Internat. J. Math. & Math. Sci.
Vol. 4 No. 3 (1981) 445-450

SEMI SEPARATION AXIOMS AND HYPERSPACES

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(Received April 21, 1980 and in revised form September 4, 1980)

ABSTRACT. In this paper examples are given to show that s-regular and s-normal are independent; that s-normal, and s-regular are not semi topological properties; and that (S(X),E(X)) need not be semi-T_1 even if (X,T) is compact, s-normal, s-regular, semi-T_2, and T_0. Also, it is shown that for each space (X,T), (S(X),E(X)), ($S(X_0)$,$E(X_0)$), and ($S(X_{S0})$,$E(X_{S0})$) are homeomorphic, where (X_0, $Q(X_0)$) is the T_0-identification space of (X,T) and (X_{S0},$Q(X_{S0})$) is the semi-T_0-identification space of (X,T), and that if (X,T) is s-regular and R_0, then (S(X), E(X)) is semi-T_2.

KEY WORDS AND PHRASES. *Semi open sets, semi topological properties, and hyperspaces.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. *54A10, 54B20.*

1. INTRODUCTION.

Semi open sets were first defined and investigated by Levine [1] in 1963.

DEFINITION 1.1. Let (X,T) be a space and let $A \subseteq X$. Then A is semi open, denoted by $A \in SO(X,T)$, iff there exists $U \in T$ such that $U \subset A \subset \overline{U}$.

Since 1963 semi open sets have been used to define and investigate many new topological properties. Maheshwari and Prasad [2], [3], and [4] generalized T_i, i = 0, 1, 2, regular, and normal to semi-T_i, i = 0, 1,2, s-regular, and s-normal, by replacing the word open in the definitions of T_i, i = 0, 1, 2, regular, and normal by semi open, respectively. Except for s-normal and s-regular, the relationships between these separation axioms have been determined. In this paper, the relationship between s-normal and s-regular is determined, and semi topological properties and hyperspaces are further investigated.

2. s-REGULAR, s-NORMAL, AND SEMI TOPOLOGICAL PROPERTIES.

Maheshwari and Prasad [4] gave an example showing that s-normal does not imply s-regular. That example can be combined with the following example to show that s-regular and s-normal are independent.

EXAMPLE 2.1 Let N denote the natural numbers, let T be the discrete topology on N, let e be the embedding map of (N,T) into $\pi\{I_f \mid f \in C^*(N,T)\}$, and let $(\beta N, W) = (\overline{e(N)}, e)$ denote the Stone-Čech compactification of (N,T). From Willard's book [5], $(\beta N, W)$ is extremely disconnected, e(N) is open in βN, and $B = \beta N - e(N)$ is infinite. For each $p \in N$ let $N_p = \{n \in N \mid n \leq p\}$. Since for each $p \in N$, there exists a function $f_p: N_p \to B \times W$ such that (1) if $i \in \{2,\ldots,p\}$, then f_i is an extension of f_{i-1}, (2) $x_i \in 0_i$ for all $i \in N_p$, (3) if $i, j \in N_p$, then $\overline{0_i} \cap \overline{0_j} \neq \emptyset$ iff i = j, and (4) $B - \bigcup_{i=1}^{p} \overline{0_i}$ is infinite, then there exists a sequence $\{(x_n, 0_n)\}_{n \in N} \subset B \times W$ such that $x_n \in 0_n$ for all $n \in N$ and $\overline{0_m} \cap \overline{0_n} \neq \emptyset$ iff m = n. Let $\{a_n\}_{n \in N}$ be a sequence such that $\{a_n \mid n \in N\} \cap \beta N = \emptyset$ and $a_n = a_m$ iff n = m, let $V = \{x_n \mid n \in N\} \cup \left(\bigcup_{n \in N}\{U_n = 0_n \cap e(N)\}\right) \subset \beta N$, let W_1 be the relative topology on V, and let $X = V \cup \{a_n \mid n \in N\}$. Since U_1 is countably infinite, then $U_1 = \{y_n\}_{n \in N}$, where $y_i = y_j$ iff i = j. For each $i \in N$, let $\mathcal{B}_i = \{0 \subset X - (\{x_n \mid n \in N\} \cup \{a_n \mid n \neq i\}) \mid 0 \cap U_n \in W_1$ for all $n \in N$, $x_n \in \overline{0 \cap U_n}$ except for finitely many $n \in N$, and $a_i, y_i \in 0\}$, and let $W_2 = \bigcup_{i \in N} \mathcal{B}_i$. Then $W_1 \cup W_2$ is a base for a topology S on X, (X,S) is s-regular, semi-T_2, and T_0, and (X,S) is not s-normal since $A = \{a_n \mid n \in N\}$ and $C = \{x_n \mid n \in N\}$ are disjoint closed sets and there do not exist disjoint semi open sets containing A and C, respectively.

Semihomeomorphisms and semi topological properties were first introduced and investigated by Crossley and Hildebrand [6].

DEFINITION 2.1. A 1-1 function from one space onto another space is a semihomeomorphism iff images of semi open sets are semi open and inverses of semi open sets are semi open. A property of topological spaces preserved by semihomeomorphisms is called a semi topological property.

Example 1.5 in [6], which was used to show that normal and regular are not semi topological properties, also shows that s-normal and s-regular are not semi topological properties.

Clearly, semi-T_i, $i = 0, 1, 2,$ are semi topological properties.

3. HYPERSPACES AND SEMI SEPARATION AXIOMS

DEFINITION 3.1. Let (X,T) be a topological space, let $A \subset X$, and define $S(X)$, $S(A)$, and $I(A)$ as follows: $S(X) = \{F \subset X \mid F$ is nonempty and closed$\}$, $S(A) = \{F \in S(X) \mid F \subset A\}$, and $I(A) = \{F \in S(X) \mid F \cap A \neq \emptyset\}$. Denote by $E(X)$ the smallest topology on $S(X)$ satisfying the conditions that if $G \in T$, then $S(G) \in E(X)$ and $I(G) \in E(X)$. Then $(S(X), E(X))$ is called a hyperspace [7].

Michael [8] showed that for a space (X,T), $\mathcal{B} = \{<G_1,\ldots,G_p> \mid p \in N$ and $G_i \in T$ for all $i \in N_p = \{1,\ldots,p\}\}$ is a base for $E(X)$, where N is the natural numbers and $<G_1,\ldots,G_p> = <G_i>_{i=1}^{p} = \{F \in S(X) \mid F \subset \bigcup_{i=1}^{p} G_i$ and $F \cap G_i \neq \emptyset$ for all $i \in N_p\}$, and observed that for each space (X,T), $(S(X), E(X))$ is T_0. Since T_0 implies semi-T_0, then for each space (X,T), $(S(X), E(X))$ is semi-T_0. The following example shows that $(S(X),E(X))$ need not be semi-T_1 even if (X,T) is compact, s-normal, s-regualr, semi-T_2, and T_0.

EXAMPLE 3.1 Let $X = \{a,b,c,d\}$ and $T = \{X,\phi,\{b\},\{d\},\{b,d\},\{a,b,d\},\{b,c,d\}\}$. Then $(S(X), E(X))$ is not semi-T_1 since $\{a,b,c\}, X \in S(X)$ such that $\{a,b,c\} \neq X$ and there does not exist a semi open set containing $\{a,b,c\}$ and not X.

In Willard's book [5], T_0-identification spaces are discussed.

DEFINITION 3.2 Let R be the equivalence relation on a space (X,T) defined by xRy iff $\overline{\{x\}} = \overline{\{y\}}$. Then the T_0-identification space of (X,T) is $(X_0, Q(X_0))$, where X_0 is the set of equivalence classes of R and $Q(X_0)$ is the decomposition topology on X_0, which is T_0.

This author [9] used T_0-identification spaces to show that hyperspaces of R_0 spaces, spaces which were first defined and investigated by Davis [10], are T_1.

DEFINITION 3.3. A space (X,T) is R_0 iff for each $0 \in T$ and $x \in 0$, $\overline{\{x\}} \subset 0$.

Since T_1 implies semi-T_1, then the hyperspace of each R_0 space is semi-T_1.

Semi open sets were used by Crossley and Hildebran [11] to define and investigate semi closed sets and semi closure.

DEFINITION 3.4. Let (X,T) be a space and let A, $B \subset X$. Then A is semi closed iff X-A is semi open and the semi closure of B, denoted by scl B, is the intersection of all semi closed sets containing B.

This author [12] used semi closure to define and investigate semi-T_0-identification spaces.

DEFINITION 3.5. Let R be the equivalence relation on a space (X,T) defined by xRy iff scl{x} = scl{y}. Then the semi-T_0-identification space of (X,T) is $(X_{S0}, Q(X_{S0}))$, where X_{S0} is the set of equivalence classes of R and $Q(X_{S0})$ is the decomposition topology on X_{S0}, which is semi-T_0.

This author [13] and [12] showed that the natural map P: $(X,T) \to (X_0, Q(X_0))$ is continuous, closed, open, onto, and $P^{-1}(P(0)) = 0$ for all $0 \in T$ and that the natural map P_S: $(X,T) \to (X_{S0}, Q(X_{S0}))$ is continuous, closed, open, onto, and $P_S^{-1}(P_S(0)) = 0$ for all $0 \in SO(X,T)$. These results are used to obtain the following result.

<u>THEOREM 3.1</u>. For a space (X,T),(S(X),E(X)), $(S(X_0),E(X_0))$, and $(S(X_{S0}),E(X_{S0}))$ are homeomorphic.

PROOF: Let f: $(S(X),E(X)) \to (S(X_0),E(X_0))$ and let f_S: $(S(X),E(X)) \to (S(X_{S0}),E(X_{S0}))$ defined by f(F) = P(F) and $f_S(F) = P_S(F)$. Then f and f_s are homeomorphisms.

<u>THEOREM 3.2</u>. If (X,T) is R_0, $G \in T$, and $F \in S(X)$ such that $F \cap \overline{G} \neq \emptyset$, then $S(\overline{G}) = \overline{S(G)}$ and $F \in \overline{I(G)}$.

PROOF: Since $S(G) \subset S(\overline{G})$, which is closed, then $\overline{S(G)} \subset S(\overline{G})$. Let $A \in S(\overline{G})$. Let $\langle B_i \rangle_{i=1}^{P} \in \mathcal{B}$ such that $A \in \langle B_i \rangle_{i=1}^{P}$. Then $A \subset \overline{G}$ and

$\emptyset \neq A \cap B_i \subset \overline{G} \cap B_i$ for all $i \in N_p$, which implies $G \cap B_i \neq \emptyset$ for all $i \in N_p$. For each $i \in N_p$ let $x_i \in G \cap B_i$. Then $\overline{\{x_i\}} \subset G \cap B_i$ for all $i \in N_p$ and $\bigcup_{i \in N_p} \overline{\{x_i\}} \in S(G) \cap \langle B_i \rangle_{i=1}^{p}$. Thus $A \in \overline{S(G)}$ and $S(\overline{G}) \subset \overline{S(G)}$, which implies $S(\overline{G}) = \overline{S(G)}$.

Let $\langle U_i \rangle_{i=1}^{m} \in \mathcal{B}$ such that $F \in \langle U_i \rangle_{i=1}^{m}$. Then $F \subset \bigcup_{i \in N_m} U_i \in T$ and $F \cap \overline{G} \neq \emptyset$, which implies $G \cap \left(\bigcup_{i \in N_m} U_i \right) \neq \emptyset$. Let $y \in G \cap \left(\bigcup_{i \in N_m} U_i \right)$ and for each $i \in N_m$ let $y_i \in B_i$. Then $\overline{\{y\}} \cup \left(\bigcup_{i \in N_m} \overline{\{y_i\}} \right) \in I(G) \cap \langle U_i \rangle_{i=1}^{m}$. Hence, $F \in \overline{I(G)}$.

THEOREM 3.3. If (X,T) is s-regular and R_0, then $(S(X),E(X))$ is semi-T_2.

PROOF: Let $A, B \in S(X)$ such that $A \neq B$. Then $A - B \neq \emptyset$ or $B - A \neq \emptyset$, say $B - A \neq \emptyset$. Let $x \in B - A$. Then there exists disjoint semi open sets O and W such that $x \in O$ and $A \subset W$. Let $U, V \in T$ such that $U \subset O \subset \overline{U}$ and $V \subset W \subset \overline{V}$. Then $I(U)$ and $S(V)$ are disjoint open sets, $B \in \overline{I(U)}$, and $A \in S(\overline{V}) = \overline{S(V)}$, which implies $S(V) \cup \{A\}$ and $I(U) \cup \{B\}$ are disjoint semi open sets.

Maheshwari and Prasad [4] showed that every s-normal R_0 space is s-regular. This result can be combined with Theorem 3.3 to obtain the following corollary.

COROLLARY 3.1. If (X,T) is s-normal and R_0, then $(S(X),E(X))$ is semi-T_2.

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