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ALMOST-CONTINUOUS PATH CONNECTED SPACES

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ABSTRACT. M. K. Singal and Asha Rani Singal have defined an almost-continuous function $f:X \to Y$ to be one in which for each $x \in X$ and each regular-open set V containing f(x), there exists an open U containing x such that $f(U) \subset V$. A space Y may now be defined to be almost-continuous path connected if for each $y_o, y_1 \in Y$ there exists an almost-continuous $f:I \to Y$ such that $f(0) = y_o$ and $f(1) = y_1$. An investigation of these spaces is made culminating in a theorem showing when the almost-continuous path connected components coincide with the usual components of Y.

KEY WORDS AND PHRASES. *Almost-continuous functions, path connected spaces.*

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1. INTRODUCTION.

The concept of an almost continuous function $f:X \to Y$ has been defined in [1] as one in which for each $x \in X$ and each regular-open V containing f(x) there exists an open set U containing x such that $f(U) \subset V$. Using this concept we make the following two definitions:

DEFINITION 1. The function $f:I \to Y$ is an almost-continuous path (a.c. path) from y_o to y_1 if f is almost continuous, $f(0) = y_o$ and $f(1) = y_1$.

DEFINITION 2. The space Y is a.c. path connected if for each y_o, $y_1 \in Y$, there exists an a.c. path from y_o to y_1.

The regular-open sets in Y may be used as a base to form the semi-regular topology T_s on Y from which $f:X \to (Y,T)$ is almost-continuous if and only if $f:X \to (Y,T_s)$ is continuous. Thus, Definition 2 may be restated as (Y,T) is a.c. path connected if for each y_o, y_1 in Y, there is a continuous $f:I \to (Y,T_s)$ such that $f(0) = y_o$ and $f(1) = y_1$. In view of this observation, many of the known results for path connected spaces in the usual sense also apply to a.c. path connected spaces. For example, if $y_o \in Y$, then Y is a.c. path connected if and only if for each $y \in Y$, there is an a.c. path from y to y_o. Furthermore, slight variations in known results may sometimes be made to easily produce statements concerning a.c. path connected spaces. An example is that if $f:X \to Y$ is an almost-continuous surjection and X is path connected, then Y is a.c. path connected.

2. MAIN RESULTS.

THEOREM 1. Every a.c. path connected space Y is connected.

PROOF. Assume $Y = U \cup V$ where U and V are open in (Y,T) and $U \cap V = \emptyset$. Then U and V are regular-open so that $U \cup V$ is a separation of (Y,T_2). But (Y,T_s) is path connected, hence connected. The contradiction implies Y is connected.

Let R be the reals with the usual topology, Q the set of rational numbers and R_Q the reals with the topology generated by the usual open intervals together with Q as a subbase. Since the semi-regular topology associated with R_Q is the usual topology on R, it follows that $f:X \to R_Q$ is almost-continuous if and only if $f:X \to R$ is continuous. Similarly, $f:I \to R_Q \times R_Q$ is almost-continuous if and only if $f:I \to R \times R$ is continuous. These observations lead to the following example which shows the converse of Theorem 1 is false.

EXAMPLE 1. Let $Y = \{(x,y) : y = \sin(1/x),\ 0 < x \leq 1\} \subset R \times R$. Then $\overline{Y} = Y \cup \{(x,y) : -1 \leq y \leq 1\}$ in $R \times R$ as well as $R_Q \times R_Q$. Hence $\overline{Y}$ is connected, but not a.c. path connected.

Example 1 also shows that a space may be a.c. path connected but its closure may not be a.c. path connected.

For a given space Y, define xRY to mean there is an a.c. path from x to y. We see immediately that R is an equivalence relation on Y. The resulting equivalence classes are referred to as <u>a.c. path connected components</u> of Y. Consideration of the semi-regular topology reveals that each a.c. path connected component in Y is open (and therefore closed) if and only if each point of Y has an a.c. path connected neighborhood.

THEOREM 2. A space Y is a.c. path connected if and only if it is connected and each $y \in Y$ has an a.c. path connected neighborhood.

PROOF. If Y is an a.c. path connected space, then Y is connected by Theorem 1 and each $y \in Y$ has an a.c. path connected neighborhood by the remarks preceeding Theorem 2.

Conversely, the hypothesis and the remarks preceeding Theorem 2 show that the only a.c. path connected component of Y is Y itself. Therefore, Y is a.c. path connected.

THEOREM 3. Let Y be a space. If (a) each a.c. path connected component in Y is open or (b) if each point $y \in Y$ has an a.c. path connected neighborhood, then the a.c. path connected components of Y coincide with the usual components of Y.

PROOF. The remarks preceeding Theorem 2 show that conditions (a) and (b) are equivalent. So if we assume that each a.c. path connected component of Y is open, then each point of Y has an a.c. path connected neighborhood. In particular, the a.c. path connected component [y] is an a.c. path connected neighborhood of y. Thus, [y] is connected by Theorem 1. It follows that $[y] \subset C(y)$, where C(y) is the usual component of $y \in Y$. Since [y] is both open and closed in Y, [y] is both open and closed in C(y). But C(y) connected implies [y] = C(y).

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