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NOTE ON A ROLE FOR ENTIRE FUNCTIONS OF THE CLASSES P AND P*

C.L. PRATHER
Department of Mathematics
Virginia Tech
Blacksburg, Virginia 24061 U.S.A.

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ABSTRACT. We use the B and B* operators of Levin on the Classes P and P* and a comparison principle to prove a Gauss-Lucas Theorem for differential operators. The connection with the determination of final sets for differential operators is then clarified.

KEY WORDS AND PHRASES. *Final set, differential operators, comparison principle, Gauss-Lucas Theorem.*

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1. INTRODUCTION.

The point of this note is to indicate another value of the classes P and P* of the entire functions as introduced by B. Ya. Levin (see e.g. Levin [1, Chap. 9] or Boas [2, Chap. 11]). As we shall see, they are the appropriate classes with which to obtain Gauss-Lucas and final set results for differential operators of the types B and B*. In writing this, the author is prompted in part by the remark of Efimov, Krein and Ostrowski in a recent tribute to Levin [3, pg. 142]: "Apparently the role of the class P in the theory of entire functions is not completely clarified". While the author has no claims as to what "the" role is, the comparison principle presented here has new applications, some of which have been recently worked out [4, 5, 6, 7].

First, we recall some definitions [2, Chap. 11].

Definition 1. A function ϕ is in the class P if it is entire of exponential type, has no zeros in $y < 0$, or satisfies the equivalent condition $h(-\pi/2) \geq h(\pi/2)$, where h is the indicator function of ϕ

Definition 2. A B-operator is an additive, homogeneous operator that transforms functions of exponential type into functions of exponential type and leaves the class P invariant.

The class of B-operators is fairly large [2], some examples are: $D = d/dz$; $D-\beta I$, for $\mathrm{Im}(\beta) \leq 0$; $P(D)$, where P is a polynomial with roots in the lower half plane $\mathrm{Im}(z) \leq 0$ and more generally $\phi(D) = \sum_k c_k D^k$, where $\bar{\phi}(z) = \sum_k \bar{c}_k z^k$ is in the class P; $\exp(-cD^2)\,\phi(D)$ with $c > 0$ and $\bar{\phi} \in P$; the operator $(\phi(D)f)(z) = \int_{z-i\lambda}^{z+i\lambda} f(t)dt$ for $\lambda > 0$.

Functions of the class P are characterized by their infinite product expansions [2, Theorem 7.8.3]: A function $\omega(z)$ of exponential type belongs to the class P if and only if it has the form $\omega(z) = Az^m \exp(cz)\prod_{n=1}^{\infty}(1 - z/z_n)\exp(z\mathrm{Re}(1/z_n))$, where $\mathrm{Im}(a_n) \geq 0$ and $2\,\mathrm{Im}(c) = h(-\pi/2) - h(\pi/2) \geq 0$, and h is the indicator function of ω.

Well known is the fact that Bernstein's theorem has been generalized in various directions using B-operators on the class P[1,2]. In particular, inequalities of Bernstein type have been demonstrated for the class of so called asymmetric entire functions as given in [9,10,11,12].

The new applications [4,6,7] of B-operators and the class P are based on an extension of an idea due to Pólya [13,14].

Definition 3. Let L be a differential operator and f a function analytic on a domain D. We say that a point z_0 lies in the final set $S = S(L,f)$ of f relative to L when, by considering the iterates $L^n f$, $n = 0, 1, 2, 3, \ldots$, $L^0 = I$, every neighborhood of z_0 contains zeros of infinitely many of the iterates $L^n f$.

Final sets for derivatives of meromorphic functions and certain restricted classes of entire functions have been given by Pólya [13,14] and Edrei [15,16].

Definition 4. a) Let $f(z)$ be a trigonometric sum $f(z) = \Sigma_k a_k \exp(\lambda_k z)$, where $\lambda_k \in C$, lie in a bounded set, and $\Sigma_k |a_k| < \infty$. Let H be the convex hull of the λ_k and suppose that H is either a line segment or is polygonal. We say that f is balanced when the following conditions hold:

Letting $\delta = \sup_k |\lambda_k|$, the circle $|z| = \delta$ contains at least two exponents $\lambda_{k_1},\ldots,\lambda_{k_p}$ and the coefficients corresponding to these exponents are all nonzero.

b) We say that the finite Fourier integral $f(z) = \int_{-\delta}^{\delta} e^{izt}\psi(t)dt$ is strongly balanced when $\psi(t)$ is bounded and measurable and $\psi(t) \to L_1, L_2$ as $t \to -\delta, +\delta$, respectively with $L_1 L_2 \neq 0$.

When $H = [-i\delta, +i\delta]$, for $\delta > 0$, the class of trigonometric sums described in a) coincides with the class $[\delta]$ as defined by Levin [1, Chapter 6] and accordingly its zeros lie in some strip $|y| < h$ [1, Chap. 6, Theorem 3]. As we shall see, not only is it true that the zeros of the successive derivatives lie in $|y| < h$ as discussed in [9,10], but the zeros of more general B-operators on these functions lie in this strip.

Final sets for the B-operator $\phi(D)$, where $D = d/dz$ and $\phi(z)$ is an entire function of genus ≤ 1 having only real zeros times the factor $\exp(-az^2)$, $a \geq 0$ were determined in [4,5,6] on both balanced exponential sums $\Sigma_k C_k \exp(i\lambda_k z)$, where the λ_k are real, and strongly balanced finite Fourier transforms. For the case of exponential sums, the exponents are allowed to accumulate to the endpoints $\pm i\delta$, say, provided they do so at a certain rate. The result is that the final set consists of a discrete set of points on the horizontal line $\mathrm{Im}(z) = -(2\delta)^{-1}\ln|C_{-\delta}/C_{\delta}|$ or the whole line, the conditions by which each occurs being given. A similar result is give for balanced exponential sums $\Sigma_k C_k \exp(\lambda_k z)$, $\lambda_k \in C$, for derivatives [6].

The fact that the functions we are considering are balanced is important when ϕ has real zeros, for Boas [8] has shown that for an asymmetric entire function f bounded on the real axis with $h_f(-\pi/2) = 0$, every half plane $y > a \geq 0$ contains zeros of infinitely many derivatives of f, making a final set result unlikely. However, as shown in [6], it is possible to obtain a nonempty final set with asymmetric exponential sums for operators $\phi(D)$, where ϕ has nonreal zeros. Another reason to start with balanced exponential sums is that the extremal functions for theorems of Bernstein type are balanced.

We also consider B*-operators on functions of the class P* [1, Chap. 9]. It is well known that a B*-operator is a B-operator and $P \subset P^*$. Moreover, f belongs to the class P* if and only if it is a uniform limit on compact sets of a sequence of polynomials whose zeros lie in the upper half plane [1, Chap. 8].

The elementary yet useful comparison principle [9,10] is the key to obtaining final sets for B*-operators.

COMPARISON PRINCIPLE: Let K be a closed subset of the complex plane C and M a complex linear space of meromorphic functions with poles in K. Let N be that part of M consisting of functions having no zeros outside K and L a linear operator from M to M. Then

a) the inequality $|f(z)| \le |g(z)|$, $z \in C/K$, $f,g \in M$ implies the inequality $|(Lf)(z)| \le |(Lg)(z)|$

if and only if

b) $L(N) \subset N$.

This principle is, in essence, a Gauss-Lucas Theorem for B* (and hence B) operators, simple cases of which were given by Genchev [9,10,17]. Wide applications have been made of Gauss-Lucas Theorems for derivatives; e.g., Marden [18] and Rubel [19]. Gauss-Lucas Theorems for operators are the point of Pólya's papers [20,21] the lengthy proofs of which occur in the paper of Benz [22]. However, the above principle yields a very simple proof of this type theorem. For example, we demonstrate the following:

THEOREM. Let $\phi(D)$ be a B*-operator, where $\phi(z)$ is an entire function of genus ≤ 1 with real zeros times the factor $\exp(-az^2)$, $a \ge 0$. Let f be an entire function of order < 2 with zeros in the strip $T = \{z: A \le Im(z) \le B\}$. Then $(\phi(D)f)(z)$ also has its zeros in T.

PROOF. If $A \ge 0$, then $f \in P^*$. If $A < 0$, then $f(z+Ai) \in P^*$, so it suffices to let $A \geq 0$. As $\phi(z)$ belongs to the class P*, $\phi(z)$ can be uniformly approximated on compact sets by polynomials $P_n(z)$ whose zeros lie on the real axis, by a characterization of the class P* stated earlier. Letting $L_n = P_n(D)$, L_n is a B*-operator. By taking M to be the linear span of the class P* and N the sub-class of M consisting of those functions in M whose zeros lie in T, since by definition B*-operators preserve the class P*, $L_n(N) \subset N$. By the comparison principle, $|f(z)| > 0$ for $z \in C/T$ (by hypothesis) implies $|(L_n f)(z)| > 0$. By Hurwitz's Theorem, $|(\phi(D)f)(z)| > 0$ for $z \in C/T$. Hence $\phi(D)f$ has all its zeros in T.

Using this idea, final sets have been determined for certain B*-operators on classes of entire functions of order < 2 representable by a Fourier integral having only real zeros [7]. One is lead to suspect that other final set results are obtainable for B*-operators by asymptotic methods. A study of the zeros of the successive iterates of multiplier sequence operators (which includes B-operators) on analytic functions has been submitted for publication.

Under not too restrictive assumptions related to the continuity of the operator, Levin [1, Chap. 9] obtains the general form of B and B*-operators.

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