AXIAL SHEAR MODULUS OF A FIBER-REINFORCED COMPOSITE WITH RANDOM FIBER CROSS-SECTIONS

S.K. BOSE

Department of Mathematics
Regional Engineering College
Durgapur-713209, India
and
L. DEBNATH

Mathematics and Physics Departments
East Carolina University
Greenville, North Carolina 27834, U.S.A.

ABSTRACT. A study is made of the effective axial shear modulus of a fiber rein-
forced material with random fiber cross-sections so that tne micromechanics is
governed by stochastic differential equations. A coarse-graining procedure is
adopted to investigate the macroscopic behavior of the material. This analysis
leads to the formula for the effective axial shear modulus,

$$\mu^* = \mu_1 / \{ 1 - 2c \ (\mu_2 - \mu_1)/(\mu_2 + \mu_1)\} \ ,$$

where μ_1 and μ_2 are the shear modulus of the matrix and fibers respectively and c
is the concentration of the fibers less that 0.5. For c > 0.5, the fiber and
matrix moduli are to be interchanged and c is to be replaced by 1-c. The results
of this study are compared with those of the theory of fibre reinforced materials.
Finally, a numerical example is presented with graphical representation.

KEY WORDS AND PHRASES. Axial shear modulus, Fiber-Reinforced composites, Random
distributions, Random cross-sections, Coarse-graining, Bounds on moduli.

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1. INTRODUCTION.

From the view point of statistical continuum mechanics, a few theories have

been developed for the effective viscosity and the effective thermal conductivity
in random suspensions. Einstein's formula [1] is well known as the first theo-
retical result for the effective viscosity in dilute suspensions of shperical
particles. In recent years, Beran [2] and Miller [3] have used perturbation meth-
ods combined with Green's functions to derive the expression for the effective
thermal conductivity in suspensions. On the other hand, Hori [4], Hori and
Yonezawa [5-8] have obtained the rigorous formula for the effective thermal con-
ductivity in a random medium using diagrammatic expansions from a field-theoretical
view point. In spite of this progress, no results are available for the effective
viscosity in more concentrated suspensions.

In a very recent paper, Nagatani [9] has presented a general theory of the
macroscopic motions of slow viscous flows in suspensions from the view point of
statistical continuum mechanics. With the assumption that the characteristic
length scale is much larger than the size of the suspended particles, he derived
the coarse-grained equations from the governing equations using the formal pertur-
bation methods and coarse-graining procedures. The macroscopic motions on much
larger scale than the size of the suspended particles. It has been shown that this
formula reduces to the Einstein formula for dilute suspensions. Nagatani's formula
for the effective viscosity has a simple closed form and has been found to agree
well with experimental results.

In the derivation of the formula, the suspension with its variable physical
properties on a microscipic scale, is governed by stochastic differential equations.
The macroscopic behavior of the system could be obtained by ensemble averaging the
equations. However, this procedure is usually difficult and an alternative "coarse-
graining" procedure is adopted. The coarse-graining means downgrading the spatial
resolution to a much larger scale than that of a particle. This is achieved by
looking at the particles as random perturbations and representing the field vari-
ables by Fourier series in the space co-ordinates. Modes with low wave numbers
(lower than the reciprocal of the size of a particle) are retained while those
with high wave numbers are repeatedly eliminated through a formal perturbation
technique. This yields the expanded form of the effective viscosity, which when

formally resumed gives the formula.

We adopt this procedure to determine the effective axial shear modulus of a random composite, with unidirectional fibers. This analysis leads to the formula for the effective axial shear modulus, $\mu^* = \mu_1/ \{ 1 - 2c\ (\mu_2 - \mu_1)/(\mu_2 + \mu_1)\}$ where μ_1 and μ_2 are the shear moduli of the matrix and fibers respectively and c is the concentration of the fibers less than 0.5. The fibers with their random cross-sections are looked upon as random perturbations over the matrix material, and as such the concentration c of the fibers can not exceed 0.5 (if c exceeds 0.5 we can interchange the roles of the fibers and the matrix) - a range which is important from the practical point of view. For c = 0, it reduces to the shear modulus of the matrix and for c = 0.5 it yields the mean of the fiber and matrix moduli. This shows the consistency of the result, since the fiber geometry plays no role in the method, and is random. An important feature of the formula is its simple structure and can be of practical use. Even when the fibers are completely identical (which is difficult to ensure in practice) the formula would be approximately true.

The formula is compared with the bounds obtained by Hashin [10] and it is shown to exceed his lower bound. For higher values of the concentration, it, however, exceeds his upper bound also.

2. BASIC STOCHASTIC EQUATIONS ON THE MICROSCOPIC SCALE.

We consider a cylindrical specimen of the material with side surface parallel to the fibers. Let the specimen be in a state of shear in the axial direction. If we introduce co-ordinate axes OX_1 and OX_2 in a cross-section, the shear stress T_j on it is given by

$$T_j = \mu\ \frac{\partial w}{\partial x_j}\ ,\qquad (j = 1,2) \qquad\qquad (2.1)$$

where $w(\underline{x})$, $\underline{x} = (x_1, x_2)$ is the accompanying displacement in the axial direction and μ is the shear modulus of the material. If μ_1 and μ_2 are respectively the shear moduli of the materials of the matrix and the fibers, μ takes these values in the two phases. Looking upon the fibers as random perturbations over the matrix,

we write

$$\mu = \mu_1 + (\mu_2 - \mu_1) \, \xi(\underset{\sim}{x}) \qquad (2.2)$$

where

$$\xi(\underset{\sim}{x}) = 0, \text{ in the matrix}$$
$$= 1, \text{ in the fibers.} \qquad (2.3ab)$$

The function $\xi(\underset{\sim}{x})$ is a random variable with mean

$$\langle \xi \rangle = \frac{1}{S} \int_S \xi(\underset{\sim}{x}) \, d\underset{\sim}{x} = c \qquad (2.4)$$

where S is the area of the cross-section and c is the 'concentration' of the fibers.

In the statistical analysis of the problem, we shall also encounter the mean of

$\{\xi(\underset{\sim}{x})\}^n$. Noting from equation (2.3) that its value is that of $\xi(\underset{\sim}{x})$, we also have

$$\langle \xi^n \rangle = c \qquad (2.5)$$

The stress equation of equilibrium of the material, with equation (2.1)

becomes

$$\frac{\partial}{\partial x_j} \left(\mu \, \frac{\partial w}{\partial x_j} \right) = 0 \qquad (2.6)$$

which apparantly is a stochastic differential equation. On the microscale the

displacement and stresses must be continuous across the interfaces of the two

phases.

3. COARSE-GRAINED EQUATIONS.

In order to obtain the macroscopic behavior from the stochastic equations

(2.1) - (2.6), we subject them to the coarse-graining procedure. This envisages

downgrading the spatial resolution to a scale much larger than the cross-section of

a fiber. To this end we represent the variables in equations (2.1) - (2.6) in

Fourier series:

$$T_j = S^{-\frac{1}{2}} \sum_{\underset{\sim}{k}} U_{j\underset{\sim}{k}} \, e^{i\underset{\sim}{k} \cdot \underset{\sim}{x}} \qquad (3.1)$$

$$w = S^{-\frac{1}{2}} \sum_{\underset{\sim}{k}} W_{\underset{\sim}{k}} \, e^{i\underset{\sim}{k} \cdot \underset{\sim}{x}} \qquad (3.2)$$

$$\xi = S^{-\frac{1}{2}} \sum_{\underset{\sim}{k}} X_{\underset{\sim}{k}} \, e^{i\underset{\sim}{k} \cdot \underset{\sim}{x}} . \qquad (3.3)$$

The Fourier components in the above are given by

$$U_{jk} = S^{-\frac{1}{2}} \int_S T_j \, e^{-i k \cdot x} \, dx$$

$$W_k = S^{-\frac{1}{2}} \int_S w \, e^{-i k \cdot x} \, dx$$

$$X_k = S^{-\frac{1}{2}} \int_S \xi \, e^{-i k \cdot x} \, dx$$

If we use equations (2.2), (3.1), (3.2) and (3.3) in (2.1), we obtain

$$U_{jk} = \mu_1 i k_j W_k + (\mu_2 - \mu_1) S^{-\frac{1}{2}} \sum_{k'} i(k_j - k_j') X_{k'} W_{k-k'} \, .$$

It is apparent that, in the summation on the right hand side, all the wave numbers k' contribute to the component U_{jk}. If from this summation we separate out the term corresponding to the lowest wave number $k' = 0$, we get

$$U_{jk} = i k_j [\mu_1 + (\mu_2 - \mu_1) \langle \xi \rangle] W_k$$
$$+ (\mu_2 - \mu_1) S^{-\frac{1}{2}} \sum_{k' \neq 0} i(k_j - k_j') X_{k'} W_{k-k'} \, , \qquad (3.4)$$

where we have used the relation

$$X_0 = S^{\frac{1}{2}} \langle \xi \rangle . \qquad (3.5)$$

Adopting the same procedure for equation (2.6) with (2.2), (3.2) and (3.3), we get

$$W_k = - \frac{\mu_2 - \mu_1}{\mu_1 k^2} S^{-\frac{1}{2}} \sum_{k'} k_m(k_m - k_m') X_{k'} W_{k-k'} \, . \qquad (3.6)$$

In the coarse-graining procedure we repeatedly use equation (3.6) in the right hand side of (3.4) and remove the term corresponding to the high wave numbers $k' \neq 0$ at each step. Physically this means downgrading the spatial resolution of the composite. As the wave numbers should occur with the diameter of a fiber in a natural way, the resolution would be on a scale much larger that the cross-section of a fiber. At the first step we have

$$U_{j\underline{k}} = ik_j \, [\mu_1 + (\mu_2 - \mu_1) \, \langle \xi \rangle \,] \, W_{\underline{k}}$$

$$- \frac{(\mu_2 - \mu_1)^2}{\mu_1} \, s^{-1} \, W_{\underline{k}} \cdot \sum_{\underline{k}' \neq 0} ik_m \, \frac{(k_j - k_j')(k_m - k_m')}{(\underline{k} - \underline{k}')^2} \, X_{\underline{k}'} \, X_{-\underline{k}'}$$

$$- \frac{(\mu_2 - \mu_1)^2}{\mu_1} \, s^{-1} \, \sum_{\underline{k}' \neq 0} \sum_{\underline{k}'' \neq -\underline{k}'} \frac{i(k_j - k_j')(k_m - k_m')(k_m - k_m' - k_m'')}{(\underline{k} - \underline{k}')^2} \, X_{\underline{k}'} \, X_{\underline{k}''} \, W_{\underline{k} - \underline{k}' - \underline{k}''}$$

$$\tag{3.7}$$

If we assume that the composite is statistically isotropic, the second order tensor $(k_j - k_j')(k_m - k_m')$ is approximately an isotropic tensor of order two with

$$\frac{(k_j - k_j')(k_m - k_m')}{(\underline{k} - \underline{k}')^2} \approx \frac{1}{2} \, \delta_{jm}$$

The second term on the right hand side of (3.7) thus becomes

$$- \frac{(\mu_2 - \mu_1)^2}{\mu_1} \, s^{-1} \, W_{\underline{k}} \, ik_j \, [\, \frac{1}{2} \sum_{\underline{k}'} X_{\underline{k}'} \, X_{-\underline{k}'} - X_0^2 \,]$$

$$= - \frac{(\mu_2 - \mu_1)^2}{\mu_1} \, ik_j \, W_{\underline{k}} \, [\, \frac{1}{2} \langle \xi^2 \rangle - \langle \xi \rangle^2 \,] \, , \tag{3.8}$$

where we have used the relation (3.5) and the equation (3.3) to obtain

$$\sum_{\underline{k}'} X_{\underline{k}'} \, X_{\underline{k} - \underline{k}'} = \int_S \xi^2 \, (\underline{x}) \, e^{-i\underline{k} \cdot \underline{x}} \, d\underline{x}$$

This leads to the result

$$\sum_{\underline{k}'} X_{\underline{k}'} \, X_{-\underline{k}'} = s \, \langle \xi^2 \rangle \qquad \text{for} \quad \underline{k} = 0$$

At the next step of the coarse-graining procedure, we again replace $W_{\underline{k} - \underline{k}' - \underline{k}''}$ by the approximate expression from (3.6) and remove the term corresponding to $\underline{k}' \neq 0$. The contribution of this term is

$$\frac{(\mu_2 - \mu_1)^3}{\mu_1^2} \, ik_j \, W_{\underline{k}} \, [\, \frac{1}{4} \langle \xi^3 \rangle - \langle \xi \rangle \langle \xi^2 \rangle + \langle \xi \rangle^3 \,] . \tag{3.9}$$

Continuing this process, we get from (3.7)

$$U_{j\underline{k}} = \mu^* \, ik_j \, W_{\underline{k}} \, , \tag{3.10}$$

where

$$\mu^* = \mu_1 \left[1 + \frac{\mu_2 - \mu_1}{\mu_1} \langle \xi \rangle - \frac{(\mu_2 - \mu_1)^2}{\mu_1^2} \{ \frac{1}{2} \langle \xi^2 \rangle - \langle \xi \rangle^2 \} \right.$$

$$\left. + \frac{(\mu_2 - \mu_1)^3}{\mu_1^3} \{ \frac{1}{4} \langle \xi^3 \rangle - \langle \xi \rangle \langle \xi^2 \rangle + \langle \xi \rangle^3 \} - \ldots \right] . \qquad (3.11)$$

With equations (3.1) and (3.2), the result (3.10) yields

$$T_j = \mu^* \frac{\partial w}{\partial x_j} . \qquad (3.12)$$

The above being the mean stress-strain relation, μ^* is the effective axial shear modulus of the composite.

4. EFFECTIVE AXIAL SHEAR MODULUS.

If we use equations (2.4) and (2.5) in (3.11), we get the expanded form of μ^* in terms of c. If the infinite series is summed formally, we get the closed form

$$\frac{\mu^*}{\mu_1} = \frac{1}{1 - \left\{ \frac{2(\mu_2 - \mu_1)}{\mu_2 + \mu_1} c \right\}} \qquad (4.1)$$

The expression (4.1) gives the correct value μ_1 when c = 0. When $\mu_2 > \mu_1$, it increases with c. However, when c approaches $(\mu_2 + \mu_1)/[2(\mu_2 - \mu_1)] > \frac{1}{2}$, it becomes infinitely large. However, we note that we have looked upon the fibers as random perturbations over the matrix material and as such c should not exceed 0.5. For c = 0.5 we get $\mu^* = (\mu_1 + \mu_2)/2$, which is a consistent result, as the fiber geometry is random. If c exceeds 0.5 we should interchange the roles of the fibers and the matrix and we have

$$\frac{\mu^*}{\mu_2} = \frac{1}{1 - \left\{ \frac{2(\mu_1 - \mu_2)}{\mu_1 + \mu_2} (1 - c) \right\}} .$$

This case is, however, not very important from a practical point of view.

Hashin [10] has given lower and upper bounds for μ^*. If these are denoted by μ_L^* and μ_U^*, then in our notations

$$\frac{\mu_L^*}{\mu_1} = \frac{\mu_2 + \mu_1 + c(\mu_2 - \mu_1)}{\mu_2 + \mu_1 - c(\mu_2 - \mu_1)} , \qquad (4.2)$$

$$\frac{\mu_U^*}{\mu_1} = \frac{\mu_2}{\mu_1}\left[\frac{\mu_2 + \mu_1 - (1 - c)(\mu_2 - \mu_1)}{\mu_2 + \mu_1 + (1 - c)(\mu_2 - \mu_1)}\right] . \tag{4.3}$$

It easily follows from (4.1) and (4.2) that $\mu^* > \mu_L^*$. Also the rate of increase of μ^* is higher than that of μ_L^* . For higher concentrations the deviation from the lower bound will then be pronounced. This behavior is generally in agreement with experiments. In relation to the upper bound however, $\mu^* > \mu_U^*$ for $c > (\mu_2 - \mu_1)/(2\mu_2) < \frac{1}{2}$.

5. NUMERICAL EXAMPLE.

We consider Boron fibers in Aluminum matrix. For such a composite $\mu_2/\mu_1 = 25/3.87$. We can easily compute the normalized shear moduli given by equations (4.1), (4.2) and (4.3) for different values of the concentration c. The results are graphically presented in Fig. 1:
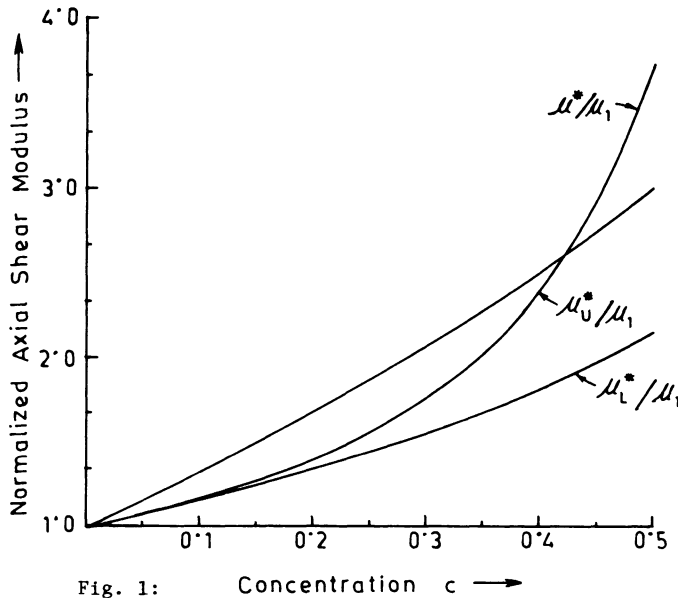

Fig. 1:   Concentration c ⟶

The figure indicates that μ^* is closer to the lower bound for values of c up to 0.35; thereafter, it is closer to the upper bound.

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