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ON ELATIONS IN SEMI-TRANSITIVE PLANES

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ABSTRACT. Let π be a semi-transitive translation plane of even order with reference to the subplane π_0. If π admits an affine elation fixing π_0 for each axis in π_0 and the order of π_0 is not 2 or 8, then π is a Hall plane.

KEY WORDS AND PHRASES. *Elations, Semi-transitive planes.*

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1. INTRODUCTION.

Kirkpatrick [9] and Rahilly [10] have characterized the Hall planes as those generalized Hall planes of order q^2 that admit $q+1$ central involutions.

In [7] the author has shown that the derived semifield planes of characteristic $\neq 3$ and order q^2 are Hall planes precisely when they admit $q+1$ central involutions. This extends Kirkpatrick and Rahilly's work as generalized Hall planes are certain derived semifield planes.

If a translation plane π of order q^2 admits $q+1$ affine elations with distinct axes then the generated group $\mathcal{G}$ contains $SL(2,q)$, $S_Z(q)$ or contains a normal subgroup N of odd order and index 2 (Hering [5]). In the latter case, little is known about $\mathcal{G}$ except that it is usually dihedral.

In this article, we study semi-transitive translation planes of order q^2 that admit $q+1$ affine elations.

In [8], the author introduces the concept of the generalized Hall planes of type i. These are derivable translation planes that admit a particular collineation group which is transitive on the components outside the derivable net. In this situation the group is generated by Baer collineations.

More generally, Jha [6] has considered the "semi-transitive" translation planes.

(1.1) Let π be a translation plane with subplane π_0. If there is a collineation group $\mathcal{S}$ such that

1) $\mathcal{S}$ fixes $\pi_0 \cap \ell_\infty$ pointwise,

2) leaves π_0 invariant, and

3) acts transitively on $\ell_\infty - \pi_0 \cap \ell_\infty$,

then π is said to be a <u>semi-transitive translation plane with reference to</u> π_0 <u>and with respect to</u> $\mathcal{S}$.

Our main result is that semi-transitive planes of order not 16 or 64 that admit elations with axis $\mathcal{L}$ fixing π_0 for every component $\mathcal{L}$ of π_0 are Hall planes. We also give a necessary and sufficient condition that a translation plane of order $q^2 \neq 64$ admitting $q+1$ elations with distinct axes is derivable.

2. TRANSLATION PLANES OF EVEN ORDER q^2 ADMITTING $q+1$ ELATIONS.

(2.1) THEOREM. Let π be a translation plane of even order $q^2 \neq 64$ that admits $q+1$ affine elations with distinct axes. Let $\mathcal{N}$ denote the net of degree $q+1$ that is defined by the elation axes and assume the group D generated by these elations leaves $\mathcal{N}$ invariant. Then $\mathcal{N}$ is derivable if and only if D is either isomorphic to $SL(2,q)$ or is dihedral of order $2(q+1)$ where the cyclic stem fixes at least two components.

PROOF. If D is isomorphic to $SL(2,q)$ then $\mathcal{N}$ is derivable and actually π is Desarguesian by Foulser-Johnson-Ostrom [3].

Let $D = \langle \sigma, \chi \mid \sigma^2 = \chi^{q+1} = 1, \sigma\chi = \chi^{-1}\sigma \rangle$. If $\langle \chi \rangle$ fixes the components $X = \mathcal{O}$, $Y = \mathcal{O}$ then we may choose coordinates so that σ is $(x,y) \rightarrow (y,x)$ and χ is $(x,y) \rightarrow (xT, yT^{-1})$ for some matrix T of order $q+1$.

By Ostrom [11], Theorem 3, there is a Desarguesian plane Σ containing the two χ-fixed components and $\mathcal{N}$. Clearly $\mathcal{N}$ is an André net in Σ and thus derivable in π.

Conversely, suppose $\mathcal{N}$ is derivable. Since each elation fixes $\mathcal{N}$, D must fix each Baer subplane of $\mathcal{N}$ incident with $\mathcal{O}$. By Foulser [2], Theorem 3, $D \le GL(2,q)$ in its action on π so that $D \le SL(2,q)$ (each elation is then in $SL(2,q)$). By Gleason [4], D is transitive on the elation axes so $q+1 \mid |D|$. Thus, D is clearly $SL(2,q)$ or is dihedral of order $2(q+1)$. Moreover, if $\mathcal{N}$ is derivable then χ fixes at least two infinite points of $\pi - \mathcal{N}$. Let $\overline{\mathcal{N}}$ replace $\mathcal{N}$ so $\mathcal{G}$ fixes $\overline{\mathcal{N}}$ componentwise in the derived plane $\overline{\pi}$. Let $\langle\overline{\chi}\rangle \lhd \langle\chi\rangle$ such that $|\overline{\chi}|$ is a prime 2-primitive divisor of q^2-1 (one exists since $q^2 \ne 64$). Then $\overline{\chi}$ fixes at least two infinite points of $\overline{\pi} - \overline{\mathcal{N}}$ so there is a unique Desarguesian plane Σ containing the $\overline{\chi}$-fixed components of $\overline{\pi}$ (see Ostrom [11], Cor. to Theorem 1—uniqueness comes from the fact that the degree of $\Sigma \cap \overline{\pi}$ is greater than $q+1$). Since $\mathcal{G}$ permutes the components of $\Sigma \cap \overline{\pi}$ (i.e., $\langle\overline{\chi}\rangle$ is characteristic in $\langle\chi\rangle$), $\mathcal{G}$ is a collineation group of Σ. The collineation χ has the form $(x,y) \longrightarrow (x^{\phi}a, y^{\phi}a)$ where ϕ is an automorphism of $GF(q^2)$ and $a \in GF(q^2)$. (Note χ fixes $\overline{\mathcal{N}}$ componentwise.) Since $q+1$ is odd, $\langle\chi^2\rangle = \langle\chi\rangle$. Choosing coordinates so that the components of $\overline{\mathcal{N}}$ are $X = \mathcal{O}$, $Y = \mathcal{O}$, $y = x\alpha$, $\alpha \in GF(q^2)$ then χ fixes $y = x\alpha$ for all $\alpha \in GF(q^2)$ if and only if $\alpha^{\phi} = \alpha$. Since $\langle\chi^2\rangle = \langle\chi\rangle$, we may assume $\phi = 1$. Thus, χ fixes ℓ_∞ of Σ pointwise. Since Σ and π share at least two components (those fixed by $\overline{\chi}$), χ must fix at least two components of π.

3. SEMI-TRANSITIVE TRANSLATION PLANES OF EVEN ORDER.

Let π be a translation plane of even order q^2 that admits $q+1$ elations as in section 2. Then, π is a derivable plane provided the generated group D is dihedral and the cyclic stem fixes at least 2 points or $SL(2,q)$. In any case let $\mathcal{N}$ denote the net defined by the elation axes. Let $\mathcal{G}$ be a collection group that commutes with D. Then clearly, $\mathcal{G}$ must fix $\mathcal{N} \cap \ell_\infty$ pointwise.

(3.1) THEOREM. Let π be a translation plane of even order $q^2 \ne 64$ that admits $q+1$ elations with distinct axes. Assume the group D generated by these

$q+1$ elations leaves the net $\mathcal{N}$ of the elation axes invariant. Let $\mathcal{G}$ be a collineation group which commutes with D and is transitive on $\ell_\infty - \mathcal{N} \cap \ell_\infty$. Then π is a Hall plane.

PROOF. Since $q^2 \neq 64$, there is a prime 2-primitive divisor m of q^2-1. By Gleason [4], $q+1 \,\big|\, |D|$. Clearly, $m \,\big|\, q+1$. Let χ be an element of D of order m. χ acts on the $q(q-1)$ points of $\ell_\infty - \mathcal{N} \cap \ell_\infty$ so must fix at least two points of $\ell_\infty - \mathcal{N} \cap \ell_\infty$. Since $\mathcal{G}$ commutes with χ and $\mathcal{G}$ is transitive on $\ell_\infty - \mathcal{N} \cap \ell_\infty$, χ must fix $\ell_\infty - \mathcal{N} \cap \ell_\infty$ pointwise.

By the corollary to Theorem 1, Ostrom [11], there is a Desarguesian plane Σ such that the components fixed by χ in π are exactly the common components of Σ and π. Let $\pi = \mathcal{N} \cup \mathcal{M}$ where $\mathcal{M}$ is the net complementary to $\mathcal{N}$ in π. Then $\Sigma = \overline{\mathcal{N}} \cup \mathcal{M}$ for some net $\overline{\mathcal{N}}$ of degree $q+1$. So Σ and π are two extensions of a net $\mathcal{M}$ of critical deficiency (see Ostrom [12]). Then π must be Hall since Σ and π must be related by derivation (i.e., π cannot be itself Desarguesian) by Ostrom [12].

The conditions of (3.1) are close to giving the definition of a "semi-transitive" translation plane (see (1.1)). In (3.1), it is possible that $\mathcal{G}$ may not satisfy condition 2. Also, it is not clear that a semi-transitive translation plane is derivable. However, Jha [6] shows if π has order not 16 and there is a nontrivial kern homology in π then π is derivable and π_0 is a Baer subplane.

We may overcome this restriction on the kern in our situation:

(3.2) THEOREM. Let π be a semi-transitive translation plane of even order with respect to a collineation group $\mathcal{G}$ and with reference to a subplane π_0. Let π admit an affine elation for each axis in π_0.

1) If the order of π_0 is not 8 then π is derivable.

2) If the order of π_0 is not 2 or 8 then π is a Hall plane.

PROOF. Following Jha's [6] ideas, let π_1 be a minimal subplane of π properly containing π_0. Clearly, the stabilizer $\mathcal{G}_{\pi_1}$ of π_1 is a semi-transitive collineation group of π_1 with reference to π_0. Moreover, a sylow 2-subgroup of $\mathcal{G}_{\pi_1}$ must leave π_0 pointwise fixed since $\mathcal{G}$ fixes π_0 and fixes $\pi_0 \cap \ell_\infty$ pointwise. (Note $|\mathcal{G}_{\pi_1}|$ is divisible by $(2^r+1)-(2^s+1)$ for some r,s.) Clearly, π_0 is a Baer subplane of π_1.

Every elation which leaves π_0 invariant must also leave any superplane invariant. So the group D generated by the elations leaves π_1 invariant and, clearly, $\mathcal{S}$ commutes with D since $\mathcal{S}$ fixes $\pi_0 \cap \ell_\infty$ pointwise ($\mathcal{S}$ must commute with each central collineation fixing π_0).

By (3.1), if the order of π_0 is not 8 then π_1 is a Hall plane and π_1 is derivable. We may now directly use Jha [6] to show that if the order of π_0 is not 2 then $\pi_1 = \pi$ (that is, Jha uses the hypothesis that there is a kern homology to show that π_1 is derivable).

Actually, our proof of (3.2) proves the following more general theorem for arbitrary order.

(3.3) THEOREM. Let π be a semi-transitive translation plane with reference to π_0 and with respect to $\mathcal{S}$ and order p^r. Let χ be a collineation generated by central collineations leaving π_0 invariant such that $|\chi|$ is a prime p-primitive divisor of (order $\pi_0)^2$-1 (where the order of π_0 is not 2). Then π is a Hall plane.

Note that a semi-transitive plane of odd order p^{2r} must admit Baer p-elements (see Jha [6]). By Foulser [1], we could then not have both Baer p-elements and elations so we could restate our Theorem (3.2) without reference to order.

(3.2)2) is also valid if the order π_0 is 8. The arguments supporting this will appear in a related article.

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