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ON BELLMAN-BIHARI INTEGRAL INEQUALITIES

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ABSTRACT. Integral inequalities of the Bellman-Bihari type are established for integrals involving an arbitrary number of independent variables.

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1. INTRODUCTION.

In a number of recent papers, Dhongade and Deo [1] and Pachpatte [2,3,4] have generalized the well known Bellman inequality [5] and Bihari's generalization of it [6] in several different directions. Although the results concern only functions of a single variable, it was shown in [7] that corresponding inequalities also hold for functions of several independent variables. The purpose of this note is to show that the technique employed in [7] can be profitably utilized to establish more general integral inequalities of the Bellman-Bihari type in any number of independent variables. We present here some of the results along this line.

As in [7] we assume that all the functions under discussion are defined in a bounded domain R of E^n which, for convenience, is assumed to contain the origin. The symbol $x < y$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are any two points of R, means $x_i < y_i$ for $i = 1, \ldots, n$. We also adopt the notation

$$\int_0^x f(s)ds = \int_0^{x_n}\cdots\int_0^{x_1} f(s_1,\ldots,s_n)ds_1..ds_n$$

2. MAIN RESULTS.

Our first result is a variation of Theorem 3 of [7].

THEOREM 1. Let u, f, and g be continuous and nonnegative in R and let a be continuous, positive and nondecreasing in R. Let $W: [0,\infty) \to [0,\infty)$ be continuously differentiable and nondecreasing such that

$$v^{-1}W(u) \leq W(v^{-1}u), \quad u \geq 0, \; v > 0 \tag{2.1}$$

Then the inequality

$$u(x) \leq a(x) + \int_0^x f(s)[u(s) + \int_0^s g(t)W(u)dt]ds \tag{2.2}$$

implies

$$u(x) \leq a(x)[1 + \int_0^x f(s)G^{-1}(G(1) + \int_0^s f(t)dt)ds] \tag{2.3}$$

if $g(x) \leq f(x)$ or

$$u(x) \leq a(x)[1 + \int_0^x f(s)G^{-1}(G(1) + \int_0^s g(t)dt)ds] \tag{2.4}$$

if $f(x) \leq g(x)$, where G^{-1} is the inverse of the function

$$G(w) = \int_{w_o}^{w} \frac{dr}{r+W(r)}, \quad w > w_o > 0 \tag{2.5}$$

provided $G(1) + \int_0^x f(t)dt$ lies in the domain of G^{-1}.

PROOF. Since $a > 0$, $W \geq 0$ and both are nondecreasing, and by (2.1), we may rewrite (2.2) in the form

$$m(x) \leq 1 + \int_0^x f(s)[m(s) + \int_0^s g(t)W(m)dt]ds \tag{2.6}$$

where $m(x) \leq u(x)/a(x)$. If we set $v(x)$ equal to the right hand side of (2.6) and differentiate, we find

$$D_1 \ldots D_n v(x) = f(x)(m(x) + \int_0^x g(t)W(m)dt) \tag{2.7}$$

$$\leq f(x)(v(x) + \int_0^x g(t)W(v)dt)$$

where D_i indicates differentiation with respect to x_i, $i = 1,\ldots,n$.

Let us define

$$w(x) = v(x) + \int_0^x g(t)W(v)dt \tag{2.8}$$

and assume $g(x) \leq f(x)$. Then, by differentiating (2.8) and using (2.7), we obtain

$$\begin{aligned} D_1 \ldots D_n w(x) &= D_1 \ldots D_n v(x) + g(x)W(v) \\ &\leq f(x)w(x) + g(x)W(w) \\ &\leq f(x)(w(x) + W(w)) \end{aligned} \tag{2.9}$$

Set $S(x) = w(x) + W(w)$. Following the technique in [7], we observe from (2.9) that

$$\frac{S(x)D_1 \ldots D_n w(x)}{S(x)^2} \leq f(x) + \frac{D_1 S(x) D_2 \ldots D_n w(x)}{S(x)^2}$$

or

$$D_1\left(\frac{D_2 \ldots D_n w(x)}{S(x)}\right) \leq f(x)$$

Note that, from the hypotheses, it follows that $D_i(w(x) + W(w)) \geq 0$, for $i = 1,2,\ldots,n$. Hence, integrating with respect to x_1 from 0 to x_1, we find

$$\frac{D_2 \ldots D_n w(x)}{S(x)} \leq \int_0^{x_1} f(s_1,x_2,\ldots,x_n)ds_1 \tag{2.10}$$

Similarly, since

$$\frac{D_2 S(x)(D_3 \ldots D_n w(x))}{S(x)^2} \geq 0$$

the left hand side of (2.10) can be replaced by

$$D_2\left(\frac{D_3 \ldots D_n w(x)}{S(x)}\right) \leq \int_0^{x_1} f(s_1,x_2,\ldots,x_n)ds_1$$

By integrating this from 0 to x_2, we obtain

$$\frac{D_3 \ldots D_n w(x)}{S(x)} \leq \int_0^{x_2} \int_0^{x_1} f(s_1,s_2,x_3,\ldots,x_n)ds_1 ds_2$$

Continuing in this manner, we have after (n-1) steps

$$\frac{D_n w(x)}{S(x)} \leq \int_0^{x_{n-1}} \cdots \int_0^{x_1} f(s_1,\ldots,s_{n-1},x_n)ds_1 \ldots ds_{n-1} \tag{2.11}$$

With the function $G(w)$ defined in (2.5), we note that $D_nG(w) = G'(w)D_nw(x) = D_nw(x)/(w(x) + W(w))$. Hence, integration of (2.11) from 0 to x_n yields

$$G(w(x_1,\dots,x_n)) - G(w(x_1,\dots,x_{n-1},0)) \leq \int_0^x f(s)\,ds$$

or

$$w(x) \leq G^{-1}\left(G(1) + \int_0^x f(s)ds\right) \tag{2.12}$$

since $w(x) = v(x) = 1$ when $x_i = 0$ for any i, $1 \leq i \leq n$.

From (2.7) and (2.8) we have

$$D_1\dots D_nv(x) \leq f(x)w(x) \tag{2.13}$$

Substituting for $w(x)$ from (2.12) and integrating (2.13), we finally obtain

$$v(x) \leq 1 + \int_0^x f(s)G^{-1}\left(G(1) + \int_0^s f(t)dt\right)ds \tag{2.14}$$

The inequality (2.3) follows from (2.6), (2.14), and the fact that $m(x) = u(x)/a(x)$.

If $f(x) \leq g(x)$, then we need only replace f by g in the last line of (2.9) to obtain again (2.12) with f replaced by g. The result (2.4) then follows in the same fashion.

Our next theorem combines the feature of Theorems 1 and 2 of [7].

THEOREM 2. Let u, f, g, and h be continuous and nonnegative functions in R, and let a be continuous, positive, and nondecreasing in R. Let $Z: [0,\infty) \to [0,\infty)$ satisfy the same conditions as W in Theorem 1 such that Z is submultiplicative. If u satisfies

$$u(x) \leq a(x) + \int_0^x f(s)\left[u(s) + \int_0^s g(t)u(t)dt\right]ds + \int_0^x h(s)Z(u)ds \tag{2.15}$$

then

$$u(x) \leq a(x)p(x)H^{-1}\left(H(1) + \int_0^x h(s)Z(p)ds\right) \tag{2.16}$$

where

$$p(x) = 1 + \int_0^x f(s)\exp\int_0^s (f(t) + g(t))dtds \tag{2.17}$$

and H^{-1} is the inverse of the function

$$H(v) = \int_0^v \frac{dr}{Z(r)}, \quad v > v_o > 0 \tag{2.18}$$

The proof of this theorem makes use of the following result which we state as a lemma. This was established in [7] as Theorem 1.

LEMMA. Under the hypotheses of Theorem 2, the inequality

$$u(x) \leq a(x) + \int_0^x f(s)[u(s) + \int_0^s g(t)u(t)dt]ds$$

implies

$$u(x) \leq a(x)[1 + \int_0^x f(s)\exp \int_0^s (f(t) + g(t))dtds].$$

PROOF of Theorem 2. As in Theorem 1 we rewrite (2.15) in the form

$$m(x) \leq 1 + \int_0^x f(s)[m(s) + \int_0^s g(t)m(t)dt]ds + \int_0^x h(s)Z(m)ds \tag{2.19}$$

If we set

$$v(x) = 1 + \int_0^x h(s)Z(m)ds \tag{2.20}$$

then (2.19) becomes

$$m(x) \leq v(x) + \int_0^x f(s)[m(s) + \int_0^s g(t)m(t)dt]ds.$$

Hence, by the lemma, we have

$$\begin{aligned} m(x) &\leq v(x)(1 + \int_0^x f(s)\exp \int_0^s (f(t) + g(t))dtds) \\ &\leq v(x)p(x) \end{aligned} \tag{2.21}$$

Since Z is submultiplicative, we note that $Z(m) \leq Z(v)Z(p)$. Therefore, differentiating (2.20) with respect to $x_1,\ldots,x_n$, we find

$$\begin{aligned} D_1\ldots D_n v(x) &= h(x)Z(m) \\ &\leq h(x)Z(v)Z(p) \end{aligned}$$

or
$$\frac{D_1 \dots D_n v(x)}{Z(v)} \leq h(x)Z(p) \tag{2.22}$$

By the same argument as in the proof of Theorem 1, we can integrate (2.22) to obtain

$$H(v(x_1,\dots,x_n)) - H(v(x_1,\dots,x_{n-1},0)) \leq \int_0^x h(s)Z(p)ds$$

where H(v) is defined by (2.18). This gives

$$v(x) \leq H^{-1}(H(1) + \int_0^x h(s)Z(p)ds) \tag{2.23}$$

The substitution of (2.23) in (2.21) yields the inequality (2.16) since $m(x) = u(x)/a(x)$.

When $g(x) = 0$, Theorem 2 reduces to Theorem 3 of [7].

By combining Theorems 1 and 2, we finally have

THEOREM 3. Let u, a, f, g, h, and Z be as in Theorem 2 and let W be as in Theorem 1. If u satisfies

$$u(x) \leq a(x) + \int_0^x f(s)[u(s) + \int_0^s g(t)W(u)dt]ds \tag{2.24}$$

$$+ \int_0^x h(s)Z(u)ds, \text{ where } g(x) \leq f(x)$$

then

$$u(x) \leq a(x)q(x)H^{-1}(H(1) + \int_0^x h(s)Z(q)ds) \tag{2.25}$$

where

$$q(x) = 1 + \int_0^x f(s)G^{-1}(G(1) + \int_0^s f(t)dt)ds \tag{2.26}$$

G^{-1} is the inverse of the function defined in (2.5) and H^{-1} is the inverse of the function defined in (2.18).

PROOF. We rewrite (2.24) in the form

$$m(x) \leq v(x) + \int_0^x f(s)[m(s) + \int_0^s g(t)W(m)dt]ds \tag{2.27}$$

where

$$v(x) = 1 + \int_0^x h(s)Z(m)ds \tag{2.28}$$

with $m(x) = u(x)/a(x)$. Then according to Theorem 1, we have

$$m(x) \leq v(x)[1 + \int_0^x f(s)G^{-1}(G(1) + \int_0^s f(t)dt)ds] \tag{2.29}$$

$$\leq v(x)q(x)$$

Since $Z(m) \leq Z(v)Z(q)$, we obtain from (2.28)

$$D_1 \dots D_n v(x) = h(x)Z(m) \leq h(x)Z(v)Z(q)$$

With $H(v)$ defined by (2.18), we obtain as in the proof of Theorem 2

$$v(x) \leq H^{-1}(H(1) + \int_0^x h(s)Z(q)ds)$$

The substitution of this for $v(x)$ in (2.29) leads to the desired inequality (2.25).

Observe that, when $h(x) = 0$, (2.25) reduces to (2.3); when $W = u$, it agrees with (2.16) with g replaced by f in view of the condition $g \leq f$.

We remark that our Theorems 1, 2, and 3 correspond respectively to Theorems 4, 2, and 5 of [4]. From the argument presented above, we readily see that other more general integral inequalities can also be established for n independent variables along the lines considered in [1] and [4].

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