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ON LINEAR ALGEBRAIC SEMIGROUPS III

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ABSTRACT. Using some results on linear algebraic groups, we show that every connected linear algebraic semigroup S contains a closed, connected diagonalizable subsemigroup T with zero such that E(T) intersects each regular $\mathcal{J}$-class of S. It is also shown that the lattice $(E(T),\leq)$ is isomorphic to the lattice of faces of a rational polytope in some $\mathbb{R}^n$. Using these results, it is shown that if S is any connected semigroup with lattice of regular $\mathcal{J}$-classes $\mathcal{U}(S)$, then all maximal chains in $\mathcal{U}(S)$ have the same length.

KEY WORDS AND PHRASES. *Linear algebraic semigroup, idempotent, polytope.*

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0. INTRODUCTION.

Throughout this paper, $\mathbb{R}$, Z, Q, $\mathbb{R}^+$, Z^+, Q^+ will denote the sets of reals, integers, rationals, positive reals, positive integers and positive rationals respectively. If X is a set then $|X|$ denotes the cardinality of X. If X is a subset of a semigroup, then $\langle X\rangle$ denotes the subsemigroup generated by X. If $(P,\leq)$ is a partially ordered set and $\{\alpha_1 < \alpha_2 < \dots < \alpha_n\}$ is a finite chain in P, then we define the length of the chain to be n - 1. K will denote a fixed algebraically closed field, $K^n = K \times \dots \times K$ the affine n-space. $\mathcal{M}_n(K)$ will denote the set of all n×n matrices and GL(n,K) the group of units of $\mathcal{M}_n(K)$. $\mathcal{F}(X_1,\dots,X_n)$ will denote the free commutative semigroup in the variables $X_1,\dots,X_n$ and $K[X_1,\dots,X_n]$

the free commutative algebra over K in the variables $X_1,\dots,X_n$. We use the notation of [6,7] for algebraic semigroups. Let S be an algebraic monoid with identity element 1 and group of units G. If $g \in G$, then the maps $x \to xg$, $x \to gx$, $x \to g^{-1}xg$ are all automorphisms of the variety S. The last one is also a semigroup automorphism. If we let $\tilde{G} = \{(a,b) \mid a,b \in S,\ ab = 1\}$, then $\tilde{G}$ becomes an algebraic group. Actually, with more general notions of varieties [5], G itself can be viewed as an algebraic group. By [6, Theorem 1.1], we can assume that S is a closed submonoid of some $M_n(K)$. Then clearly $G = GL(n,K) \cap S$ and $S \backslash G$ is closed. If S_1 is closed submonoid of S with group of units G_1, then $G_1 = G \cap S_1$. If H is a closed subgroup of G, then H is the group of units of $\overline{H}$. If S is connected, then clearly so is G, $\overline{G} = S$ and dim S = dim G. If S is not connected, then [7; Lemma 1.9], 1 lies in a unique irreducible component S_1 of S and S_1 is a closed connected submonoid of S. We say that S is trigonalizable if S is *-isomorphic to a closed semigroup of lower triangular matrices. If S is connected, then since $\overline{G} = S$, it follows from the Lie-Kolchin Theorem [5; Theorem 17.6] that S is trigonalizable if and only if G is solvable. S is a d-semigroup if S is *-isomorphic to a closed subsemigroup of $(K^p,\cdot)$ for some $p \in Z^+$. If S is connected, then since $\overline{G} = S$, we see that S is a d-semigroup if and only if G is a torus. By [7; Corollary 3.15], a connected d-semigroup with zero can be characterized as a connected Clifford semigroup with zero. If $X,Y \subseteq S$, then X is conjugate to Y if $g^{-1}Xg = Y$ for some $g \in G$.

1. CONNECTED SEMIGROUPS

LEMMA 1.1. Let S be a connected monoid, $e \in E(S)$, $e \neq 1$. Then there exists a closed connected submonoid S' of S such that 1, $e \in S'$ and e is the zero of S'.

PROOF. Let G denote the group of units of S and set $V = S \backslash G$. Then $V = V_1 \cup \dots \cup V_r$ where $V_1,\dots,V_r$ are closed and irreducible. Let $m_i = \dim V_i$, $i = 1,\dots,r$. Then $m_i \leq n-1$, where n = dim S. Let $\phi: S \to eS$ be given by $\phi(x) = ex$. Let $q = \dim eS$, $\phi_i = \phi|_{V_i}$, $W_i = \overline{\phi(V_i)} \subseteq eS$. Let $i \in \{1,\dots,r\}$. Suppose $W_i = eS$. Then ϕ_i is dominant. So by [5; Theorem 4.3], there exists a non-empty open set O_i of eS such that $O_i \subseteq \phi(V_i)$ and so that $\dim \phi_i^{-1}(x) = m_i - q < n - q$ for all $x \in O_i$. So

$$\dim(V_i \cap \phi^{-1}(x)) < n-q \text{ for all } x \in O_i \tag{1}$$

Next suppose $W_i \neq eS$. Then set $O_i = eS \setminus W_i$. Then

$$V_i \cap \phi^{-1}(x) = \emptyset \text{ for all } x \in O_i. \tag{2}$$

Let $O = O_1 \cap O_2 \cap \ldots \cap O_r$. Since eS is connected, $O \neq \emptyset$. Let $x \in O$. Then $x \in \phi^{-1}(x)$. Let D be an irreducible component of $\phi^{-1}(x)$ such that $x \in D$. Then [5; Theorem 4.1] $\dim D \geq n-q$. We claim that $D \not\subseteq V$. For suppose $D \subseteq V$. Then $D \subseteq V_i$ for some i. Since $x \in \phi^{-1}(x) \cap O_i \cap V_i$, (2) is ruled out. So by (1) $\dim D < n-q$, a contradiction. Hence $D \not\subseteq V$. So $D \cap G \neq \emptyset$. Let $g \in D \cap G$. So $\phi(g) = x$. Thus $eg = x$, $xg^{-1} = e$. Let $Y = Dg^{-1}$. Then Y is closed and irreducible. Let $y \in Y$. Then $yg \in D$. So $eyg = x$ and $ey = xg^{-1} = e$. Hence $ey = e$ for all $y \in Y$. Since $g \in D$, $1 = gg^{-1} \in Y$. Since $x \in D$, $e = xg^{-1} \in Y$. Let $S_1 = \{a \mid a \in S,\ ea = e\}$. Then S_1 is a closed submonoid of S and $Y \subseteq S_1$. Let S_2 be the (unique) irreducible component of 1 in S_1. Then $Y \subseteq S_2$ and S_2 is a closed connected submonoid of S. Thus $1, e \in S_2$ and $ea = e$ for $a \in S_2$. By the dual of the above argument, there exists a closed connected submonoid S_3 of S_2 such that $e \in S_3$ and $ae = e$ for all $a \in S_3$. So $ae = ea = e$ for all $a \in S_3$.

FACT 1.2. Let $\mathcal{A} \subseteq M_n(K)$ such that $AB = BA$ for all $A, B \in \mathcal{A}$. Suppose also that each $A \in \mathcal{A}$ is lower triangular and diagonalizable. Then there exists a lower triangular, invertible matrix P such that $P^{-1}AP$ is diagonal.

PROOF. We prove by induction on n. Let $\mathcal{A} = \{A_\alpha \mid \alpha \in \Omega\}$, $A_\alpha = \begin{bmatrix} a_\alpha & 0 \\ B_\alpha & C_\alpha \end{bmatrix}$. C_α is $(n-1) \times (n-1)$, $a_\alpha \in K$. Clearly $C_\alpha C_\beta = C_\beta C_\alpha$ for all α, β. Since

$$\text{minimum polynomial of } C_\alpha \mid \text{minimum polynomial of } A_\alpha,$$

minimum polynomial of C_α has no multiple roots. So each C_α is diagonalizable. So there exists, by induction, an invertible, lower triangular $(n-1) \times (n-1)$ matrix M_1 such that $M_1^{-1} C_\alpha M_1$ is diagonal for all α. Let $M = \begin{bmatrix} 1 & 0 \\ 0 & M_1 \end{bmatrix}$. Then

$$D_\alpha = M^{-1}A_\alpha M = \begin{bmatrix} a_\alpha & 0 \\ F_\alpha & G_\alpha \end{bmatrix}, \quad G_\alpha \text{ is } (n-1) \times (n-1) \text{ and diagonal.}$$

Let $E_\alpha = D_\alpha - a_\alpha I$, $\alpha \in \Omega$. Then each E_α is diagonalizable and $E_\alpha E_\beta = E_\beta E_\alpha$ (all α,β). Moreover,

$$E_\alpha = \begin{bmatrix} 0 & 0 & & & \\ b_2^{(\alpha)} & c_2^{(\alpha)} & & 0 & \\ \cdot & & \cdot & & \\ \cdot & & & \cdot & \\ \cdot & & 0 & & \cdot \\ b_n^{(\alpha)} & & & & c_n^{(\alpha)} \end{bmatrix}$$

Since $E_\alpha E_\beta = E_\beta E_\alpha$,

$$c_i^{(\alpha)} b_i^{(\beta)} = b_i^{(\alpha)} c_i^{(\beta)}, \quad i = 2,\ldots,n, \text{all } \alpha,\beta \tag{3}$$

Also, since E_α is diagonalizable,

$$c_i^{(\alpha)} = 0 \text{ implies } b_i^{(\alpha)} = 0, \text{ all } i,\alpha \tag{4}$$

Let $i \in \{2,\ldots,n\}$. If there exists α such that $c_i^{(\alpha)} \neq 0$, let $u_i = -b_i^{(\alpha)}/c_i^{(\alpha)}$. By (3), u_i is independent of the choice of α. If there is no such α, let $u_i = 0$.

Let $u_1 = 1$ and set $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$. By (4), $E_\alpha u = 0$ for all α. Let e_i be the column with 1 in i^{th} component and 0 elsewhere. Then $u, e_2,\ldots,e_n$ is a linearly independent set of eigenvectors of E_α for all $\alpha \in \Omega$. Let $R = [u, e_2,\ldots,e_n]$. Then R is lower triangular and invertible. Clearly $R^{-1}E_\alpha R$ is diagonal for all α. So $R^{-1}D_\alpha R$ is diagonal for all α. Let $P = MR$.

LEMMA 1.3. Let S be a connected monoid with identity element 1, zero e. Let G denote the group of units of S. Suppose G is solvable. Then for any maximal torus T of G, $e \in \overline{T}$.

PROOF. We can assume that S is a closed submonoid of $\mathcal{M}_n(K)$. By the Lie-Kolchin theorem [5; Theorem 17.6] there exists $P \in GL(n,K)$ such that $P^{-1}GP$ is lower triangular. Since $\overline{G} = S$, $P^{-1}SP$ is lower triangular. So we can assume that S is lower triangular. Let T be a maximal torus of G and set $X = T \cup \{e\}$. Then X satisfies the hypothesis of Fact 1.2. So there exists a lower triangular $R \in GL(n,K)$ such that $R^{-1}XR$ is diagonal. Clearly $R^{-1}SR$ remains lower triangular. So we can assume that X is diagonal. If $a \in S$, then let $\phi(a)$ be the $n \times n$ diagonal matrix, with the diagonal being that of a. Then $\phi(X) = X$. Clearly ϕ is a $*$-homomorphism of S into $M_n(K)$ and $\phi(G)$ is a torus. By [5; Corollary 21.3C], $\phi(G) = \phi(T) = T$. So $\phi(G) \subseteq S$. Since $\overline{G} = S$, $\phi(S) \subseteq S$. Let $W = \phi(S)$. Then $W = \{a \mid a \in S,\ \phi(a) = a\}$ is closed. Since $\phi(S) = W$, W is a closed connected submonoid of S. Let H denote the group of units of W. Then $T \subseteq H \subseteq G$ and H is a torus. So $T = H$ and $W = \overline{T}$. Clearly $e = \phi(e) \in W = \overline{T}$.

THEOREM 1.4. Let S be a connected monoid with group of units G. Let B be a Borel subgroup of G. Then $S = \bigcup_{x \in G} x\overline{B}x^{-1}$.

PROOF. We can assume that S is a closed submonoid of $W = M_n(K)$. Let $G_1 = \{(a,a^{-1}) \mid a \in G\}$ Then G_1 is a closed subset of $W \times W$. If $(a,b),(c,d) \in G_1$, then define $(a,b)(c,d) = (ac,db)$. Then G_1 is an algebraic group $*$-isomorphic to G. Let $B_1 = \{(a,a^{-1}) \mid a \in B\}$. Then B_1 is a Borel subgroup of G_1. Now [5; Theorem 21.3], G_1/B_1 is a projective variety. Let $\phi: G_1 \to G_1/B_1$ be the natural projection $\phi(a) = aB_1$. Let $V = W \times G_1$, $Y = W \times G_1/B_1$. By [1; Theorem 6.8], G_1/B_1 is smooth and hence a normal variety. The same is true for W. So $Y = W \times G_1/B_1$ is normal [1; p. 77]. Let $\psi: V \to Y$ be given by $\psi(a,b) = (a,\phi(b))$. Then ψ is a surjective morphism. Clearly each fibre of ψ is irreducible and has dimension equal to that of B_1. So [1; Proposition 18.4], ψ is an open map. Let $X = \{(a,g,g^{-1}) \mid a \in S,\ g \in G, g^{-1}ag \in \overline{B}\}$. Then X is closed in V. So $\psi(\sim X)$ is open in Y. Hence $\sim\psi(\sim X)$ is closed in Y. Clearly $\sim\psi(\sim X) \subseteq \psi(X)$. Suppose $\omega \in \psi(X)$, $\omega \in \psi(\sim X)$. Then $\omega = \psi(x) = \psi(y)$ for some $x \in X$, $y \in \sim X$. So $x = (a,g,g^{-1})$, $y = (a,h,h^{-1})$ for some $a \in S$, g, $h \in G$. Now $g^{-1}ag \in \overline{B}$. Since $\psi(x) = \psi(y)$, $\phi(g,g^{-1}) = \phi(h,h^{-1})$. So $gB = hB$. Thus $h = gb$ for some $b \in B$. So $h^{-1}ah = b^{-1}(g^{-1}ag)b \in b^{-1}\overline{B}b = \overline{B}$, a contradiction. So $\sim\psi(\sim X) = \psi(X)$ and $\psi(X)$ is closed. Let $\theta: Y = W \times G_1/B_1 \to W$ denote the projection of Y onto W. Then since G_1/B_1 is projective, θ is a closed map [5; Theorem 6.2]. Hence $\theta(\psi(X))$ is closed in W. Clearly $\theta(\psi(X)) = \bigcup_{g\in G} g\overline{B}g^{-1} \subseteq S$. By [5; Theorem 22.2], $G \subseteq \theta(\psi(X))$. Since $\overline{G} = S, \theta(\psi(X)) = S$. This proves the theorem.

COROLLARY 1.5. Let S be a connected monoid with zero e and T a maximal torus in the group of units G of S. Then $e \in \overline{T}$.

PROOF. Now $T \subseteq B$ for some Borel subgroup B of G. By Theorem 1.4, $e \in x\overline{B}x^{-1}$ for some $x \in G$. So $e = x^{-1}ex \in \overline{B}$. Hence e is the zero of $\overline{B}$. By Lemma 1.3, $e \in \overline{T}$.

COROLLARY 1.6. Let S be a connected monoid with group of units G. Let $e_1,\ldots,e_k \in E(S)$ such that $e_1 > e_2 > \ldots > e_k$. Then there exists a maximal torus T of G such that $e_1,\ldots,e_k \in \overline{T}$.

PROOF. We prove by induction on k. If k = 1, we are done by Lemma 1.1 and Corollary 1.5. So assume k > 1. By Lemma 1.1, there exist closed connected submonoids $S_1,\ldots,S_k$ of S such that e_i is the zero of S_i. Then $ae_k = e_ka = e_k$ for all $a \in S_i$, $i = 1,\ldots,k$. Let $V = \{a | a \in S, ae_k = e_ka = e_k\}$. Then V is a closed submonoid of S and $S_1,\ldots,S_k \subseteq V$. Let W be the (unique) irreducible component of 1 in V. Then $S_1,\ldots,S_k \subseteq W$ and W is a closed connected submonoid of S. So e_k is the zero of S. Let G_1 denote the group of units of W. By our induction hypothesis, there exists a maximal torus T_1 and G_1 such that $e_1,\ldots,e_{k-1} \in \overline{T}_1$. By Corollary 1.5, $e_k \in \overline{T}_1$. Let $T_1 \subseteq T$ where T is a maximal torus of G. Then $e_1,\ldots,e_k \in \overline{T}$.

By [7; Lemma 1.3] we have,

LEMMA 1.7. Let S be a semigroup, $J_1,\ldots,J_k \in \mathcal{U}(S)$, $J_1 > J_2 > \ldots > J_k$. Then there exists $e_1,\ldots,e_k \in E(S)$ such that $e_i \in J_i$, $i = 1,\ldots,k$ and $e_1 > e_2 > \ldots > e_k$.

THEOREM 1.8. Let S be a connected monoid with group of units G. Then

(1) All maximal closed connected d-submonoids of S are conjugate.

(2) All maximal closed connected d-submonoids with zeroes, of S, are conjugate.

(3) Let Y be a maximal closed connected d-submonoid with zero, of S. Then $\bigcup_{g\in G} gE(Y)g^{-1} = E(S)$. In particular $E(Y) \cap J \neq \emptyset$ for all $J \in \mathcal{U}(S)$. More-

PROOF. Since the group of units of a maximal closed connected d-submonoid of S is a maximal torus in G, (1) follows from [5; Corollary 21.3A]

(2) Let S_1, S_2 be two maximal closed connected d-submonoids with zeroes of S. Let e_i be the zero of S_i, $i = 1, 2$. Let H_i be the group of units of S_i, $i = 1, 2$. Then $H_i \subseteq T_i$, T_i a maximal torus of G, $i = 1, 2$. Let $V_i = \overline{T}_i$, $i = 1, 2$. Let f_i be the minimum idempotent of V_i. Then $e_i \geq f_i$. Let $W_i = \{a \mid a \in V_i,\ af_i = f_i\}$, U_i the (unique) irreducible component of 1 in W_i. Since V_1, V_2 are conjugate by (1), so are W_1, W_2. Hence U_1, U_2 are conjugate. Since $S_i \subseteq W_i$, $S_i \subseteq U_i$, $i = 1, 2$. By Lemma 1.1, $f_i \in U_i$, $i = 1, 2$. By the maximality of S_i, $S_i = U_i$, $i = 1, 2$.

(3) Let $e \in E(S)$. By Corollary 1.6, $e \in S_1$ for some closed connected d-submonoid S_1 of S. By [7; Theorem 3.16], there exists a closed connected d-submonoid with zero, S_2 of S_1 such that $e \in S_2$. By (2) $xS_2x^{-1} \subseteq Y$ for some $x \in G$. So $xex^{-1} \in Y$. Hence $E(Y) \cap J_e \neq \emptyset$. Next let $J_1 > J_2 > \dots > J_k$ be a maximal chain in $\mathcal{U}(S)$. By Lemma 1.7, there exist $e_i \in E(J_i)$ such that $e_1 > e_2 > \dots > e_k$. Clearly

$$e_1 > e_2 > \dots > e_k \tag{5}$$

is a maximal chain in E(S). For if $e_i > f > e_{i+1}$, $f \in E(S)$, then $J_{e_i} > J_f > J_{e_{i+1}}$, a contradiction. By Corollary 1.6, $e_1, \dots, e_k \in M_1$ for some closed connected d-submonoid M_1 of S. By [7; Theorem 3.16], $e_1, \dots, e_k \in M_2$ for some closed connected d-submonoid with zero, M_2 of M_1. So $e_1, \dots, e_k \in M_3$ for some maximal connected d-submonoid with zero, M_3 of S. Since (5) is maximal in E(S), it is maximal in $E(M_3)$. By [7; Theorem 3.17], $\dim M_3 = k - 1$. By (2) $\dim M_3 = \dim Y$.

THEOREM 1.9. Let S be a connected semigroup. Then all maximal chains in $\mathcal{U}(S)$ have the same length.

PROOF. $\mathcal{U}(S)$ has maximum element J_0. Let $e \in E(J_0)$. By [7; Lemma 1.3, 1.7]. $\mathcal{U}(eSe) = \{J \cap eSe \mid J \in \mathcal{U}(S)\} \cong \mathcal{U}(S)$. Now eSe is a connected monoid. We are done by Theorem 1.8 (3).

THEOREM 1.10. Let S be a connected monoid such that for all $a, b \in S$, $a \mid b$

implies $a^2|b^i$ for some $i \in Z^+$. Let Y be a maximal closed connected d-submonoid with zero, of S. Then $J \cap Y$ is a subgroup of Y for all $J \in \mathcal{U}(S)$. In particular $\mathcal{U}(Y) = \{J \cap Y | J \in \mathcal{U}(S)\}$ and $(\mathcal{U}(S), \leq) \cong (E(Y), \leq)$.

PROOF. The hypothesis implies by [8] that J is a subsemigroup of S for all $J \in \mathcal{U}(S)$. Let $J \in \mathcal{U}(S)$. Then $J \cap Y \neq \emptyset$ by Theorem 1.8. Let a, $b \in J \cap Y$. Then $a\mathcal{H}e$, $b\mathcal{H}f$ in Y for some e, $f \in E(Y)$. So e, $f \in J$. Since e, $f \in Y$, $ef = fe$. Since J is completely simple, $e = f$. So $a\mathcal{H}b$ in Y and $J \cap Y$ is a subgroup of Y.

Now applying the proof of Theorem 1.9 we have,

COROLLARY 1.11. Let S be a connected semigroup such that for all a, $b \in S$, $a|b$ implies $a^2|b^i$ for some $i \in Z^+$. Then $(\mathcal{U}(S), \leq) \cong (E(Y), \leq)$ for some connected d-monoid with zero, Y.

THEOREM 1.12. Let S be a connected monoid such that the group of units G of S is nilpotent. Then E(S) is finite.

PROOF. By [5; Proposition 19.2], G has a unique maximal torus T. So $\overline{T}$ is the unique maximal closed connected d-submonoid of S. By Theorem 1.8, $E(S) \subseteq \overline{T}$. So E(S) is finite.

EXAMPLE 1.13. $S = \left\{ \begin{bmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{bmatrix} \middle| a, b, c, d \in K \right\}$ is an example of a connected monoid satisfying the hypothesis of Theorem 1.12.

CONJECTURE 1.14. Let S be a connected monoid with zero such that E(S) is finite. Then the group of units of S is solvable.

EXAMPLE 1.15. Let $S = \left\{ \begin{bmatrix} a & b \\ 0 & a^2 \end{bmatrix} \middle| a, b \in S \right\}$. Then S is a connected monoid with zero and $|E(S)| = 2$. The group of units of S is solvable but not nilpotent.

2. CONNECTED d-SEMIGROUPS WITH ZEROS

Let S be a connected d-semigroup with zero, dim S > 0. Then S is a monoid [7; Theorem 2.7]. By a character of S, we mean a *-homomorphism $\chi: S \to K$ such that $\chi(1) = 1$, $\chi(0) = 0$. We let $\Phi(S)$ denote the commutative semigroup of all characters of S with pointwise multiplication. It is clear that if S_1, S_2 are connected d-semigroups with zeros, $\dim S_i > 0$, then S_1 *-isomorphic to S_2 implies $\Phi(S_1) \cong \Phi(S_2)$.

A commutative semigroup W is said to be <u>totally cancellative</u> if W is cancellative and for all $a, b \in W$, $n \in Z^+$, $a^n = b^n$ implies $a = b$. We will need the following result of Grillet [3; Theorem 2.2].

THEOREM A [Grillet]. Let W be a finitely generated commutative semigroup. Then W can be embedded in a free commutative semigroup if and only if W is totally cancellative and idempotent-free.

LEMMA 2.1. Let S be a connected d-semigroup with zero 0 and identity 1, $\dim S > 0$. Then

(1) $\Phi(S) \neq \emptyset$.

(2) If $e \in E(S)$, $e \neq 0$, then there exists $\chi \in \Phi(S)$ such that for all $g \in E(S)$, $g \geq e$ implies $\chi(g) = 1$, $g \not\geq e$ implies $\chi(g) = 0$.

(3) $\Phi(S)$ is idempotent-free and totally cancellative.

(4) $\Phi(S)$ is linearly independent in the vector space of all functions from S into K.

PROOF. Let G denote the group of units of S. We can assume that S is a closed submonoid of $M_n(K)$ for some $n \in Z^+$. If $a \in S$, let $\alpha(a) = \det a$. Then $\alpha \in \Phi(S)$, $\alpha(f) = 0$ for $f \in E(S)$, $f \neq 1$. So $\Phi(S) \neq \emptyset$. Let $e \in E(S)$, $e \neq 0$. Then by the above, there exists $\beta \in \Phi(eS)$ such that $\beta(f) = 0$ for all $f \in E(eS)$ with $f \neq e$. Define $\chi: S \to K$ as $\chi(a) = \beta(ea)$. This proves (2).

Let $\chi \in \Phi(S)$ such that $\chi^2 = \chi$. Then $\chi(S) = \{1,0\}$ contradicting the fact that S is connected. So $\Phi(S)$ is idempotent-free. Let $\chi_1, \chi_2, \chi_3 \in \Phi(S)$ such that $\chi_1\chi_2 = \chi_1\chi_3$. If $a \in G$, then $\chi_1(a) \neq 0$ and so $\chi_2(a) = \chi_3(a)$. So $\chi_2 = \chi_3$ on G. Since $\overline{G} = S$, $\chi_2 = \chi_3$ on S. So $\Phi(S)$ is cancellative. Now let $\chi_1, \chi_2 \in \Phi(S)$, $m \in Z^+$ such that $\chi_1^m = \chi_2^m$. Let $Y = \{\xi \mid \xi \in K, \xi^m = 1\}$. Then Y is finite. If $\xi \in Y$, let $S_\xi = \{a \mid a \in S, \chi_1(a) = \xi\chi_2(a)\}$. If $a \in G$, then $\chi_i(a) \neq 0$ for $i = 1, 2$. So $a \in S_\xi$ for some $\xi \in Y$. Thus $G \subseteq \bigcup_{\xi \in Y} S_\xi$. Since $\overline{G} = S$, $S = \bigcup_{\xi \in Y} S_\xi$. Since S is connected, $S = S_\xi$ for some $\xi \in Y$. In particular, $1 = \chi_1(1) = \xi\chi_2(1) = \xi$. So $\chi_1 = \chi_2$ and $\Phi(S)$ is totally cancellative. This proves (3).

Now let $\chi_1, \ldots, \chi_m \in \Phi(S)$ be distinct characters of S which are linearly dependent. Let ψ_i denote the restriction of χ_i to G. Then ψ_i, $i = 1,\ldots,m$ are linearly dependent homomorphism of G. So [5; Lemma 16.1], $\psi_i = \psi_j$ for some $i \neq j$. Since $\overline{G} = S$, $\chi_i = \chi_j$, contradiction. This proves the lemma.

LEMMA 2.2. Let S be a connected d-semigroup with zero, dim $S > 0$. Then

(1) S is *-isomorphic to a closed submonoid S' of $(K^n, \cdot)$ for some $n \in Z^+$ such that $0 = (0,\ldots,0) \in S'$.

(2) $\Phi(S)$ is finitely generated.

(3) If a, b $\in$ S, s $\neq$ b, then there exists $\chi \in \Phi(S)$ such that $\chi(a) \neq \chi(b)$.

PROOF. First we prove (1). We can assume that S is closed subsemigroup of $(K^n, \cdot)$ for some $n \in Z^+$, n minimal. Let e denote the zero of S and set $S_1 = \{a-e \mid a \in S\}$. Then $a \longleftrightarrow a-e$ represents a *-isomorphism between S and S_1, $0 = (0,\ldots,0)$ is the zero of S_1. So without loss of generality we can assume that $e = (0,\ldots,0)$. Let $f = (\alpha_1,\ldots,\alpha_n)$ denote the identity of S. So $\alpha_i^2 = \alpha_i$, $i = 1,\ldots,n$. Suppose some $\alpha_i = 0$, say $i = 1$. Then $n > 1$ and $S \subseteq \{0\} \times K^{n-1}$. So S is *-isomorphic to a closed subsemigroup of $(K^{n-1}, \cdot)$, contradicting the minimality of n. So $\alpha_i \neq 0$, $i = 1,\ldots,n$. So $\alpha_i = 1$, $i = 1,\ldots,n$ and S is a closed submonoid of $(K^n, \cdot)$. This proves (1).

Let S be a closed submonoid of $(K^n, \cdot)$ with identity $1 = (1,\ldots,1)$, zero $0 = (0,\ldots,0)$. Let χ_i denote the i^{th} projection of S into K. Then clearly $\chi_1,\ldots,\chi_n \in \Phi(S)$. Let $\chi \in \Phi(S)$. Since $\chi(0) = 0$, χ does not have a constant term. So there exist $\omega_1,\ldots,\omega_t \in F(X_1,\ldots,X_n)$, $\alpha_1,\ldots,\alpha_n \in K$ such that

$$\chi(a) = \sum_{i=1}^{t} \alpha_i \omega_i(a) \text{ for all } a \in S.$$

So $\chi = \sum_{i=1}^{t} \alpha_i \omega_i(\chi_1,\ldots,\chi_n)$. By Lemma 2.1(4), $\chi = \omega_i(\chi_1,\ldots,\chi_n)$ for some i. So $\Phi(S) = \langle \chi_1,\ldots,\chi_n \rangle$. This proves (2). Let a, b $\in$ S such that $\chi(a) = \chi(b)$ for all $\chi \in \Phi(S)$. Then $a = (\chi_1(a),\ldots,\chi_n(a)) = (\chi_1(b),\ldots,\chi_n(b)) = b$. This proves (3).

LEMMA 2.3. Let S be a closed connected submonoid of $(K^n, \cdot)$ with zero $0 = (0,\ldots,0)$. Then there exist $u_1,\ldots,u_t, v_1,\ldots,v_t \in F(X_1,\ldots,X_n)$ such that for

$a \in K^n$, $a \in S$ if and only if $u_i(a) = v_i(a)$, $i = 1,\dots,t$.

PROOF. Let χ_i denote the i^{th} projection of S into K. Then $\chi_1,\dots,\chi_n \in \Phi(S)$. Let $I = \{f \mid f \in K[X_1,\dots,X_n],\ f(S) = 0\}$. Let $D = \{f \mid f \in I,\ f = u - v$ for some u, $v \in F(X_1,\dots,X_n)\}$. We claim that $I = (D)$. Suppose not. Then there exists $f \in I$, $f \notin (D)$. Since $f(0) = 0$, there exist $\omega_1,\dots,\omega_r \in F(X_1,\dots,X_n)$, $\alpha_1,\dots,\alpha_r \in K \setminus \{0\}$ such that $f = \sum_{i=1}^{r} \alpha_i\omega_i$. Of all such f choose one with r minimal. So $\sum_{i=1}^{r} \alpha_i\omega_i(a) = f(a) = 0$ for all $a \in S$. So

$$\sum_{i=1}^{r} \alpha_i\omega_i(\chi_1,\dots,\chi_n) = 0$$

By Lemma 2.1(4), $\omega_p(\chi_1,\dots,\chi_n) = \omega_q(\chi_1,\dots,\chi_n)$ for some $p, q \in \{1,\dots,r\}$, $p \neq q$. Assume $p = 1$, $q = 2$. So $\omega_1(a) = \omega_2(a)$ for all $a \in S$. Thus $\omega_1 - \omega_2 \in D$. Now

$$(\alpha_1 + \alpha_2)\omega_2 + \sum_{i=3}^{r} \alpha_i\omega_i = f - \alpha_1(\omega_1 - \omega_2) \in I$$

By minimality of r, $f - \alpha_1(\omega_1 - \omega_2) \in (D)$. So $f \in (D)$, a contradiction. Thus $I = (D)$. By the Hilbert Basis Theorem there exist $f_1,\dots,f_t \in I$ such that $I = (f_1,\dots,f_t)$. Since $I = (D)$, $f_1,\dots,f_t \in (D_1)$ for some finite subset D_1 of D. So $(D_1) = I$. This proves the lemma.

Let S_1, S_2 be connected d-semigroups with zeros, $\dim S_i > 0$, $i = 1, 2$. Let $\phi: S_1 \to S_2$ be a *-homomorphism such that $\phi(1) = 1$, $\phi(0) = 0$. Then define $\phi^*: \Phi(S_2) \to \Phi(S_1)$ by $\phi^*(\chi) = \chi \circ \phi$. Next assume that $\psi: \Phi(S_2) \to \Phi(S_1)$ is a homomorphism. Then we claim:

$$\text{for all } a \in S_1, \text{ there exists unique } b \in S_2 \text{ such that} \quad \chi(b) = (\psi(\chi))(a) \text{ for all } \chi \in \Phi(S_2) \tag{6}$$

Assume (6). Then we can define $\bar{\psi}: S_1 \to S_2$ as $\bar{\psi}(a) = b$. Then

$$\chi(\bar{\psi}(a)) = (\psi(\chi))(a) \text{ for all } a \in S_1,\ \chi \in \Phi(S_2) \tag{7}$$

Next we claim,

$$\bar{\psi} \text{ is a *-homomorphism}, \quad \bar{\psi}(1) = 1,\ \bar{\psi}(0) = 0 \tag{8}$$

We now prove (6), (8). Note that the uniqueness of b in (6) follows from Lemma 2.2(3). By Lemma 2.2(1) we can assume that S_2 is a closed submonoid of $(K^n, \cdot)$

with zero $0 = (0,\dots,0)$. Let χ_i denote the i^{th} projection of S_2 into K. Then by Lemma 2.2, $\chi_1,\dots,\chi_n \in \Phi(S_2)$ and $\Phi(S_2) = < \chi_1,\dots,\chi_n >$. By Lemma 2.3, there exist $u_1,\dots,u_t$, $v_1,\dots,v_t \in F(X_1,\dots,X_n)$ such that for $b \in K^n$, $b \in S_2$ if and only if $u_i(b) = v_i(b)$, $i = 1,\dots,t$. So $u_i(\chi_1,\dots,\chi_n) = v_i(\chi_1,\dots,\chi_n)$, $i=1,\dots,t$. Hence $u_i(\psi(\chi_1),\dots, \psi(\chi_n)) = v_i(\psi(\chi_1),\dots,\psi(\chi_n))$, $i = 1,\dots,t$. Thus $u_i((\psi(\chi_1))(a),\dots,(\psi(\chi_n))(a)) = v_i((\psi(\chi_1))(a),\dots,(\psi(\chi_n))(a))$ for all $a \in S_1$, $i=1,\dots,t$. So $((\psi(\chi_1))(a),\dots,(\psi(\chi_n))(a)) \in S_2$ for all $a \in S_1$. Define $\overline{\psi}:S_1 \to S_2$ as $\overline{\psi}(a) = ((\psi(\chi_1))(a),\dots,((\psi(\chi_n))(a))$. So $\overline{\psi}$ is a $*$-homomorphism, $\overline{\psi}(1)=1$, $\overline{\psi}(0) = 0$. Clearly $\chi_i(\overline{\psi}(a)) = ((\psi(\chi_i))(a)$ for all $a \in S$, $i=1,\dots,t$. Since $\Phi(S_2) = < \chi_1,\dots,\chi_n >$, (7) and hence (6) is true. It is clear from (7) that

$$\overline{\psi}^* = \psi \tag{9}$$

Now let $\phi:S_1 \to S_2$ be a $*$-homomorphism, $\phi(1) = 1$, $\phi(0) = 0$. Then for all $\chi \in \Phi(S_2)$, $a \in S_1$, by (7),

$$\chi((\overline{\phi^*})(a)) = (\phi^*(\chi))(a) = \chi(\phi(a))$$

By Lemma 2.2(3),

$$\overline{\phi^*} = \phi \tag{10}$$

THEOREM 2.4. Let S_1, S_2, S_3 be connected d-semigroups with zeros, $\dim S_i > 0$, $i = 1, 2, 3$. Then

(1) If $\phi:S_1 \to S_2$ is a $*$-homomorphism with $\phi(0) = 0$, $\phi(1) = 1$, then $\phi^*: \Phi(S_2) \to \Phi(S_1)$ is a homomorphism and $\overline{\phi^*} = \phi$.

(2) If $i:S_1 \to S_1$ is the identity map then $i^*:\Phi(S_1) \to \Phi(S_1)$ is the identity map.

(3) If $\psi:\Phi(S_2) \to \Phi(S_1)$ is a homomorphism, then $\overline{\psi}:S_1 \to S_2$ is a $*$-homomorphism with $\overline{\psi}(0) = 0$, $\overline{\psi}(1) = 1$. Moreover $\overline{\psi}^* = \psi$.

(4) If $i:\Phi(S_1) \to \Phi(S_1)$ is the identity map, then $\overline{i}:S_1 \to S_1$ is the identity map.

(5) If $\phi_1:S_1 \to S_2$, $\phi_2:S_2 \to S_3$ are $*$-homomorphism with $\phi_i(0) = 0$, $\phi_i(1) = 1$, $i = 1, 2$, then $(\phi_2 \circ \phi_1)^* = \phi_1^* \circ \phi_2^*$.

(6) If $\psi_1:\Phi(S_2)\to\Phi(S_1)$, $\psi_2:\Phi(S_3)\to\Phi(S_2)$ are homomorphisms, then $\overline{\psi_1 \circ \psi_2} = \overline{\psi_2} \circ \overline{\psi_1}$.

(7) S_1 is *-isomorphic to S_2 if and only if $\Phi(S_1)$ is isomorphic to $\Phi(S_2)$.

PROOF. (1), (3) follow from the equations (6)-(10). (2), (4) are trivial. (7) follows from (2), (4), (5), (6). So we need only prove (5), (6). First we prove (5). Let $\chi \in \Phi(S_3)$. Then for all $a \in S_1$,

$$\begin{aligned}((\phi_2 \circ \phi_1)^*(\chi))(a) &= \chi(\phi_2(\phi_1(a)))\\ &= (\phi_2^*(\chi))(\phi_1(a))\\ &= (\phi_1^*(\phi_2^*(\chi)))(a)\\ &= ((\phi_1^* \circ \phi_2^*)(\chi))(a)\end{aligned}$$

So $(\phi_2 \circ \phi_1)^* = \phi_1^* \circ \phi_2^*$.

Next we prove (6). Let $a \in S$, $\chi \in \Phi(S_3)$. Then by equation (7),

$$\begin{aligned}\chi(\overline{\psi_1 \circ \psi_2}(a)) &= ((\psi_1 \circ \psi_2)(\chi))(a)\\ &= (\psi_1(\psi_2(\chi)))(a)\\ &= (\psi_2(\chi))(\overline{\psi}_1(a))\\ &= \chi(\overline{\psi}_2(\overline{\psi}_1(a)))\\ &= \chi(\overline{\psi}_2 \circ \overline{\psi}_1(a))\end{aligned}$$

By Lemma 2.2(3), $\overline{\psi_1 \circ \psi_2} = \overline{\psi_2} \circ \overline{\psi_1}$, proving the theorem.

THEOREM 2.5. Let $\omega_1,\dots,\omega_n \in F(X_1,\dots,X_m)$. Let $V = \{(\omega_1(a),\dots,\omega_n(a)) \mid a \in K^m\} \subseteq K^n$. Set $S = \overline{V}$. Then S is a closed connected d-submonoid with zero, of $(K^n,\cdot\,)$. Moreover $\Phi(S) \cong \langle \omega_1,\dots,\omega_n \rangle$.

PROOF. Define $\theta:(K^m,\cdot)\to(K^n,\cdot)$ as $\theta(a) = (\omega_1(a),\dots,\omega_n(a))$. Then θ is a *-homomorphism with image V. So $S = \overline{V}$ is connected. Clearly $1 = \theta(1)$, $0=\theta(0)\in S$. Let χ_i denote the i^{th} projection of S into K. Then by Lemma 2.2, $\Phi(S) = \langle \chi_1,\dots,\chi_n \rangle$. Let $u, v \in F(Y_1,\dots,Y_n)$. Suppose $u(\chi_1,\dots,\chi_n) = v(\chi_1,\dots,\chi_n)$. Then $u(b) = v(b)$ for all $b \in S$. So

$$u(\omega_1(a),\ldots,\omega_n(a)) = v(\omega_1(a),\ldots,\omega_n(a)) \text{ for all } a \in K^m \tag{11}$$

Since K is infinite, $u(\omega_1,\ldots,\omega_n) = v(\omega_1,\ldots,\omega_n)$ in $F(X_1,\ldots,X_m)$. Conversely suppose $u(\omega_1,\ldots,\omega_n) = v(\omega_1,\ldots,\omega_n)$ in $F(X_1,\ldots,X_m)$. Then (11) is true. So $u(b) = v(b)$ for all $b \in V$. Since $\overline{V} = S$, $u(b) = v(b)$ for all $b \in S$. So $u(\chi_1,\ldots,\chi_n) = v(\chi_1,\ldots,\chi_n)$. It follows that $\Phi(S) = \langle \chi_1,\ldots,\chi_n \rangle \cong \langle \omega_1,\ldots,\omega_n \rangle$.

By Theorem A, Lemmas 2.1, 2.2, Theorems 2.4, 2.5, we have.

THEOREM 2.6. Let N_1 be the category of connected d-semigroups with zeros of dimension > 0 with morphism being *-homomorphisms ϕ with $\phi(0) = 0$, $\phi(1) = 1$. Let N_2 be the category of finitely generated, commutative, idempotent free, totally cancellative semigroups with morphisms being semigroup homomorphisms. Then $(\Phi, *)$ is a contravariant equivalence between N_1 and N_2.

THEOREM 2.7. Let S be a closed connected submonoid of $(K^n,\cdot)$ with zero $0 = (0,\ldots,0)$. Then for some $m \in Z^+$, $\omega_1,\ldots,\omega_n \in F(X_1,\ldots,X_m)$, $S = \overline{V}$ where $V = \{(\omega_1(a),\ldots,\omega_n(a)) \mid a \in K^m\}$.

PROOF. Let χ_i denote the i^{th} projection of S into K. Then by Lemma 2.2, $\Phi(S) = \langle \chi_1,\ldots,\chi_n \rangle$. By Theorem A, $\Phi(S) \cong \langle \omega_1,\ldots,\omega_n \rangle$ for some $m \in Z^+$, $\omega_1,\ldots,\omega_n \in F(X_1,\ldots,X_m)$ with $\chi_i \leftrightarrow \omega_i$. Let $V = \{(\omega_1(a),\ldots,\omega_n(a)) \mid a \in K^m\}$ and set $S_1 = \overline{V}$. Then $1 = (1,\ldots,1), 0 = (0,\ldots,0) \in S_1$. Let $u, v \in F(Y_1,\ldots,Y_n)$. Suppose $u(c) = v(c)$ for all $c \in S$. Then $u(\chi_1,\ldots,\chi_n) = v(\chi_1,\ldots,\chi_n)$. So $u(\omega_1,\ldots,\omega_n) = v(\omega_1,\ldots,\omega_n)$. Thus $u(b) = v(b)$ for all $b \in V$. Since $\overline{V} = S_1$, $u(b) = v(b)$ for all $b \in S_1$. Conversely suppose $u(b) = v(b)$ for all $b \in S_1$. Then

$$u(\omega_1(a),\ldots,\omega_n(a)) = v(\omega_1(a),\ldots,\omega_n(a)) \text{ for all } a \in K^m$$

Since K is infinite, $u(\omega_1,\ldots,\omega_n) = v(\omega_1,\ldots,\omega_n)$. So $u(\chi_1,\ldots,\chi_n) = v(\chi_1,\ldots,\chi_n)$. Thus $u(c) = v(c)$ for all $c \in S$. By Lemma 2.3, $S = S_1$.

COROLLARY 2.8. Let S be a closed connected submonoid of $(K^n,\cdot)$ with zero $0 = (0,\ldots,0)$, $\dim S = 1$. Then there exist $i_1,\ldots,i_n \in Z^+$ such that $S = \{(a^{i_1},\ldots,a^{i_n}) \mid a \in K\}$. Conversely, for any $i_1,\ldots,i_n \in Z^+$, S defined as above is a closed connected submonoid of $(K^n,\cdot)$ with zero $0 = (0,\ldots,0)$ and $\dim S = 1$.

$V = \{(\omega_1(a),\ldots,\omega_n(a)) \mid a \in K^m\}$ and $S = \overline{V}$. Let $V_1 = \{(\omega_1(a,\ldots,a),\ldots,\omega_n(a,\ldots,a)) \mid a \in K\}$, $S_1 = \overline{V}_1$. Then $S_1 \subseteq S$, $\dim S_1 = 1$. So $S = S_1$. So there exist $i_1,\ldots,i_n \in Z^+$ such that $V_1 = \{(a^{i_1},\ldots,a^{i_n}) \mid a \in K\}$. Define $\theta: K \to S$ as $\theta(a) = (a^{i_1},\ldots,a^{i_n})$. Then it is easy to see that θ is a finite morphism in the usual sense of [5; Section 4.2]. By [5; Proposition 4.2], $\theta(K) = S$.

THEOREM 2.9. Let S be a connected monoid with zero, $\dim S = 1$. Then S is *-isomorphic to a semigroup of the type given in Corollary 2.8.

PROOF. By Corollary 1.5, S is a d-semigroup. We are now done by Lemma 2.2 and Corollary 2.8.

THEOREM 2.10. Let S be a connected semigroup, $e, f \in E(S)$, $e > f$. Then there exists a closed connected subsemigroup S_1 of S such that e is the identity of S_1, f is the zero of S_1 and $\dim S_1 = 1$.

PROOF. We can assume that e is the identity element of S (otherwise we work with eSe). By Lemma 1.1 we are reduced to the case when f is the zero of S. By Corollary 1.5, we are reduced to the case when S is also a d-semigroup.

$\Lambda(S)$ = {All prime ideals of S} $\cup \{\emptyset\}$.

$X(S)$ = $\{S \setminus I \mid I \in \Lambda(S)\}$.

$\Omega(S)$ = Maximal semilattice image of S.

It is easy to see that $(\Lambda(S), \subseteq) \cong (\Lambda(\Omega(S)), \subseteq)$ is a complete lattice. If S is finitely generated, then $\Omega(S)$ is finite and so $(\Lambda(S), \subseteq)$ is a finite lattice.

THEOREM 3.1. Let S be a connected d-semigroup with zero. Define $\alpha: I(S) \to \Gamma(\Phi(S))$ as $\alpha(I) = \{\chi \mid \chi \in \Phi(S),\ \chi(a) = 0 \text{ for all } a \in I\}$. Define $\beta: \Gamma(\Phi(S)) \to I(S)$ as $\beta(W) = \{a \mid a \in S,\ \chi(a) = 0 \text{ for all } \chi \in W\}$. Then α, β are inclusion reversing bijections and $\beta = \alpha^{-1}$. Moreover $\alpha(A(S)) = \Lambda(\Phi(S))$.

PROOF. Clearly α,β are inclusion reversing. Let $I \in A(S)$. Then $I = eS$ for some $e \in E(S)$. So $\alpha(I) = \{\chi \mid \chi \in \Phi(S),\ \chi(e) = 0\}$. It follows that $\alpha(I) \in \Lambda(\Phi(S))$. Clearly $I \subseteq \beta(\alpha(I))$. We claim that $I = \beta(\alpha(I))$. Suppose not. Then there exists $a \in \beta(\alpha(I))$ such that $a \notin I$. Let $a\,\mathcal{H} f$, $f \in E(S)$. Then $f \notin I$, $f \in \beta(\alpha(I))$. So $e \not\geq f$. By Lemma 2.1(2), there exists $\chi \in \Phi(S)$ such that $\chi(f) = 1$, $\chi(e) = 0$. So

$\chi \in \alpha(I)$ and $f \notin \beta(\alpha(I))$, a contradiction. So

$$\text{for all } I \in A(S),\ \ \alpha(I) \in \Lambda(\Phi(S)) \text{ and } \ \beta(\alpha(I)) = I \tag{12}$$

Let $P \in \Lambda(\Phi(S))$. We claim that $\beta(P) \in A(S)$ and $\alpha(\beta(P))=P$. By Lemma 2.1, this is true for $P = \Phi(S)$. So assume $P \neq \Phi(S)$. Then $F = \Phi(S)\setminus P$ is a subsemigroup of $\Phi(S)$. By Lemma 2.2 we can assume that S is a closed submonoid of some $(K^n,\cdot)$, $0=(0,\ldots,0) \in S$ and that $\Phi(S) = <\chi_1,\ldots,\chi_n>$ where χ_i is the i^{th} projection of S into K, $i=1,\ldots,n$. Let $A = \{\chi_i | \chi_i \in F\}$. Then $<A> = F$. Let $e=(e_1,\ldots,e_n)$ where $e_i=1$ if $\chi_i \in A$, $e_i=0$ if $\chi_i \notin A$. We claim that $e \in S$. Suppose not. Then by Lemma 2.3, there exist $u, v \in F(X_1,\ldots,X_n)$ such that $u(a)=v(a)$ for all $a \in S$ and $u(e) \neq v(e)$. Since $u(e)^2 = u(e)$ and $v(e)^2 = v(e)$ we can assume that $u(e) = 1$, $v(e) = 0$. Clearly $u(\chi_1,\ldots,\chi_n) = v(\chi_1,\ldots,\chi_n)$. Since $u(e)=1$, $u(X_1,\ldots,X_n)$

By Lemma 2.2 and Theorem 2.7, we can assume that S is as in Theorem 2.7, with $e = (1,\ldots,1)$, $f=(0,\ldots,0)$. Let $V_1 = \{(\omega_1(a,\ldots,a),\ldots,\omega_n(a,\ldots,a)) | a \in K\}$, $S_1 = \overline{V}_1$. Then $e, f \in S_1$, $\dim S_1 = 1$, $S_1 \subseteq S$. Define $\theta: K \to S_1$ as $\theta(a) = (\omega_1(a,\ldots,a),\ldots,\omega_n(a,\ldots,a))$. Then θ is a *-homomorphism. So S_1 is connected. This proves the theorem.

3. POLYTOPES

If $X \subseteq \mathbb{R}^n$, then we let $C(X)$ denote the convex hull of X (see [4]). The convex hull of a finite set in $\mathbb{R}^n$ is called a polytope [4]. If the vertices of P are rational, then P is said to be a rational polytope. If $X \subseteq P$, then X is said to be a face of P [4; p. 35] if for all $a, b \in P$, $\alpha \in (0,1)$, $\alpha a + (1-\alpha)b \in X$ if and only if $a, b \in X$. Let $X(P)$ denote the set of all faces of P. Then [4; p. 21], $(X(P),\subseteq)$ is a finite lattice. Dimension of P is defined to be the dimension of the affine hull of P [4; p. 3]. Then dimension of P = (length of any maximal chain in $X(P)) - 1$. Two polytopes P_1, P_2 have the same combinatorial type if $X(P_1) \cong X(P_2)$ (see [4; p. 38]). By [4; p. 244], every polytope of dimension ≤ 3 has the same combinatorial type as some rational polytope. However this is not true in general [4: p. 94]. If $u = (\alpha_1,\ldots,\alpha_n)$, $v = (\beta_1,\ldots,\beta_n) \in \mathbb{R}^n$ then let $u \cdot v = \sum_{i=1}^n \alpha_i\beta_i$ denote the inner product of u and v.

Let S be a semigroup. An ideal I of S is said to be <u>semiprime</u> if for all $a \in S$, $a^2 \in I$ implies $a \in I$. I is <u>prime</u> if for all a, $b \in S$, $ab \in I$ implies $a \in I$ or $b \in I$. Let

$$I(S) = \{\text{All ideals of } S\}$$

$$A(S) = \{\text{All principal ideals of } S\}$$

$$\Gamma(S) = \{\text{All semiprime ideals of } S\} \cup \{\emptyset\}$$

involves only those X_1's with $\chi_i \in F$. So $u(\chi_1,\ldots,\chi_n) \in F$. Since $v(e) = 0$, $v(X_1,\ldots,X_n)$ involves at least one X_i with $\chi_i \notin F$. So $v(\chi_1,\ldots,\chi_n) \in P$. This contradiction shows that $e \in S$. Clearly $\chi(e) = 1$ for $\chi \in F$, $\chi(e) = 0$ for $\chi \in P$. Hence $P = \{\chi | \chi \in \Phi(S),\ \chi(e) = 0\} = \alpha(eS)$. By (12), $\beta(P) = \beta(\alpha(eS)) = eS \in A(S)$, $\alpha(\beta(P)) = \alpha(eS) = P$. So

$$\text{for all } P \in \Lambda(\Phi(S)),\ \beta(P) \in A(S) \text{ and } \alpha(\beta(P)) = P \tag{13}$$

Clearly

$$\alpha(I_1 \cup I_2) = \alpha(I_1) \cap \alpha(I_2) \text{ for all } I_1, I_2 \in I(S) \tag{14}$$

Let $W_1, W_2 \in \Gamma(\Phi(S))$. Then clearly $\beta(W_i) \subseteq \beta(W_1 \cap W_2)$, $i = 1, 2$. So $\beta(W_1) \cup \beta(W_2) \subseteq \beta(W_1 \cap W_2)$. Let $a \in \beta(W_1 \cap W_2)$. Suppose $a \notin \beta(W_i)$, $i = 1, 2$. Then there exist $\theta_i \in W_i$, $i = 1, 2$ such that $\theta_i(a) \neq 0$, $i = 1, 2$. So $\theta = \theta_1\theta_2 \in W_1 \cap W_2$, $\theta(a) \neq 0$. So $a \notin \beta(W_1 \cap W_2)$, a contradiction. Thus

$$\beta(W_1 \cap W_2) = \beta(W_1) \cup \beta(W_2) \text{ for all } W_1, W_2 \in \Gamma(\Phi(S)) \tag{15}$$

Clearly $A(S)$ is finite. Let $I \in I(S)$. Then $I = I_1 \cup I_2 \cup \ldots \cup I_k$ for some $I_1,\ldots,I_k \in A(S)$. By (12), $\beta(\alpha(I_r)) = I_r$, $r = 1,\ldots,k$. By (14), (15),

$$\begin{aligned} \alpha(I) &= \alpha(I_1) \cap \ldots \cap \alpha(I_k) \\ \beta(\alpha(I)) &= \beta(\alpha(I_1)) \cup \ldots \cup \beta(\alpha(I_k)) \\ &= I_1 \cup \ldots \cup I_k \\ &= I \end{aligned}$$

So

$$\beta(\alpha(I)) = I \text{ for all } I \in I(S) \tag{16}$$

Since $\Phi(S)$ is finitely generated, $\Lambda(\Phi(S))$ is finite. Let $W \in \Gamma(\Phi(S))$. By [2; p. 125, Exercise 9], $W = W_1 \cap \ldots \cap W_k$ for some $W_1,\ldots,W_k \in \Lambda(\Phi(S))$. Then by (13)

$\alpha(\beta(W_r)) = W_r$ for $r = 1,\ldots,k$. Then by (14), (15),

$$\begin{aligned}
\beta(W) &= \beta(W_1) \cup \ldots \cup \beta(W_k) \\
\alpha(\beta(W)) &= \alpha(\beta(W_1)) \cap \ldots \cap \alpha(\beta(W_k)) \\
&= W_1 \cap \ldots \cap W_k \\
&= W
\end{aligned}$$

So

$$\alpha(\beta(W)) = W \text{ for all } W \in \Gamma(\Phi(S)) \tag{17}$$

By (16) and (17), $\alpha^{-1} = \beta$. By (12), (13), $\alpha(A(S)) = \Lambda(\Phi(S))$. This proves the theorem.

REMARK. The classical Hilbert's Nullstellensatz yields a 1 - 1 correspondence between the closed subsets of K^n and the radical ideals of $K[X_1,\ldots,X_n]$. Moreover this restricts to a 1 - 1 correspondence between the closed irreducible subsets of K^n and the prime ideals of $K[X_1,\ldots,X_n]$. Analagously, Theorem 3.1 yields a 1 - 1 correspondence between the ideals of a connected d-semigroup with zero S and the semiprime ideals of its character semigroup $\Phi(S)$. Moreover this correspondence restricts to a correspondence between the principal ideals of S and the prime ideals of $\Phi(S)$.

THEOREM 3.2. Let S be a connected d-semigroup with zero. Then $(\mathcal{U}(S), \leq) \cong (E(S), \leq) \cong (\mathcal{X}(\Phi(S)), \subseteq)$.

PROOF. Clearly $(A(S), \subseteq) \cong (\mathcal{U}(S), \leq) \cong (E(S), \leq)$. By Theorem 3.1, $(A(S), \subseteq)$ is anti-isomorphic to $(\Lambda(\Phi(S)), \subseteq)$. Clearly $(\Lambda(\Phi(S)), \subseteq)$ is anti-isomorphic to $(\mathcal{X}(\Phi(S)), \subseteq)$. This proves the theorem.

Let $Q^{m\times n}$ denote the set of all m×n matrices over Q. The following result is well known. However, we include a proof here for the convenience of the reader.

FACT 3.3. Let $A \in Q^{m\times n}$, $u = (\alpha_1,\ldots,\alpha_m) \in \mathbb{R}^m$ such that $uA = 0$. Then there exists $v = (\beta_1,\ldots,\beta_m) \in Q^m$ such that $vA = 0$ and for $i = 1,\ldots,m$, $\alpha_i > 0$ implies $\beta_i > 0$, $\alpha_i < 0$ implies $\beta_i < 0$.

PROOF. Let $N = \{X \mid X \in \mathbb{R}^m, XA = 0\}$ denote the left null space of A. Since $A \in Q^{m\times n}$, there exist $u_1,\ldots,u_t \in Q^m$ such that $u_1,\ldots,u_t$ is a basis of N. So

$u = \sum_{j=1}^{t} \varepsilon_j u_j$ for some $\varepsilon_1,\ldots,\varepsilon_t \in \mathbb{R}$. Let $\varepsilon \in \mathbb{R}^+$, $\varepsilon_1',\ldots,\varepsilon_t' \in Q$. Then

$v = \sum_{j=1}^{t} \varepsilon_j' u_j \in N \cap Q^m$. For $\Sigma|\varepsilon_j' - \varepsilon_j|$ small enough, $|u-v| < \varepsilon$. For ε small enough, the conclusion of the lemma clearly holds.

COROLLARY 3.4. Let $u_1,\ldots,u_m$, $v_1,\ldots,v_n \in Q^d$, $\alpha_1,\ldots,\alpha_m$, $\beta_1,\ldots,\beta_n \in \mathbb{R}^+$ such that $\sum_{i=1}^{m} \alpha_i u_i = \sum_{j=1}^{n} \beta_j v_j$. Then there exist $\alpha_1',\ldots,\alpha_m'$, $\beta_1',\ldots,\beta_n' \in Z^+$ such that $\sum_{i=1}^{m} \alpha_i' u_i = \sum_{j=1}^{n} \beta_j' v_j$.

PROOF. By Fact 3.3 we can choose $\alpha_1',\ldots,\alpha_m'$, $\beta_1',\ldots,\beta_n' \in Q^+$ such that $\sum_{i=1}^{m} \alpha_1' u_i = \sum_{j=1}^{n} \beta_j' v_j$. Then for some $s \in Z^+$, $\alpha_i'' = s\alpha_i$, $\beta_j'' = s\beta_j \in Z^+$, $i = 1,\ldots,m$, $j = 1,\ldots,n$. Clearly $\sum_{i=1}^{m} \alpha_i'' u_i = \sum_{j=1}^{n} \beta_j'' v_j$.

THEOREM 3.5. The classes $\{X(S) \mid S$ is a finitely generated, commutative, idempotent-free, totally cancellative smeigroup$\}$ and $\{X(P) \mid P$ is a rational polytope in $\mathbb{R}^n$ for some $n \in Z^+\}$ are identical to within lattice isomorphisms.

PROOF. Let S be a finitely generated, commutative, idempotent-free, totally cancellative semigroup. By Theorem A we can assume that $S = \langle u_1,\ldots,u_n \rangle \subseteq (Z^d,+)$, $0 \notin S$. Let $C = C(u_1,\ldots,u_n)$. By Fact 3.3, $0 \notin C$. So $C \cap -C = \emptyset$. By [4; p. 11], there exists $u \in \mathbb{R}^d$ such that $u \cdot a > 0$ for all $a \in C$. So $u \cdot u_i > 0$, $i = 1,\ldots,n$. If $v \in \mathbb{R}^d$, then $|u \cdot u_i - v \cdot u_i| = |(u-v) \cdot u_i| \le \|u-v\| \; \|u_i\|$. So for $\|u-v\|$ small enough, $v \cdot u_i > 0$ for $i = 1,\ldots,n$. So, without loss of generality, we can assume that $u \in Q^d$, If $a \in S$, then let $\theta(a) = \frac{a}{u \cdot a} \in Q^d$. Let $a_1,\ldots,a_k \in S$, $\alpha_1,\ldots,\alpha_k \in Z^+$ and set $a = \alpha_1 a_1 + \ldots + \alpha_k a_k$. Then

$$\begin{gathered}\theta(a) = \sum_{i=1}^{k} \beta_i \theta(a_i) \in C(\theta(a_1),\ldots,\theta(a_k)), \\ \sum_{i=1}^{k} \beta_i = 1 \text{ where } \beta_i = \frac{\alpha_i a_i \cdot u}{a \cdot u} > 0, \quad i = 1,\ldots,k.\end{gathered} \tag{18}$$

So $P = \mathcal{C}(\theta(S)) = \mathcal{C}(\theta(u_1),\ldots,\theta(u_n))$ is a rational polytope. If $X \in \mathcal{X}(S)$, then let $\phi(X) = \mathcal{C}(\theta(X)) \subseteq P$. If $F \in \mathcal{X}(P)$, then let $\psi(F) = \{a \mid a \in S,\ \theta(a) \in F\} \subseteq S$.

Let $X \in \mathcal{X}(S)$. Let $x, y \in P$, $\alpha \in (0,1)$ such that $\alpha x + (1-\alpha)y = z \in \phi(X)$. There exist $a_1,\ldots,a_p$, $b_1,\ldots,b_q \in S$, $c_1,\ldots,c_r \in X$, $\alpha_1,\ldots,\alpha_p$, $\beta_1,\ldots,\beta_q$, $\gamma_1,\ldots,\gamma_r \in (0,1)$ such that $x = \Sigma\alpha_i\theta(a_i)$, $y = \Sigma\beta_j\theta(b_j)$, $z = \Sigma\gamma_k\theta(c_k)$, $\Sigma\alpha_i = \Sigma\beta_j = \Sigma\gamma_k = 1$. So there exist $\alpha_1',\ldots,\alpha_p'$, $\beta_1',\ldots,\beta_q'$, $\gamma_1',\ldots,\gamma_r' \in \mathbb{R}^+$ such that

$$\Sigma\alpha_i'a_i + \Sigma\beta_j'b_j = \Sigma\gamma_k'c_k$$

By Corollary 3.4 there exist $\alpha_1'',\ldots,\alpha_p''$, $\beta_1'',\ldots,\beta_q''$, $\gamma_1'',\ldots,\gamma_r'' \in Z^+$ such that

$$\Sigma\alpha_i''a_i + \Sigma\beta_j''b_j = \Sigma\gamma_k''c_k \in X$$

Since $X \in \mathcal{X}(S)$, $a_1,\ldots,a_p$, $b_1,\ldots,b_q \in X$. Since $x \in \mathcal{C}(\theta(a_1),\ldots,\theta(a_p))$ and $y \in \mathcal{C}(\theta(b_1),\ldots,\theta(b_q))$, $x, y \in \phi(X)$. Hence $\phi(X) \in \mathcal{X}(P)$. Clearly $X \subseteq \psi(\phi(X))$. Let $a \in \psi(\phi(X))$. Then $\theta(a) \in \mathcal{C}(\theta(X))$. So there exist $a_1,\ldots,a_p \in X$, α, $\alpha_1,\ldots,\alpha_p \in \mathbb{R}^+$ such that $\alpha a = \Sigma\alpha_i a_i$. By Corollary 3.4, there exist α', $\alpha_1',\ldots,\alpha_p' \in Z^+$ such that $\alpha' a = \Sigma\alpha_i'a_i \in X$. So $a \in X$. Hence

$$\text{for all } X \in \mathcal{X}(S),\quad \phi(X) \in \mathcal{X}(P) \text{ and } \psi(\phi(X)) = X \tag{19}$$

Let $F \in \mathcal{X}(P)$. By (18) $\psi(F)$ is $\emptyset$ or a subsemigroup of S. Let $a, b \in S$ such that $a + b \in \psi(F)$. By (18), $\theta(a+b) = \varepsilon\theta(a) + (1-\varepsilon)\theta(b) \in F$ for some $\varepsilon \in (0,1)$. So $\theta(a), \theta(b) \in F$. Hence $a, b \in \psi(F)$ and $\psi(F) \in \mathcal{X}(S)$. Clearly $\phi(\psi(F)) \subseteq F$. Let $x \in F$. Then $x \in P = \phi(S)$. So $x = \sum_{i=1}^{k} \varepsilon_i\theta(a_i)$ for some $a_1,\ldots,a_k \in S$, $\varepsilon_1,\ldots,\varepsilon_k \in (0,1)$ such that $\Sigma\varepsilon_i = 1$. Then $\theta(a_i) \in F$, $i = 1,\ldots,k$. So $a_1,\ldots,a_k \in \psi(F)$ and $x \in \phi(\psi(F))$. So $\phi(\psi(F)) = F$. Hence

$$\text{for all } F \in \mathcal{X}(P),\quad \psi(F) \in \mathcal{X}(S) \text{ and } \phi(\psi(F)) = F \tag{20}$$

Since ϕ, ψ are clearly inclusion preserving, it follows from (19), (20) that $(\mathcal{X}(S), \subseteq) \cong (\mathcal{X}(P), \subseteq)$.

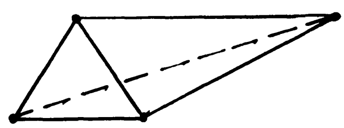
Conversely let $P \subseteq \mathbb{R}^m$ be a rational polytope. Then $P = \mathcal{C}(a_1,\ldots,a_n)$ for some $a_1,\ldots,a_n \in Q^m$. If $\alpha \in Z^+$, then clearly $(\mathcal{X}(P), \subseteq) \cong (\mathcal{X}(\alpha P), \subseteq)$. So we can assume that $a_1,\ldots,a_n \in Z^m$. Let $u_i = (a_i, 1)$, $i = 1,\ldots,n$, $d = m+1$. Then $P_1 = \mathcal{C}(u_1,\ldots,u_n) =$

$P \times \{1\} \subseteq \mathbb{R}^d$ is a rational polytope and $(\mathcal{X}(P), \subseteq) \cong (\mathcal{X}(P_1), \subseteq)$. Let $S = \langle u_1, \ldots, u_n \rangle \subseteq Z^d$. Then $0 \notin S$ and S is a finitely generated, commutative, totally cancellative, idempotent-free semigroup. Let $u = (0,1) \in Z^d$. Then $u \cdot u_i = 1$, $i = 1, \ldots, n$. So $\theta(u_i) = \dfrac{u_i}{u \cdot u_i} = u_i$, $i = 1, \ldots, n$. By the proof of the first half of this theorem, $(\mathcal{X}(S), \subseteq) \cong (\mathcal{X}(P_1), \subseteq)$. This proves the theorem.

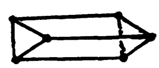
If S is a connected d-semigroup with zero, then by [7; Theorem 3.17], dim S = length of any maximal chain in E(S). By Theorems 2.6, 3.2 and 3.5 we have,

THEOREM 3.6. The classes $\{(E(S), \leq) \mid S$ is a connected d-semigroup with zero, $\dim S > 0\}$ and $\{(\mathcal{X}(P), \subseteq) \mid P$ is a rational polytope in $\mathbb{R}^n$ for some $n \in Z^+\}$ are identical to within lattice isomorphisms. Moreover, for any corresponding S and

EXAMPLE 3.7. If $S = (K^4, \cdot)$, then the corresponding polytope P is a tetrahedron



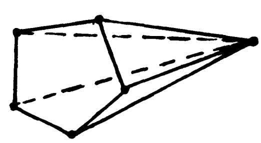
More generally if $S = (K^n, \cdot)$, then the corrsponding polytope P is the $(n-1)$-simplex.

EXAMPLE 3.8. Let $S = \{(a_1, b_1, a_2, b_2, a_3, b_3) \mid a_i, b_j \in K,\ a_i b_j = a_j b_i,\ i, j = 1, 2, 3\}$. Then by [7; Example 4.7] S is a closed connected d-submonoid with zero, of $(K^6, \cdot)$. Moreover $\dim S = 4$ and $|E(S)| = 22$. The corresponding polytope P can be shown to be the triangular prism: 

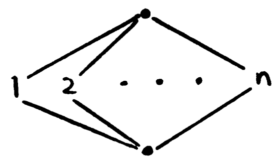
EXAMPLE 3.9. Let $S = \{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_1, \ldots, a_6 \in K,\ a_3^2 a_1 = a_5 a_2^2,\ a_2 a_5^2 = a_1 a_4^2,\ a_2 a_4^2 = a_5 a_3^2\}$. Define $\phi: K^6 \to K^6$ as

$$\phi(x_1, x_2, x_3, x_4, x_5, x_6) = (x_3^2 x_4^2 x_5^3,\ x_2^2 x_3^2 x_5,\ x_1 x_2^2 x_3,\ x_1^2 x_2 x_4,\ x_1^2 x_4^2 x_5,\ x_6)$$

Then $\phi(K^6) = S$ and so S is a closed connected d-submonoid with zero, of $(K^6, \cdot)$. Clearly $\dim S = 4$ and $|E(S)| = 24$. The corresponding polytope P can be shown to be the pentagonal pyramid:

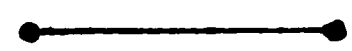


COROLLARY 3.10. Let S be a connected semigroup such that $\mathcal{U}(S)$ is the following lattice:



Then $n \leq 2$.

PROOF. By the proof of Theorem 1.9, we can assume that S is a monoid. Let S_1 be a maximal connected d-submonoid with zero, of S. Then by Theorem 1.8(3) $E(S_1) \cap J \neq \emptyset$ for all $J \in \mathcal{U}(S)$ and $\dim S_1 = 2$. By Theorem 3.6, the polytope P corresponding to S_1 has dimension 1. So P is the line



So $|E(S_1)| = 4$. Thus $|\mathcal{U}(S)| \leq |E(S_1)| = 4$. Hence $n \leq 2$.

EXAMPLE 3.11. $\mathcal{M}_2(K)$ and $(K^2, \cdot)$ show that n can be 1 or 2 in Corollary 3.10.

4. SEMILATTICES

As usual, by a semilattice, we mean a commutative, idempotent semigroup.

LEMMA 4.1. Let Ω be a finite semilattice. Then $|\mathcal{X}(\Omega)| = |\Omega| + 1$.

PROOF. We prove by induction on $|\Omega|$. If $|\Omega| = 1$ this is clear. So assume $|\Omega| > 1$. Let α be a maximal element of Ω. Then $\{\alpha\} \in \mathcal{X}(\Omega)$. Define $\phi: \mathcal{X}(\Omega) \to \mathcal{X}(\Omega_1)$ as $\phi(F) = F \cap \Omega_1$. Let $F_1 \in \mathcal{X}(\Omega_1)$ and set $P_1 = \Omega_1 \setminus F_1$. Let $p \in P_1$. We claim that $\alpha p \in P_1$. Otherwise $f = \alpha p \in F_1$. Then $f = pf \in P_1$, a contradiction. So $\alpha P_1 \subseteq P_1$. If $F_1 \in \mathcal{X}(\Omega)$, then $\phi(F_1) = F_1$. Suppose not. Then $\alpha F_1 \not\subseteq P_1$. So there exists $f_1 \in F_1$ such that $\alpha f_1 \in F_1$. Now we claim that $F_1 \cup \{\alpha\} \in \mathcal{X}(\Omega)$. Otherwise $\alpha F_1 \not\subseteq F_1$. So $\alpha f_2 \in P_1$ for some $f_2 \in F_1$. So $\alpha f_1 f_2 = (\alpha f_1) f_2 \in F_1$ and $\alpha f_1 f_2 = (\alpha f_2) f_1 \in P_1$, a contradiction. So $F_1 \cup \{\alpha\} \in \mathcal{X}(\Omega)$, $\phi(F_1 \cup \{\alpha\}) = F_1$. Thus ϕ is surjective. Let $F_1 \in \mathcal{X}(\Omega_1)$, $F_1 \neq \emptyset$, $F, G \in \mathcal{X}(\Omega)$, $\phi(F) = F_1 = \phi(G)$, $F \neq G$. We can assume that $\alpha \in F, \alpha \notin G$. So $G = F_1$, $F = F_1 \cup \{\alpha\}$. Since $\alpha \in F$, $\alpha F \subseteq F$. So $\alpha F_1 \subseteq F_1$.

Since $\alpha \notin G$, $\alpha G \subseteq \Omega \setminus G$. So $\alpha F_1 \subseteq \Omega_1 \setminus F_1$, a contradiction. Thus $|\phi^{-1}(F_1)| = 1$. Clearly $\phi^{-1}(\emptyset) = \{\emptyset, \{\alpha\}\}$. So $|X(\Omega)| = |X(\Omega_1)| + 1 = |\Omega_1| + 1 + 1 = |\Omega| + 1$.

If Ω is a semilattice, then let $Y(\Omega)$ denote the semilattice of all homomorphisms of Ω into $\Omega_0 = \{0,1\}$. Then clearly $Y(\Omega) \cong (X(\Omega), \cap)$. Let $Y^*(\Omega) = Y(\Omega) \setminus \{1,0\}$. Then $Y^*(\Omega)$ may or may not be a subsemilattice of $Y(\Omega)$.

LEMMA 4.2. Let Ω be a finite semilattice. Then $Y^*(Y(\Omega))$ is a semilattice and $\Omega \cong Y^*(Y(\Omega))$.

PROOF. Define $\theta : \Omega \to Y(Y(\Omega))$ as $\theta(\alpha)(f) = f(\alpha)$. Then θ is a homomorphism. Clearly $\theta(\alpha)(1) = 1$, $\theta(\alpha)(0) = 0$. So $\theta(\alpha) \in Y^*(Y(\Omega))$. We claim that θ is injective. We can assume that $\Omega \subseteq \Omega_0 \times \dots \times \Omega_0$. Let f_i denote the i^{th} projection of Ω into Ω_0. Then $f_i \in Y(\Omega)$. Let $\alpha, \beta \in \Omega$ such that $\theta(\alpha) = \theta(\beta)$. Then $\theta(\alpha)(f_i) = \theta(\beta)(f_i)$ for all i. So $f_i(\alpha) = f_i(\beta)$ for all i. So $\alpha = \beta$. By Lemma 4.1, $|Y^*(Y(\Omega))| = |\Omega|$. Hence $\Omega \cong Y^*(Y(\Omega))$.

COROLLARY 4.3. Let Ω_1, Ω_2 be finite semilattice such that $(X(\Omega_1), \subseteq) \cong (X(\Omega_2), \subseteq)$. Then $\Omega_1 \cong \Omega_2$.

COROLLARY 4.4. Let Ω be a finite semilattice such that $Y^*(\Omega)$ is a semilattice. Then $Y(Y^*(\Omega) \cong \Omega$.

PROOF. Let $\Omega_1 = Y^*(\Omega)$. Then by Lemma 4.2,

$$Y^*(Y(\Omega_1)) \cong \Omega_1 = Y^*(\Omega).$$

So $Y(Y(\Omega_1)) \cong Y(\Omega)$. Again by Lemma 4.2,

$$Y(\Omega_1) \cong Y^*(Y(Y(\Omega_1))) \cong Y^*(Y(\Omega)) \cong \Omega$$

If S is a finitely generated semigroup and if Ω is the maximal semilattice image of S, then clearly Ω is finite and $(X(S), \subseteq) \cong (X(\Omega), \subseteq)$. By Theorem 3.5, Lemma 4.2, Corollaries 4.3, 4.4, we have.

THEOREM 4.5. (1) Let $(L, \vee, \wedge)$ be a finite lattice. Then $L \cong X(P)$ for some rational polytope P if and only if $\Omega = Y^*(L, \wedge)$ is a semilattice and Ω is isomorphic to the maximal semilattice image of some finitely generated, commutative, idempotent free, totally cancellative semigroup.

(2) Let Ω be a finite semilattice. Then Ω is the maximal semilattice image of some finitely generated, commutaitve, idempotent free, totally cancellative semigroup if and only if $(\mathcal{X}(\Omega), \subseteq)$ is isomorphic to $(\mathcal{X}(P), \subseteq)$ for some rational polytope P.

If P is a polytope, call $\mathcal{X}(P)$, the face lattice of P. By a theorem of Tarski (see [4; p. 91]), the enumeration problem for face lattices of polytopes is solvable. However, for rational polytopes the problem is not yet solved [4; p. 92]. By Theorem 4.5, we have,

THEOREM 4.6. The enumeration problem for face lattices of rational polytopes is solvable if and only if the enumeration problem for maximal semilattice images of finitely generated, commutative, idempotent-free totally cancellative semigroups, is solvable.

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