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RESEARCH NOTES

ALMOST CONVEX METRICS AND PEANO COMPACTIFICATIONS

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ABSTRACT. Let (X,d) denote a locally connected, connected separable metric space. We say the X is S-*metrizable* provided there is a topologically equivalent metric ρ on X such that (X,ρ) has Property S, i.e., for any $\varepsilon > 0$, X is the union of finitely many connected sets of ρ-diameter less than ε. It is well-known that S-metrizable spaces are locally connected and that if ρ is a Property S metric for X, then the usual metric completion $(\tilde{X},\tilde{\rho})$ of (X,ρ) is a compact, locally connected, connected metric space; i.e., $(\tilde{X},\tilde{\rho})$ is a Peano compactification of (X,ρ). In an earlier paper, the author conjectured that if a space (X,d) has a Peano compactification, then it must be S-metrizable. In this paper, that conjecture is shown to be false; however, the connected spaces which have Peano compactificatons are shown to be exactly those having a totally bounded, almost convex metric. Several related results are given.

KEY WORDS AND PHRASES. *Almost Convex Metrics, Property S metrics, Peano spaces, Compactifications.*

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1. INTRODUCTION.

Throughout this note let (X,d) denote a metric space. We say that d is convex provided that, for any pair $x,y \in X$, there is $z \in X$ such that $d(x,z) = d(z,y) = d(x,y)/2$. It is *almost convex* if, for $x,y \in X$ and $\varepsilon > 0$, there is $z \in X$ such that $|d(x,z)-d(x,y)/2| < \varepsilon$ and $|d(z,y)-d(x,y)/2| < \varepsilon$ [1,2].

We say that X is S-*metrizable* provided there is a topologically equivalent metric ρ on X such that (X,ρ) has Property S, i.e., for any $\varepsilon > 0$, X is the union of finitely many connected sets of ρ-diameter less than ε. It is well-known that S-metrizable spaces are locally connected and that if ρ is a Property S metric for X, then the usual metric completion $(\tilde{X},\tilde{\rho})$ of (X,ρ) is a compact, locally connected, connected metric space, i.e., $(\tilde{X},\tilde{\rho})$ is a Peano compactification of (X,ρ) [3, p. 154].

It is a famous result of R. H. Bing that any continuous curve P (i.e., a compact, locally connected, connected metric space) can be assigned a convex metric [1].

In an earlier paper [4], the author conjectured that, if X is locally connected and if X has a Peano compactification, then X is S-metrizable. In this paper we show, by example, that this conjecture is false; however, we do obtain a characterization of such spaces in terms of the existence of a totally bounded, almost convex metric for X. We also obtain several related results characterizing totally bounded (S-metrizable, almost convex) metrics.

2. PEANO COMPACTIFICATIONS.

THEOREM 2.1. A connected metric space (X,d) has a Peano compactification if and only if it has a topologically equivalent totally bounded, almost convex metric.

PROOF. The necessity. Let (P,r) be a Peano compactification of X, i.e., P is a continuous curve and X is a dense subset of P. By R. H. Bing's result, there exists an equivalent metric ρ for P such that ρ is convex. It then follows that $\sigma = \rho | X$ is totally bounded and almost convex; cf. [1, Thm. 10].

The Sufficiency. Let r be an almost convex, totally bounded metric for X. Let $(\tilde{X},\tilde{r})$ be the usual metric completion of (X,r). We will argue that $(\tilde{X},\tilde{r})$ is a Peano compactification of (X,r). Clearly, $\tilde{X}$ is compact since r is totally bounded. Furthermore, $\tilde{r}$ is a convex metric for $\tilde{X}$; let $x,y \in \tilde{X}$. Since r is almost convex, there exists a sequences $x_1,x_2,\ldots,y_1,y_2,\ldots$, and $z_1,z_2,\ldots$ in X such that

$$|r(x_n,z_n) - r(x_n,y_n)/2| < 2^{-n} \text{ and } |r(z_n,y_n) - r(x_n,y_n)/2| < 2^{-n}.$$

Since r is totally bounded, without loss of generality, we may assume that each of the sequences $x_1,x_2,\ldots,y_1,y_2,\ldots$, and $z_1,z_2,\ldots$ is Cauchy in X. Then by the completeness of $(\tilde{X},\tilde{r})$, it follows that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Furthermore, if $\lim_{n\to\infty} z_n = z$, $\tilde{r}(x,z) = \tilde{r}(z,y) = \tilde{r}(x,y)/2$ since $\tilde{r}$ is continuous. Thus $\tilde{r}$ is convex and complete. It follows from Theorem 3.1 of [5] that the spheres $S_{\tilde{r}}(x,\varepsilon)$ of $\tilde{X}$ are connected sets. This implies that $\tilde{X}$ is locally connected and this completes the proof.

EXAMPLE 2.1. Let P be the square $\{(x,y)\in\mathbb{R}^2 : 0 \leq x,y \leq 1\}$ in the plane. For $n\in\mathbb{N}$, let $L_n = \{(1/n,y): 0 < y < 1\}$ and let $L_0 = \{(0,y): 0 < y < 1\}$. Set $X = P\setminus\bigcup_{n=0}^{\infty} L_n$. Then P is a Peano compactification of X; however, X is not S-metrizable. Suppose ρ is an S-metric for X and let $A = \{(x,1): 0 \leq x \leq 1\}$ and $B = \{(x,0): 0 \leq x \leq 1\}$. Then A and B are compact and hence $\rho(A,B) = \varepsilon > 0$. Now the components C_1, C_2, ... of $X\setminus(A \cup B)$ have limit points in each of A and B. Thus, any collection of connected sets of ρ-diameter less than $\varepsilon/3$ that covers a component C_n has at least one such connected subset lying entirely in C_n. This implies that ρ is not an S-metric for X; however, if d is the relative metric on X inherited from the usual metric on P, d is almost convex and totally bounded.

3. RELATED RESULTS.

A compatible normal sequence in a space Z is a sequence $\mathcal{U}_1,\mathcal{U}_2,\ldots$ of open covers of Z such that $\mathcal{U}_{n+1}$ star-refines $\mathcal{U}_n$ for $n = 1,2,\ldots$ and so, for any $x\in Z$, $\{St(x,\mathcal{U}_n): n = 1,2,\ldots\}$ is a neighborhood base for x [5].

THEOREM 3.1. [6, Prop. 23.4] A T_0-space is metrizable if and only if it has a compatible normal sequence.

COROLLARY 3.1. A metric space X is totally bounded if and only if X has a compatible normal sequence $\mathcal{U}_1,\mathcal{U}_2,\ldots$ where each $\mathcal{U}_n$ is a finite cover of X.

PROOF. Suppose (X,d) is totally bounded. It follows from the total boundedness of (X,d) that there is a finite open cover $\mathcal{U}_1$ of X such that $\delta_d(U) = 1/3$ for all $U\in\mathcal{U}_1$ where $\delta_d(U) = \sup\{d(x,y): x,y\in U\}$, the d-diameter of U. Since $\mathcal{U}_1$ is finite, there is a Lebesgue number $\varepsilon_1 < 3^{-2}$ such that, if $d(x,y) < \varepsilon_1$, then x and y be in some member of $\mathcal{U}_1$. Again, by the total boundedness of (X,d), there is a finite open cover V_1 of x such that $\delta_d(V) < \varepsilon_1$. If $\varepsilon_2 < \varepsilon_1$ is a Lebesgue number for V_1 and $\mathcal{U}_2$ is any fini open cover of X such that $\delta_d(U) < \varepsilon_2$ for any $U \in \mathcal{U}_2$, then $\mathcal{U}_2$ star-refines $\mathcal{U}_1$. Continue in this manner and obtain a compatible normal sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ for X.

On the other hand, suppose $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is a compatible normal sequence for X where each $\mathcal{U}_n$ is finite. Then, in the usual metric ρ for X that is associated with $\mathcal{U}_1, \mathcal{U}_2, \ldots$ as given by S. Willard [6], $\delta_\rho(U) < 2^{n-1}$ and $U \in \mathcal{U}_n$, $n = 2,3,\ldots$. It then follows that, since each $\mathcal{U}_n$ is finite, ρ is a totally bounded metric for X. This completes the proof.

COROLLARY 3.2. A metric space (X,d) is S-metrizable if and only if it has a compatible normal sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ where each $\mathcal{U}_n$ is a finite cover and the members of $\mathcal{U}_n$ are connected sets.

PROOF. The necessity follows from the argument above, together with the observation that the covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$ can be selected so as to consist of finitely many open and connected sets.

The sufficiency. We observe that, if $\mathcal{U}_1, \mathcal{U}_2, \ldots$ is a compatible normal sequence for X where each $\mathcal{U}_n$ is finite and the members of $\mathcal{U}_n$ are connected sets and if ρ is the usual metric associated with $\mathcal{U}_1, \mathcal{U}_2, \ldots$ as given in [6], then, for $U \in \mathcal{U}_n$, $\delta_\rho(U) < 2^{n-1}$, $n = 2,3,\ldots$ and the sets $U \in \mathcal{U}_n$ are connected. Thus, for any $\varepsilon > 0$ and $k \in \mathbb{N}$ so that $0 < 2^{-k} < \varepsilon$, $x = \cup\{U : U \in \mathcal{U}_k\}$ is a finite cover of X by connected sets of ρ-diameter less that ε. This completes the proof.

THEOREM 3.2 [2]. A connected metric space X has an almost convex metric if and only if it has a compatible normal sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ such that (i) each pair of points that is covered by either an element of $\mathcal{U}_{n+1}$ or the union of a pair of intersecting elements of $\mathcal{U}_{n+1}$ can be covered by an element of $\mathcal{U}_n$ and (ii) each pair of points that can be covered by an element of $\mathcal{U}_n$ can be covered by the union of two intersecting elements of $\mathcal{U}_{n+1}$.

It is, apparently, very difficult to combine the total boundedness (finiteness) conditions of Corollaries 3.1 and 3.2 and the intersection-type properties of Theorem 3.2. It would be very desirable to do so in light of the results of the previous section.

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