Internat. J. Math. & Math. Sci.
Vol. 4 No. 4 (1981) 819-822

RESEARCH NOTES

CONTINUITY OF MULTIPLICATION OF DISTRIBUTORS

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(Received February 20, 1981)

ABSTRACT. In a reference book for distributions [1], it is shown that the multiplication $(u,f) \longmapsto uf$ on $C^{\infty} \times \mathcal{D}'$, as well as on $\mathcal{O}_M \times \mathcal{S}'$, is hypocontinuous. We show here that in both cases it is discontinuous.

KEY WORDS AND PHRASES. *Distribution, temperate distribution, dual space, strong topology, inductive limit.*

1980 MATHEMATICS SUBJECT CLASSIFICATION CODES. *Primary 46F10, Secondary 46A09.*

1. INTRODUCTION.

The discontinuity of multiplication on $C^{\infty} \times \mathcal{D}'$, seems to be part of general folklore, but no proof has yet been published. As far as we know the second result is new. The presented proofs are simple enough that they can be included in any future textbook on distributions.

THEOREM 1. Let β be the strong and σ the weak* topology, both on $\mathcal{D}'(R^n)$, and γ the usual topology on $C^{\infty}(R^n)$. Then the multiplication $(u,f) \longmapsto uf : C^{\infty}_{\gamma} \times \mathcal{D}'_{\beta} \longrightarrow \mathcal{D}'_{\sigma}$ is not jointly continuous.

PROOF. The family $\mathcal{F}$ of all increasing sequences of positive integers is a directed set under the induced product ordering from N^N. Let Γ be the directed product $N \times \mathcal{F}$, and let $d = \partial^n/\partial x_1 \partial x_2 \ldots \partial x_n$. For each $(m,s) \in \Gamma$, put $f_{m,s}(t) = (mt^{m+1})^{-1}$ if $t \in [1, \exp(m \cdot s(m))]$, with $f_{m,s}(t) = 0$ otherwise, and $F_{m,s} : \mathcal{D} \longrightarrow C : g \longmapsto s(m)^{-\frac{1}{2}} d^m g(0)$.

If $g_{m,s}$ is the Fourier transform of $\prod_{j=1}^{n} f_{m,s}(x_j)$, then the inequalities $\int_{-\infty}^{\infty} t^k f_{m,s}(t)dt \leq m^{-1}$, $k = 1, 2, \ldots, m-1$, $\int_{-\infty}^{\infty} t^m f_{m,s}(t)dt = s(m)$ imply, respectively, $\|d^k g_{m,s}\|_\infty \leq m^{-n}$ and $d^m g_{m,s}(0) = i^{nm} s(m)^n$, where i is the imaginary unit. Thus $\lim_{(m,s)\in\Gamma} g_{m,s} = 0$ in C_γ^∞. For any $\psi \in \mathcal{D}(R^n)$, which equals 1 in some neighborhood of the origin, $\lim_{(m,s)\in\Gamma} |(g_{m,s}F_{m,s})\psi| = \lim_{(m,s)\in\Gamma} |s(m)^{-\frac{1}{2}} d_m\, g_{m,s}(0)| = \lim_{(m,s)\in\Gamma} s(m)^{n-\frac{1}{2}} = +\infty$ and $\{g_{m,s}F_{m,s}\}_{(m,s)\in\Gamma}$ does not converge to 0 in $\mathcal{D}'_\sigma$.

It remains to show that $\{F_{m,s}\}$ converges to 0 uniformly on every set $\mathcal{B}$ bounded in $\mathcal{D}$. For each such $\mathcal{B}$, there exists $r \in \mathcal{F}$ such that $|d^m g(0)| \leq r(m)$ for all $m \in N$ and $g \in \mathcal{B}$. Choose $\varepsilon > 0$ and $s \in \mathcal{F}$ such that $s(m) > \varepsilon^{-2} r^2(m)$ for all $m \in N$. Then

$$|F_{m,s}(g)| = |s(m)^{-\frac{1}{2}}\, d^m g(0)| \leq s(m)^{-\frac{1}{2}} r(m) < \varepsilon \quad \text{for all } g \in \mathcal{B}.$$

In the sequel, we need a weight function $W(x) = (1 + |x|^2)^{\frac{1}{2}}$, and Hilbert spaces $H_k = \{f : R^n \to C;\ \|f\|_k^2 = \sum_{|\alpha| \leq k} \int_{R^n} |W^{k-|\alpha|} D^\alpha f|^2 dx < +\infty\}$, $k \in N$. The space $\mathcal{S}$ of rapidly decreasing functions equals the proj lim H_k. For every $p, q \in N$, the space $\mathcal{O}_{p,q} = \{u : R^n \to C;\quad f \to uf : H_p \to H_q$ continuous$\}$ equipped with the operator norm $\|\ \|_{p,q}$ is Banach. If $\mathcal{O}_q = \operatorname{ind\,lim}_{p\to\infty} \mathcal{O}_{p,q}$, then the space $\mathcal{O}_M$ of rapidly increasing functions equals proj lim $\mathcal{O}_q$, [4]. Finally, denote by $\|\ \|_\infty$ the supremum norm of $L^\infty(R^n)$ and by d_ε the dilation operator $(d_\varepsilon f)(x) = f(\varepsilon x)$.

LEMMA 1. For each $k \in N$ and multi-index $\alpha \in N^n$,

$$\lim_{\varepsilon\to 0+} \|\varepsilon^{|\alpha|} W^k(x) D^\alpha \exp(-|\tfrac{x}{\varepsilon}|^2)\|_\infty = \|D^\alpha \exp(-|x|^2)\|_\infty, \text{ where } |x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2.$$

PROOF. $\lim_{\varepsilon\to 0+} \|\varepsilon^{|\alpha|} W^k(x) D^\alpha \exp(-\frac{x}{\varepsilon}^2)\|_\infty =$

$$\lim_{\varepsilon\to 0+} \|W^k(x)\, d_{\frac{1}{\varepsilon}}\, D^\alpha \exp(-|x|^2)\|_\infty = \lim_{\varepsilon\to 0+} \|W^k(\varepsilon x) D^\alpha \exp(-|x|^2)\|_\infty = \|D^\alpha \exp(-|x|^2)\|_\infty.$$

LEMMA 2. If $p, q \in N$, $0 \leq q \leq p$, and $r = 1 + [\frac{1}{2}n]$, then there is a sequence $\{f_m\}$ in $\mathcal{S}(R^n)$ such that $\sup_m \|f_m\|_{p,q} \leq 1$ and

$$\lim_{m\to\infty} \|f_m(x) \exp(-|x|^2)\|_{q+r} = \infty.$$

PROOF. By Prop. 8 of [4] and Lemma 1, there exists $A > 0$ such that

$$\limsup_{\varepsilon \to 0+} \|\varepsilon^q \exp(-|\tfrac{x}{\varepsilon}|^2)\|_{p,q} \leq \limsup_{\varepsilon \to 0+} \varepsilon^q A \sum_{|\alpha|\leq q} \|W^{q-p|\alpha|}(x) D^\alpha \exp(-|\tfrac{x}{\varepsilon}|^2)\|_\infty =$$

$$A \sum_{|\alpha|=q} \|D^\alpha \exp(-|x|^2)\|_\infty .$$

Define $h_m(x) = m^{-q} \exp(-|mx|^2)$, $m \in N$, $x \in R^n$.

Then $S = \sup_m \|h_m\|_{p,q} < +\infty$. If we put $f_m = S^{-1} h_m$, $m \in N$, then

$$\sup_m \|f_m\|_{p,q} \leq 1 \text{ and } \|f_m(x) \exp(-|x|^2)\|^2_{q+r} =$$

$$S^{-2} m^{-2q} \sum_{|\alpha|\leq q+r} \int_{R^n} |W^{q+r-|\alpha|}(x) D^\alpha \exp(-(1+m^2)|x|^2)|^2 \, dx =$$

$$S^{-2} m^{-2q} (1+m^2)^{-\frac{1}{2}n} \sum_{|\alpha|\leq q+r} (1+m^2)^{|\alpha|} \int_{R^n} \left| W^{q+r-|\alpha|}((1+m^2)^{-\frac{1}{2}} x) D^\alpha \exp(-|x|^2) \right| dx.$$

Take a multi-index β such that $|\beta| = q + r$. Then

$$\limsup_{m \to \infty} \|f_m(x) \exp(-|x|^2)\|^2_{q+r} \geq$$

$$\limsup_{m \to \infty} S^{-2} m^{-2q} (1+m^2)^{-\frac{1}{2}n+q+r} \int_{R^n} \left| D^\beta \exp(-|x|^2) \right|^2 dx =$$

$$S^{-2} \int_{R^n} \left| D^\beta \exp(-|x|^2) \right|^2 dx \cdot \limsup_{m \to \infty} m^{-2q} (1+m^2)^{q+r-\frac{1}{2}n} = +\infty.$$

THEOREM 2. Let β be the strong and σ the weak* topology on $\mathcal{S}'(R^n)$. Then the multiplication $(u,f) \longmapsto (u,f) : \mathcal{O}_M \times \mathcal{S}'_\beta \longrightarrow \mathcal{S}'_\sigma$ is not jointly continuous.

PROOF. The polar P of the singleton $\{\exp(-|x|^2)\} \subset \mathcal{S}$ is a σ-neighborhood of 0 in $\mathcal{S}'$. If the multiplication was continuous, there would be neighborhoods of 0, $U \subset \mathcal{O}_M$ and $V \subset \mathcal{S}'_\beta$, such that $UV \subset P$. For some $q \in N$, there exists a a neighborhood G of 0 in $\mathcal{O}_q$ such that $G \cap \mathcal{O}_M \subset U$, and there exists a ball $B(\varepsilon)$ of radius ε about the origin in $\mathcal{O}_{q,p}$ such that $B(\varepsilon) \subset G$. Since $\mathcal{S} \subset \mathcal{O}_M$, Lemma 2 implies existence of a sequence $\{f_m\}$ in $B(\varepsilon) \cap \mathcal{O}_M$ such that

$$\lim_{m\to\infty} \|f_m(x)\, \exp(-|x|^2)\|_{q+r} = +\infty, \text{ where } r = 1 + [\tfrac{1}{2}n].$$

For any $g \in V$, we have $f_m g \in UV \subset P$, which implies $\left|g(f_m \exp(-|x|^2))\right| = \left|(f_m g)\, \exp(-|x|^2))\right| \le 1$. Hence $f_m \exp(-|x|^2)$ is contained in the polar V^o of V. Since V^o is bounded in $\mathcal{S}$, the sequence $\{f_m \exp(-|x|^2)\}$ is bounded in $\mathcal{S}$, too; i.e., $\sup_m \|f_m \exp(-|x|^2)\|_{q+r} < +\infty$, which is a contradiction.

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