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SOLUTIONS OF TYPE r^m FOR A CLASS OF SINGULAR EQUATIONS

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ABSTRACT. We obtain all solutions of radial type for a class of singular partial differential equations of even order. The essential operators here are elliptic or ultrahyperbolic.

KEY WORDS AND PHRASES. Iterated elliptic or ultrahyperbolic equations, solutions of radial type, Lorentzian distance, hyperconoidal domain, homogeneous polynomial solutions, oscillatory solutions.

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1. INTRODUCTION.

This paper concerns solutions of type r^m for the class of partial differential equations

$$\left(\prod_{j=1}^{p} L_j^{q_j} \right) u = \left(L_1^{q_1} \cdots L_p^{q_p} \right) u = 0 \tag{1.1}$$

where p and $q_1, \ldots, q_p$ are positive integers and

$$L_j = \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\alpha_i^{(j)}}{x_i} \frac{\partial}{\partial x_i} \right) \pm \sum_{i=1}^{s} \left(\frac{\partial^2}{\partial y_i^2} + \frac{\beta_i^{(j)}}{y_i} \frac{\partial}{\partial y_i} \right) + \frac{\gamma_j}{r^2} . \tag{1.2}$$

The iterated operators $L_j^{q_j}$ are defined by the relations

$$L_j^{\upsilon+1} u = L_j [L_j^{\upsilon}(u)], \quad \upsilon = 1, \ldots, q_j - 1.$$

In (1.2), $\alpha_i^{(j)}$, $i = 1, \ldots, n$, $\beta_i^{(j)}$, $i = 1, \ldots, s$, and γ_j are any real parameters and

$$r^2 = \sum_{i=1}^{n} x_i^2 \pm \sum_{i=1}^{s} y_i^2 = x^2 \pm y^2 \tag{1.3}$$

where $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_s)$ denote points in R^n and R^s, respectively. The operators L_j are elliptic or ultrahyperbolic with the sign positive or negative, respectively. Equation (1.1) includes iterated forms of some well known classical equations such as the Laplace equation, the wave equation, and the EPD and GASPT equations. Many of these equations were studied by many authors in solving some physical problems or extending known results [1-6].

Before we find solutions of type r^m of equation (1.1), we note that if the operators L_j are ultrahyperbolic then, since the Lorentzian distance

$$r = \sqrt{x^2 + (iy)^2} = \sqrt{x^2 - y^2}\ , \quad i = \sqrt{-1}$$

is not real for $|x| < |y|$, solutions of type r^m are valid only in the hyperconoidal domain

$$D_n \times D_s = \{(x,y) \mid x \in D_n,\ y \in D_s,\ |y| < |x|\}$$

where $|x| = \sqrt{x_1^2 + \ldots + x_n^2}$, $|y| = \sqrt{y_1^2 + \ldots + y_s^2}$, and D_n and D_s define spherical domains centered at the origin in R^n and R^s, respectively. We also note that, since $r = 0$ on the hypercone

$$\partial(D_n \times D_s) = \{(x,y) \mid x \in D_n,\ y \in D_s,\ x^2 = y^2\}$$

solutions of type r^m have singularities at the points of this hypercone surface.

In the case where the operators L_j are elliptic, the regularity domain of solutions of r^m is a spherical domain centered at the origin in R^{n+s}. In this case, there is only a singularity at $r = 0$. However, we shall see that equation (1.1) has some polynomial solutions which are regular at all points.

2. SOLUTIONS OF TYPE r^m.

We first establish the following lemma.

LEMMA 1. Let $q_1,\ldots,q_p$ be any positive integers. Then

$$\Big(\prod_{j=1}^{p} L_j^{q_j}\Big)(r^m) = \prod_{j=1}^{p}\prod_{k=0}^{q_j-1}\{(m - 2[Q(p) - Q(j)] - 2k)$$

$$\cdot(m-2[Q(p) - Q(j)] - 2k + 2\phi_j) + \gamma_j\}\ r^{m-2Q(p)} \qquad (2.1)$$

where $Q(j) = q_1 + \ldots + q_j$, $1 \le j \le p$, and

$$2\phi_j = n + s - 2 + \sum_{i=1}^{n} \alpha_j^{(j)} + \sum_{i=1}^{s} \beta_i^{(j)} \qquad (2.2)$$

PROOF. From the definition of L and r, we have

$$L_j (r^m) = [m(m + 2\phi_j) + \gamma_j]\, r^{m-2} \tag{2.3}$$

Applying the operator L_j repeatedly on both sides of (2.3), we then obtain by induction the result

$$L_j^q (r^m) = \prod_{k=0}^{q-1} [(m - 2k)(m - 2k + 2\phi_j) + \gamma_j]\, r^{m-2q} \tag{2.4}$$

Now replace j and q in (2.4) by p and q_p, respectively, and apply in succession the operators

$$L_{p-1}^{q_{p-1}},\ L_{p-2}^{q_{p-2}},\dots,\ L_1^{q_1}$$

on both sides of (2.4). By induction, we readily obtain formula (2.1).

Using Lemma 1, we can now prove

THEOREM 1. Let the index set $I = \{j = 1,2,\dots,p\}$ be separated into three parts I_0, I_1 and I_2, such that

$$I_0 = \{j \in I,\ \phi_j^2 - \gamma_j = 0\}$$

$$I_1 = \{j \in I,\ \phi_j^2 - \gamma_j > 0\}$$

$$I_2 = \{j \in I,\ \phi_j^2 - \gamma_j < 0\}$$

Then, solution of the type $u = r^m$ of equation (1.1) are given by the formula

$$\begin{aligned} u(r) = {} & \sum_{j\in I_0} \sum_{k=0}^{q_j-1} r^{2[Q(p)-Q(j)]-\phi_j+2k} [C_{kj}^{(1)} + C_{kj}^{(2)} \log r] \\ & + \sum_{j\in I_1} \sum_{k=0}^{q_j-1} r^{2[Q(p)-Q(j)]-\phi_j+2k} [C_{kj}^{(1)}\, r^{\sqrt{\phi_j^2 - \gamma_j}} + C_{kj}^{(2)}\, r^{-\sqrt{\phi_j^2 - \gamma_j}}] \\ & + \sum_{j\in I_2} \sum_{k=0}^{q_j-1} \lambda_{kj}\, r^{2[Q(p)-Q(j)]-\phi_j+2k} \cos[\sqrt{\gamma_j - \phi_j^2} \log r + \omega_{kj}] \end{aligned} \tag{2.5}$$

where $C_{kj}^{(1)}$, $C_{kj}^{(2)}$, λ_{kj} and ω_{kj} are arbitrary constants.

PROOF. In Lemma 1, if we set

$$m - 2[Q(p) - Q(j)] - 2k = M, \tag{2.6}$$

then (2.1) can be written as

$$\left(\prod_{j=1}^{p} L_j^{q_j}\right)(r^m) = \prod_{j=1}^{p} \prod_{k=0}^{q_j-1} [M(M + 2\phi_j) + \gamma_j]\, r^{m-2Q(p)} \tag{2.7}$$

Since the roots of the algebraic equation

$$M(M + 2\phi_j) + \gamma_j = 0 \tag{2.8}$$

are $M = -\phi_j \pm\sqrt{\phi_j^2 - \gamma_j}$, we have from (2.6) the roots for m as

$$m_{kj}^{(1)} = 2[Q(p) - Q(j)] - \phi_j + 2k + \sqrt{\phi_j^2 - \gamma_j}$$

$$m_{kj}^{(2)} = 2[Q(p) - Q(j)] - \phi_j + 2k - \sqrt{\phi_j^2 - \gamma_j}$$

Thus, (2.7) can be written as

$$\left(\prod_{j=1}^{p} L_j^{q_j}\right)(r^m) = \prod_{j=1}^{p}\prod_{k=0}^{q_j-1} (m - m_{kj}^{(1)})(m - m_{kj}^{(2)})\, r^{m-2Q(p)} \tag{2.9}$$

from which it is easily seen that, for $j = 1,\dots,p$ and $k = 0,\dots,q_j-1$, the functions $r^{m_{kj}^{(1)}}$ and $r^{m_{kj}^{(2)}}$ satisfy the given equation. Since the given equation is linear, the sum

$$\sum_{j=1}^{p}\sum_{k=0}^{q_j-1} [C_{kj}^{(1)} r^{m_{kj}^{(1)}} + C_{kj}^{(2)} r^{m_{kj}^{(2)}}] \tag{2.10}$$

also satisfies (1.1).

If $j \in I_1$, then (2.8) has two distinct real roots. Thus, the corresponding solution of (2.10) will be

$$\sum_{j\in I_1}\sum_{k=0}^{q_j-1} r^{2[Q(p)-Q(j)]-\phi_j+2k} [C_{kj}^{(1)} r^{\sqrt{\phi_j^2-\gamma_j}} + C_{kj}^{(2)} r^{-\sqrt{\phi_j^2-\gamma_j}}] \tag{2.11}$$

If $j \in I_2$, then (2.8) has complex roots. By the properties

$$r^{\pm i\sqrt{\gamma_j - \phi_j^2}} = e^{\pm i\sqrt{\gamma_j - \phi_j^2}\log r}$$

$$= \cos(\sqrt{\gamma_j - \phi_j^2}\log r) \pm i\sin(\sqrt{\gamma_j - \phi_j^2}\log r),$$

the corresponding solution of (2.10) will be

$$\sum_{j\in I_2}\sum_{k=0}^{q_j-1} \lambda_{kj}\, r^{2[Q(p)-Q(j)]-\phi_j+2k} \cos[\sqrt{\gamma_j-\phi_j^2}\log r + \omega_{kj}] \tag{2.12}$$

where $C_{kj}^{(1)} + C_{kj}^{(2)} = \lambda_{kj}\cos\omega_{kj}$ and $C_{kj}^{(1)} - C_{kj}^{(2)} = i\,\lambda_{kj}\sin\omega_{kj}$.

If $j \in I_0$, then (2.8) has double root $m_{kj}^{(0)}$, that is,

$$m_{kj}^{(1)} = m_{kj}^{(2)} = 2[Q(p) - Q(j)] - \phi_j + 2k = m_{kj}^{(0)} .$$

So the right hand side of (2.9) has the factors $[m - m_{kj}^{(0)}]^2$ which itself and its first derivative with respect to m are zero for $m = m_{kj}^{(0)}$. Thus, each of the functions

$$r^{m_{kj}^{(0)}} \quad \text{and} \quad \frac{d}{dm}(r^m)\Big|_{m=m_{kj}^{(0)}} = r^{m_{kj}^{(0)}} \log r$$

satisfies the given equation. Hence, the corresponding solution of (2.10) will be

$$\sum_{j\in I_0} \sum_{k=0}^{q_j-1} r^{2[Q(p)-Q(j)]-\phi_j+2k} [C_{kj}^{(1)} + C_{kj}^{(2)} \log r] \tag{2.13}$$

Therefore, the sum of (2.11), (2.12), and (2.13) gives (2.5). Thus, the theorem is proved.

REMARK 1. In the special case where $\gamma_j = 0$, the algebraic equation (2.8) has the root M = 0. In this case, since the values

$$m_{kj} = 2[Q(p) - Q(j)] + 2k$$

are nonnegative integers for $j = 1,\ldots,p$ and $k = 0,\ldots,q_j-1$, the functions

$$r^{2[Q(p) - Q(j)] + 2k}$$

are homogeneous polynomial solutions of equation (1.1). From (2.5), we see that it is not possible to obtain polynomial solutions for equation (1.1) in all cases where $\gamma_j \neq 0$. It is possible, however, if the following condition is satisfied.

If the algebraic equation (2.8) has integral roots M_υ for some $j = \upsilon \in I$, such that

$$m_{k\upsilon} = 2[Q(p) - Q(\upsilon)] + 2k + M_\upsilon \geq 0$$

for $k = 0,\ldots,q_\upsilon-1$, then the functions $r^{m_{k\upsilon}}$ are homogeneous polynomials solutions of equation (1.1). It is clear that the above inequality is always satisfied if $M_\upsilon \geq 0$.

REMARK 2. From (2.5), we see that, if $j \in I_2$, that is, the algebraic equation (2.8) has complex roots, equation (1.1) then has oscillatory solutions in the regularity domain of its solutions.

THEOREM 2. All solutions of type $u = f(r)$ for equation (1.1) can be expressed by formula (2.5)

PROOF. Consider operator (1.2). By direct calculation, it is easily verified that

$$L_j = \frac{d^2}{dr^2} + \frac{1+2\phi_j}{r}\,\frac{d}{dr} + \frac{\gamma_j}{r^2} \tag{2.14}$$

which is an Euler type operator. If we set $r = e^t$ and $D = \frac{d}{dt}$, then we have

$$\frac{d}{dr} = e^{-t}\,D \quad \text{and} \quad \frac{d^2}{dr^2} = e^{-2t}\,(D^2 - D)$$

Hence,

$$L_j(u) = e^{-2t}\,(D^2 + 2\phi_j D + \gamma_j)\,u \tag{2.15}$$

If we let $F_j(D) = D^2 + 2\phi_j D + \gamma_j$, then (2.15) may be written as

$$L_j(u) = e^{-2t}\,F_j(D)\,u \tag{2.16}$$

From ordinary differential equations, we know that, for any polynomials with constant coefficients G and H and for any constant α, the following relation is valid

$$G(D)\,\{e^{-\alpha t}\,H(D)\,u\} = e^{-\alpha t}\,G(D-\alpha)\,H(D)\,u. \tag{2.17}$$

Considering the properties of (2.17) and applying the operator L_j repeatedly on both sides of (2.16), we then obtain by induction the result

$$L_j^q\,(u) = e^{-2qt}\prod_{k=0}^{q-1} F_j(D-2k)\,u. \tag{2.18}$$

We remark that the product of the operators $\prod_j F_j(D)$ are commutative. Now replace j and q in (2.18) by p and q_p, respectively, and apply in succession the operators

$$L_{p-1}^{q_{p-1}},\ L_{p-2}^{q_{p-2}},\dots,\ L_1^{q_1}$$

on both sides of (2.18). By induction, we obtain the formula

$$\left(\prod_{j=1}^{p} L_j^{q_j}\right) u = e^{-2Q(p)t}\prod_{j=1}^{p}\prod_{k=0}^{q_j-1} F_j(D - 2k - 2[Q(p)-Q(j)])\,u \tag{2.19}$$

Equating this last expression to zero, we obtain an ordinary differential equation with constant coefficients and of order $2Q(p) = 2(q_1 + \dots + q_p)$. The indical equation for this equation is

$$\prod_{j=1}^{p}\prod_{k=0}^{q_j-1} F_j(m - 2k - 2[Q(p)-Q(j)]) = 0.$$

Substituting $m - 2k - 2[Q(p)-Q(j)] = M$ in this equation, we find

$$\prod_{j=1}^{p} \prod_{k=0}^{q_j-1} [M(M + 2\phi_j) + \gamma_j] = 0. \tag{2.20}$$

This was obtained previously on the right hand side of (2.7). It is obvious that the corresponding solution for this equation is given by (2.5).

We note that, if we substitute $u = r^m$ in (2.19) then, by considering $r^m = e^{mt}$ and $e^{-2Q(p)t} = r^{-2Q(p)}$ and

$$\prod_{j=1}^{p} \prod_{k=0}^{q_j-1} F_j(D - 2k - 2[Q(p)-Q(j)])\, e^{mt} = e^{mt} \prod_{j=1}^{p} \prod_{k=0}^{q_j-1} F_j(m - 2k - 2[Q(p)-Q(j)]),$$

we see that (2.19) reduces to (2.1).

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