem 4, $a_0^{(i)} \in I_c(R)$ for each $i = 1,\ldots,n$ and so $a_0^{(i)}$ are nilpotent for each $i = 1,\ldots,n$. The relation defined between Y_is and X_is yields the following:

$$Y_i = a_0^{(i)} + \sum_{k=1}^{n} b_0^{(k)}U_k^{(i)} + \left(\sum_{k=1}^{n} V_1^{(k)}U_k^{(i)}\right) Y_1 + \ldots + \left(\sum_{k=1}^{n} V_i^{(k)}U_k^{(i)}\right) Y_i + \ldots + \left(\sum_{k=1}^{n} V_n^{(k)}U_k^{(i)}\right)Y_n. \quad (1)$$

Let $a_0^{(i)} = \sum_{k=0}^{\infty} C_k^{(i)}$ be a homogenous decomposition in $S[[Y_1,\ldots,Y_n]]$. Then since

$a_0^{(i)}$ is nilpotent, $C_k^{(i)}$ is nilpotent for each $k = 1,\ldots,n$. Let $C_1^{(i)} = c_{11}^{(i)}Y_1 + \ldots + c_{1n}^{(i)}Y_n$, then $c_{1j}^{(i)}$ is a nilpotent element of S for each $j = 1,\ldots,n$. Let $U_k^{(i)} = \sum_{j=0}^{\infty} U_{kj}^{(i)}$ and $V_k^{(i)} = \sum_{j=0}^{\infty} V_{kj}^{(i)}$ be homogeneous decompositions of elements $U_k^{(i)}$ and $V_k^{(i)}$ in $S[[Y_1,\ldots,Y_n]]$ and let $U_{k1}^{(i)} = u_{k11}Y_1 + \ldots + u_{k1n}Y_n$ and $V_{k1}^{(i)} = v_{k11}Y_1 + \ldots + v_{k1n}Y_n$. Then the Y_j coefficient of the right side of (1) is

$$c_{1j}^{(i)} + \sum_{k=1}^{n} b_0^{(k)} u_{k1j}^{(i)} + \sum_{k=1}^{n} V_{j0}^{(k)} U_{k0}^{(i)}$$

which is equal to 1 if $j = i$, otherwise, 0. Since $c_{1j}^{(i)}$ is nilpotent and $\sum_{k=1}^{n} b_0^{(k)} u_{k1j}^{(i)} \in J(S)$, $\sum_{k=1}^{\infty} V_{j0}^{(k)} U_{k0}^{(i)}$ is a unit of S if $i = j$ and it is in $J(S)$ if $i \neq j$. Let $A = (V_{j0}^{(k)})_{jk}$ and $B = (U_{k0}^{(i)})_{ki}$ be $n \times n$ matrices over S, then $AB = \left(\sum_{k=1}^{n} V_{j0}^{(k)} U_{k0}^{(i)} \right)_{ji}$ in which every diagonal entry is a unit of S and the rest of entries are elements of $J(S)$. So AB is invertible in $M_n(S)$; therefore, both A and B are invertible in $M_n(S)$. Clearly, $(W,(X_1,\ldots,X_n))$ is a complete Hausdorff space. Recall that the linear homogeneous component of $X_i = b_0^{(i)} + Y_1V_1^{(i)} + \ldots + Y_nV_n^{(i)}$ considered as an element of $S[[Y_1,\ldots,Y_n]]$ is $Y_1V_{10}^{(i)} + \ldots + Y_nV_{n0}^{(i)}$ for each $i = 1,\ldots,n$ and the $n \times n$ matrix $A = (V_{j0}^{(i)})_{ji}$ is invertible in $M_n(S)$. Then by Theorem 2 and Proposition 3, there is an S-automorphism ψ of $S[[Y_1,\ldots,Y_n]]$ such that $\psi(Y_i) = b_0^{(i)} + Y_1V_1^{(i)} + \ldots + Y_nV_n^{(i)}$ for each $i = 1,\ldots,n$. Then $S \cong S[[Y_1,\ldots,Y_n]]/(Y_1,\ldots,Y_n) \cong W/(\psi(Y_1),\ldots,\psi(Y_n)) = W/(X_1,\ldots,X_n) = R[[X_1,\ldots,X_n]]/(X_1,\ldots,X_n) \cong R$. This completes the proof.

CORROLLARY 7. If $R[t_1,\ldots,t_n]$ is the polynomial ring in indeterminates $t_1,\ldots,t_n$ over a ring R, then it is a forever-power-invariant.

It is easy to see that if R is a ring such that $R[[X_1,\ldots,X_n]]$ is power invariant for any positive integer n. Then R is forever-power-invariant. This raises the following open question: If R is a ring such that $I_c(R)$ is nil then for any positive integer, is $R[[X_1,\ldots,X_n]]$ power invariant?

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