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ON THE ASYMPTOTIC BIEBERBACH CONJECTURE

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ABSTRACT. The set S consists of complex functions f, univalent in the open unit disk, with $f(0) = f'(0) - 1 = 0$. We use the asymptotic behavior of the positive semidefinite FitzGerald matrix to show that there is an absolute constant N_o such that, for any $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ with $|a_3| \le 2.58$, we have $|a_n| < n$ for all $n > N_o$.

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1. INTRODUCTION.

Let S denote the class of all normalized univalent functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in the open unit disc D. The Bieberbach conjecture states that, for functions in S, one has $|a_n| \le n$ for all $n \in N$. It is known to be true for $n \le 6$. The best known estimate for all coefficients is $|a_n| \le (1.066)n$ (Horowitz [1]). On the other hand, Hayman's Regularity Theorem (Hayman [2]) states that $\lim_{n\to\infty} \frac{|a_n|}{n} \le 1$ for each $f \in S$, and that equality holds only for the Koebe function $K(z) = \frac{z}{(1-\eta z)^2}$, $|\eta| = 1$, for which $|a_n| = n$. This implies that $|a_n| \le n$ for $n \ge n_o(f)$.

Hayman [3] also proved that A_n/n tends to a limit, where A_n is the maximum of $|a_n|$ for all $f \in S$. It is still an open question as to whether this limit is equal to 1. The asymptotic Bieberbach conjecture asserts that $\lim_{n\to\infty} A_n/n = 1$, where $A_n = \max_{f\in S} |a_n|$.

Ehrig [4] has proved via the FitzGerald Inequality [5] that if $f \in S$ and $|a_3| \le C < 2.43$, then $|a_n| < n$ for all $n \ge N_o$, where N_o depends only on C and not (as in Hayman's Regularity Theorem) on f. This result is a proof of the Asymptotic Bieberbach Conjecture for a subclass of S.

In this paper, we apply the Asymptotic FitzGerald Inequalities to get, by elementary means, an improvement of Ehrig's result (Theorem 1) and the result in [6], (see Remark 2).

2. PRELIMINARY RESULTS.

THEOREM A. (FitzGerald Inequality, [5]). Let

$$f(z) = z + \sum_{k=2}^{\infty} a_k(f)\, z^k$$

be in S and define

$$q_{mn}(f) = q_{nm}(f) = \left(\sum_{j=1}^{n+m-1} \beta_j(m,n)\, b_j^2(f) \right) - b_m^2(f)\, b_n^2(f)$$

where $b_j(f) = |a_j(f)|$; $\beta_j(m,n) = \beta_j(n,m)$, $j \in N$, and for $m < n$:

$$\beta_j(m,n) = \begin{cases} m-|j-n| & \text{for } |j-n| < m \\ 0 & \text{if otherwise.} \end{cases}$$

Then the FitzGerald matrix

$$Q(f) = (q_{mn}(f))_{m,n \in N}$$

is positive semi-definite.

THEOREM B. (Asymptotic FitzGerald Inequalities [7]). Let $\{f_n\}, n \in N$, be a sequence of functions in S, such that

a) f_n converges locally uniformly to $f \in S$

b) $\liminf_{n\to\infty} b_n(f_n)/n \le \beta \le \limsup_{n\to\infty} b_n(f_n)/n$

c) $\alpha(f) = \lim_{n\to\infty} b_n(f)/n$

d) $d = \lim_{n\to\infty} \alpha(f_n)$.

Then $A = Q(j_1, j_2, \ldots, j_{r-1}, \alpha(f), \ldots, \alpha(j), \beta, d, \ldots, d)\ (f)$, defined below, is a positive semi-definite matrix.

Denote by E_{mn} the $m \times n$ matrix whose elements are all equal to one. Moreover, let $H_{mn}(f)$ be the m n matrix defined by its elements $h_{st}(f) = j_t^2 = b_{j_t}^2(f)$. We use the notation

$$Q(j_1,\ldots,j_{r-1})(f) = (q_{j_s j_t}(f))_{1\le s,t\le p}, M_p(x) = (m_{st}(x))_{1\le s,t\le p}$$

where

$$m_{st}(x) = \begin{cases} 7x^2/6 - x^4 & \text{for } s = t \\ x^2(1 - x^2) & \text{for } s \neq t \end{cases}$$

and $\delta = \limsup_{n\to\infty} \delta_n$ where $\delta_n = \sup_k b_n(f_k)/n$. Then matrix A has the following form:

$$\begin{bmatrix} Q(j_1,\ldots,j_{r-1})(f), & \alpha(f)H_{r-1,q-r}(f), & \beta^2H_{r-1,1}(f), & d^2H_{r-1,p-q}(f) \\ \alpha(f)H^T_{r-1,q-r}(f), & M_{q-r}(\alpha(f)), & \beta^2(1-\alpha^2(f))E_{q-r,1}, & d^2(1-\alpha^2(f))E_{q-r,p-q} \\ \beta^2H^T_{r-1,1}(f), & \beta^2(1 - \alpha^2(f))E_{1,q-r}, & (7\delta^2/6 - \beta^4)E_{1,1}, & d^2(1 - \beta^2)E_{1,p-q} \\ d^2H^T_{r-1,p-q}(f), & d^2(1 - \alpha^2(j))E_{p-q,q-r}, & d^2(1 - \beta^2)E_{p-q,1}, & M_{p-q}(d) \end{bmatrix}$$

where H^T is the transposed matrix of H.

THEOREM C. [6]. Let $f \in S$; if $|a_3| \le 2.042$, then $|a_n| < n$ for all $n \ge 2$.

3. MAIN RESULTS.

For the proof of the Theorem 1, we need the following lemmas:

LEMMA 1. Suppose that $n > 1$ and that

$$\alpha_n(H) = \sup_{f \in H} |a_n|,$$

where H is a compact subclass of S. Let $f(z) = z + a_2z^2 + \ldots$ be in H with $|a_n| = \alpha_n(H)$. Then f(z) satisfies the differential equation

$$z^2\{f'(z)\}^2 \frac{1}{a_n} \sum_{v=2}^{n} a_n^{(v)} f(z)^{-v-1} = n-1 + \sum_{v=1}^{n-1} \left(\frac{va_v}{a_n} z^{-n+v} + \frac{v\bar{a}_v}{\bar{a}_n} z^{n-v}\right). \quad (3.1)$$

Here, $a_n^{(v)}$ are the coefficients of $f(z)^v$, where

$$f(z)^v = \sum_{n=v}^{\infty} a_n^{(v)} z^n.$$

The proof of this lemma is completely similar to that of Theorem 1 in Schaeffer and Spencer ([8], p. 612).

As an application of the Lemma 1, we have the following:

LEMMA 2. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H$ is the extremal function maximizing $|a_3|$ such that $a_3 > 0$, then $2a_3 = a_2^2 + 2$.

The proof of this lemma is completely similar to that in Garabedian and Schiffer ([9], p. 118).

Hayman [2] showed that for each $f \in S$, the limits

$$\alpha(f) = \lim_{r\to 1} (1-r)^2 M_\infty(r,f) = \lim_{n\to\infty} \frac{|a_n(f)|}{n}$$

exist, where $M_\infty(r,f)$ is the maximum of $|f(z)|$ on $|z| = r$. The number $\alpha(0 \le \alpha \le 1)$ is called the Hayman index of f.

LEMMA 3. [10]. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$, and $|a_2|$ is given, then

$$\alpha(f) = \lim_{n\to\infty} \frac{|a_n(f)|}{n} \le 4b^{-2} \exp(2-4b^{-1})$$

where $b = 2 - (2 - |a_2|)^{1/2}$, and this inequality is sharp for $0 \le |a_2| \le 2$.

LEMMA 4. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ satisfies the conditions of the Lemma 2 with $|a_3| > 1$, then

$$\alpha(f) = \lim_{n\to\infty} \frac{|a_n(f)|}{n} \le 4C^{-2} \exp(2 - 4C^{-1})$$

where $C = 2 - \left[2 - \sqrt{2}(|a_3| - 1)^{1/2}\right]^{1/2}$.

PROOF. By Lemma 3, we have

$$\alpha(f) \le 4b^{-2} \exp(2 - 4b^{-1})$$

where $b = 2 - (2 - |a_2|)^{1/2}$. Since we may assume a_3 real positive (otherwise, we consider $e^{-i\theta}f(e^{i\theta}z) \in H$, where $0 \le \theta = -\frac{\arg a_3}{2} \le 2\pi$), we obtain that

$$b = 2 - (2 - |a_2|)^{1/2} = 2 - \left[2 - \sqrt{2}(|a_3| - 1)^{1/2}\right]^{1/2}$$

$$= 2 - \left[2 - \sqrt{2}(|a_3| - 1)^{1/2}\right]^{1/2} = C.$$

Hence,

$$\alpha(f) \le 4C^{-2} \exp(2 - 4C^{-1}).$$

LEMMA 5. [7]. Let $\{f_n\}$, $n \in N$, be a sequence of univalent functions in S, that converges locally uniformly to a function f in S and suppose that $\alpha(f) > 0$. Then $7\delta^2\ \alpha^2(f) \geq 6\beta^4$, where β and δ are chosen as in theorem B.

PROOF. Consider the $(q - r + 1) \times (q - r + 1)$ principal minor

$$Q(\alpha(f),\dots,\alpha(f),\beta) = \begin{bmatrix} M_{q-r}(\alpha(f)) & \beta^2(1-\alpha^2(f))E_{q-r,1} \\ \beta^2(1-\alpha^2(f))E_{1,q-r} & (7\delta^2/6-\beta^4)E_{1,1} \end{bmatrix}$$

of the matrix A in theorem B. A well-known result about positive semidefinite quadratic form is that all principal minor determinants of the matrix of the coefficients of the quadratic form are non-negative. Let $\alpha = \alpha(f)$ and $n = q-r$. If we use induction, we obtain:

$$\operatorname{Det} Q(\alpha,\dots,\alpha,\beta) = (\alpha^2/6)^n\ (1-\alpha^2)\ \left[n(7\delta^2-6\beta^4/\alpha^2)\right] + \alpha^{2n}\ 6^{-(n+1)}\ (7\delta^2-6\beta^4/\alpha^2) \geq 0;$$

hence, for $0 \leq \alpha < 1$

$$(7\delta^2\ \alpha^2 - 6\beta^4) + \frac{6(1-\alpha^2)\beta^4}{6n(1-\alpha^2)+1} \geq 0.$$

Since n is arbitrary, the result follows. The case $\alpha = 1$ is immediate.

THEOREM 1. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be in S. If $1 \leq |a_3| \leq 2.58$, then there is an absolute constant N_o (independent of f), such that $|a_n| < n$ for all $n > N_o$.

PROOF. Suppose the contrary and take a sequence $\{g_k\}$, $k \in N$, of univalent functions in S such that

i) $\{g_k\}$, $k \in N$, converges locally uniformly to a function $g_o \in S$,

ii) $1 \leq b_3(g_k) = |a_3(g_k)| \leq 2.58$,

iii) $2a_3(g_k) = a_2(g_k)^2 + 2$,

iv) $b_{n_k}(g_k) \geq n_k$ for sequence n_k going to infinity.

REMARK. The functions g_k are the extremal functions maximizing $|a_3(g_k)|$ in the compact subclass $H_k = \{g \in S;\ 1 \leq b_3(g) \leq 2.58 \text{ and } b_{n_k}(g) \geq n_k\}$ of S. Applying Lemma 2 to the subclass H_k, we obtain condition (iii).

We pick for each n_k one of the functions of

$$\{g_j\},\quad j = 0,1,\dots,$$

which maximizes b_{n_k} and denote it by f_{n_k}; precisely, let $\{f_{n_k}\}$, $k \in N$, be a sequence of the functions in

$$\{g_j\},\ j = 0,1,\ldots,$$

such that

$$\sup_j b_{n_k}(g_j) = b_{n_k}(f_{n_k}).$$

We may assume that $\{f_{n_k}\}$, $k \in N$, converges locally uniformly to a function $f \in S$. Otherwise, we pick a subsequence of $\{f_{n_k}\}$, $k \in N$. Evidently, $1 \le b_3(f) \le 2.58$ and $2a_3(f) = a_2(f)^2 + 2$. For this sequence $\{f_{n_k}\}$, $k \in N$, we have

$$\delta_{n_k} = \sup_j b_{n_k}(f_{n_j})/n_k = b_{n_k}(f_{n_k})/n_k.$$

Thus

$$\delta = \limsup_{k\to\infty} b_{n_k}(f_{n_k})/n_k > 1.$$

We take $\beta = \delta$ in Theorem B. First we show that $\alpha(f) > 0$. In fact, the determinant of the 2×2 submatrix $Q(\alpha(f),\delta)$ of $A = Q(j_1,\ldots,j_{r-1},\ \alpha(f),\ldots,\alpha(f),\beta,d,\ldots,d)(f)$ is

$$(7\alpha^2(f) - 6\alpha^4(f))\ (7\delta^2 - 6\beta^4)/36 - \delta^4(1 - \alpha^2(f))^2 \ge 0.$$

This excludes $\alpha(f) = 0$ because $\delta \ge 1$. By Lemma 5, we have

$$7\alpha^2(f)\delta^2 - 6\delta^4 \ge 0 \quad \text{or} \quad \alpha^2(f) \ge 6\delta^2/7 \ge 6/7.$$

This implies, by Lemma 4, that $b_3(f) > 2.58$ which contradicts the assumptions.

COROLLARY. Let $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ be in S. If $|a_3| \le 2.58$, then there is an absolute constant N_o (independent of f), such that $|a_n| < n$ for all $n > N_o$.

PROOF. The proof of corollary follows immediately from Theorem 1 and Theorem C.

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